

Barotropic Modon Propagation Over Slowly Varying Topography

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A perturbation theory is developed to describe modon propagation over slowly varying topography. The theory is developed from the rigid-lid shallow-water equations on an infinite β -plane. Nonlinear hyperbolic equations are derived, based on the conservation of energy, enstrophy and vorticity, to describe the evolution of the slowly varying modon radius, translation speed and wavenumber for arbitrary finite-amplitude topography. To leading order, the modon is unaffected by meridional gradients in topography. Analytical perturbation solutions for the modon radius, translation speed and wavenumber are obtained for small-amplitude topography. The perturbations take the form of hyperbolic transients and a stationary component proportional to the topography. The solution predicts that as the modon moves into a region of shallower (deeper) fluid the modon radius increases (decreases), the translation speed decreases (increases) and the modon wavenumber decreases (increases). In addition, as the modon propagates into a region of shallower (deeper) fluid there is an amplification (diminishing) of the extrema in the streamfunction and vorticity fields. These properties suggest that the modon may be able to be topographically-captured and amplified, and thus may have application to the onset of atmospheric blocking. The general solution is applied to mid-latitude scales and a ridge-like topographic feature.

KEY WORDS: Modon theory, solitary waves, solitons, nonlinear waves, geophysical fluid dynamics, oceanography, meteorology.

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INTRODUCTION

Whitham (1965) showed that the slow modulation of nonlinear waves in a dispersive medium could be described by the slow variation of the wave parameters (such as frequency, wavenumber and amplitude) within an averaged (over one wave period) Lagrangian formulation of the governing equations. Luke (1966), Johnson (1973) and Grimshaw (1970, 1971, 1979a, 1979b, 1981) have extended Whitham's perturbation method to examine slowly varying one-dimensional solitary waves. Kaup and Newell (1978) and Karpman and Maslov (1979a, b) have developed an inverse scattering theory (IST) to describe slowly varying solitary waves when the governing equations are amenable to IST. Warn and Brasnett (1983) have applied these methods to model atmospheric blocking as the slow modulation of atmospheric solitons over variable topography. The principal purpose of this paper is to describe a leading order perturbation theory for modon propagation over slowly varying topography.

The theory developed here is a development of Grimshaw's (see the earlier citations) perturbation theory for one-dimensional solitary waves, which is based on deriving hyperbolic transport equations for the slowly varying solitary wave parameters from averaged conservation laws.

The problem is formulated in Section 2. The theory is developed from the rigid-lid shallow-water equations on an infinite β -plane, for which, in the absence of topography, the eastward-travelling modon is an exact solution. In Section 3 the transport equations for the modon parameters are derived, based on averaged conservation laws for vorticity, energy and enstrophy. To leading order the modon parameters depend only parametrically on any meridional structure in the topography due to the scaling demand that the planetary vorticity gradient dominates topographic steering. The transport equations are valid for finite-amplitude topography. An exact solution to these nonlinear equations was not found.

In Section 4 analytical perturbation solutions are found for the transport equations in the limit of small-amplitude topography. For the initial-value problem, in which the modon is moving from a region of flat topography into a region of variable topography, the solutions for the modon parameters take the form of hyperbolic

transients and a “stationary” component proportional to the topography. The transients act to adjust the modon to the sudden imposition of variable topography at time zero, and the “stationary” component simply describes the “steady-state” evolution of the modon as it propagates over the variable topography.

In Section 5, the theory is applied for mid-latitude oceanic mesoscales and a ridge-like topographic feature. As the modon propagates into a region of deeper (shallower) fluid the modon contracts (dilates) and speeds up (slows down). The calculations also indicate that as the modon enters a region of shallower (deeper) fluid, the absolute values of the extrema in the streamfunction and vorticity fields increase (decrease). Thus the modon is amplified or diminished in shallower or deeper fluid, respectively. These calculations suggest that for finite-amplitude topography the modon may become topographically-captured and amplified, and thus may have some application to the formation of atmospheric blocks, such as that described in McWilliams (1980) or Warn and Brasnett (1983). The paper is summarized in Section 6.

2. PROBLEM FORMULATION

Following previous work [e.g. Veronis (1966), Rhines (1969a, b) and Clarke (1971), among others], the effects of variable topography on barotropic planetary waves will be assumed to be modelled by the nonlinear rigid-lid shallow water equations on an infinite β -plane, which in nondimensional form can be written as (LeBlond and Mysak, 1978; Sec. 20)

$$\nabla \cdot (H^{-1} \nabla \psi_t) + J[\psi, H^{-1} \nabla \cdot (H^{-1} \nabla \psi) + fH^{-1}] = 0, \quad (2.1a)$$

where $J[* , \cdot]$ is the Jacobian determinant $\partial(*, \cdot) / \partial(x, y)$ with x, y and t the usual east, north and time coordinates, respectively and where $\nabla = (\partial_x, \partial_y)$. The transport streamfunction, denoted ψ , is given by

$$-Hu = \psi_y, \quad Hv = \psi_x, \quad (2.1b)$$

where u and v are the (positive) eastward and (positive) northward velocity components, respectively. The space, time and velocity

scalings are a_0 , a_0/c_0 and c_0 , respectively, with a_0 and c_0 the initial values of the modon radius and translation speed, respectively.

The slowly varying topography is written

$$H(\varepsilon x, \varepsilon y) = 1 - \mu h(\varepsilon x, \varepsilon y),$$

where $\varepsilon = a_0/(\text{length scale of topography}) \ll 1$ and μ is the topographic amplitude parameter, which is the maximum absolute value of the height of the topography divided by the mean fluid depth. It will be assumed for now that $0 < \varepsilon \ll \mu \leq 1$. The leading order evolution equations which will be derived will be valid for finite-amplitude topography [i.e. $\mu \simeq O(1)$], although the smallness of μ (i.e., $0 < \varepsilon \ll \mu \ll 1$) will be eventually demanded in order to obtain analytical perturbation solutions for the slowly varying modon.

The nondimensional Coriolis parameter $f = (r_0)^{-1} + \delta^2 y$, where r_0 and δ^2 are the Rossby number $c_0(f_0 a_0)^{-1}$ and the planetary vorticity factor $\beta(a_0)^2/c_0$, respectively, with f_0 and β the local Coriolis parameter and local northward gradient in the Coriolis parameter, respectively. The effects of planetary vorticity dominate topographic steering if $(\delta^2 r_0)^{-1} O(\nabla H/H) \simeq \varepsilon (\delta^2 r_0)^{-1} < 1$ (LeBlond and Mysak, 1978; Sec. 20). Typical slopes of the ocean floor away from mid-ocean ridges, large seamounts and escarpments suggest $\varepsilon \simeq 10^{-3}$, thus $\varepsilon (\delta^2 r_0)^{-1} \simeq 10^{-1}$ (for $a_0 \simeq 10^5$ m and $c_0 \simeq 10^{-1}$ m s $^{-1}$ in mid-latitudes). Larger bottom slopes can be considered as r_0 is allowed to increase (e.g. atmospheric applications or in equatorial regions). The free surface effect has not been included in (2.1) since the external deformation radius is approximately 2000 km $\gg a_0$.

Following Grimshaw's analyses of slowly varying one-dimensional solitary waves, a perturbation solution to (2.1) will be constructed in the form

$$\begin{aligned} \psi = H(X, Y) \{ & A(X, Y, T) + \psi^{(0)}(\xi, y; X, Y, T) \\ & + \varepsilon \psi^{(1)}(\xi, y; X, Y, T) + \dots \}, \end{aligned} \quad (2.2a)$$

with the related perturbation velocity field

$$u = u^{(0)}(\xi, y; X, Y, T) + \varepsilon u^{(1)}(\xi, y; X, Y, T) + \dots, \quad (2.2b)$$

$$v = v^{(0)}(\xi, y; X, Y, T) + \varepsilon v^{(1)}(\xi, y; X, Y, T) + \dots, \quad (2.2c)$$

where

$$\xi = \varepsilon^{-1} \Theta(X, Y, T) \quad (2.2d)$$

is a rapidly varying phase of the slow variables

$$X = \varepsilon x, \quad Y = \varepsilon y, \quad T = \varepsilon t,$$

whose derivatives are given by

$$\xi_t = -c(X, Y, T)l(X, Y, T), \quad \xi_x = l(X, Y, T), \quad \xi_y = \varepsilon k(X, Y, T),$$

with the ‘‘conservation of crests’’ compatibility conditions

$$l_T + (cl)_X = 0, \quad \varepsilon k_T + (cl)_Y = 0, \quad l_Y = \varepsilon k_X. \quad (2.3)$$

The parameters $l(X, Y, T)$, $k(X, Y, T)$ and $c(X, Y, T)$ are the slowly varying *zonal* and *meridional* wavenumbers, and the zonal translation speed of the modon, respectively. Note that the slowly varying meridional wavenumber $\xi_y \simeq O(\varepsilon)$ in consequence of the original scaling demand that the planetary vorticity gradient dominates vorticity changes due to variable topography.

With the above scalings, derivatives are rewritten

$$\partial_x \rightarrow l \partial_\xi + \varepsilon \partial_X, \quad \partial_t \rightarrow -cl \partial_\xi + \varepsilon \partial_T, \quad \partial_y \rightarrow \partial_y + \varepsilon k \partial_\xi + \varepsilon \partial_Y.$$

Equation (2.1a) becomes at $O(1)$ simply:

$$J[\psi^{(0)} + cy, \Delta\psi^{(0)} + \delta^2 y] = 0, \quad (2.4)$$

where $J[* , \cdot] = \partial(*, \cdot) / \partial(\xi, y)$ and $\nabla = (l \partial_\xi, \partial_y)$, etc. The solution of (2.4) is taken to be the eastward-travelling barotropic modon (Flierl *et al.*, 1980)

$$\begin{aligned} \psi^{(0)} &= -caK_1(\delta c^{-1/2} r) \sin(\theta) / K_1(\delta ac^{-1/2}), \\ \Delta\psi^{(0)} &= -\delta^2 a K_1(\delta c^{-1/2} r) \sin(\theta) / K_1(\delta ac^{-1/2}), \\ \Delta\psi^{(0)} &= (\delta^2/c)\psi^{(0)}, \quad r > a; \end{aligned} \quad (2.5a)$$

$$\begin{aligned}\psi^{(0)} &= \delta^2 \kappa^{-2} a J_1(\kappa r) \sin(\theta) / J_1(\kappa a) - (\delta^2 + \kappa^2 c) \kappa^{-2} r \sin(\theta), \\ \Delta \psi^{(0)} &= -\delta^2 a J_1(\kappa r) \sin(\theta) / J_1(\kappa a), \\ \Delta \psi^{(0)} &= -\kappa^2 \psi^{(0)} - (\delta^2 + c \kappa^2) r \sin(\theta) \quad r < a;\end{aligned}\tag{2.5b}$$

where J_1 and K_1 are the ordinary and modified Bessel functions of order one, with the polar coordinates r and θ defined by $r^2 = [\xi - \xi_0(X, Y, T)]^2 / l^2 + y^2$ and $\tan(\theta) = ly / [\xi - \xi_0(X, Y, T)]$ and where $a(X, Y, T)$ and $\kappa(X, Y, T)$ are the slowly varying modon radius and modon wavenumber, respectively.

The term $\xi_0(X, Y, T)$ is an $O(\varepsilon)$ phase shift (in comparison to the leading order phase ξ) and is determined by first-order perturbation energy considerations (Ko and Kuehl, 1978; Grimshaw, 1979a, b; Kodama and Ablowitz, 1981). For the $O(1)$ analysis presented here it remains undetermined and is chosen as a constant (the x -coordinate of the wave centre at $T=0$). The y -coordinate of the wave center is taken to be $y=0$.

The ground-state modon wavenumber κ in (2.5b) is the first nonzero solution of the dispersion relation (obtained by requiring the continuity of $\nabla \psi^{(0)}$ on $r=a$)

$$-\delta J_2(\kappa a) K_1(\delta a c^{-1/2}) = c^{1/2} \kappa J_1(\kappa a) K_2(\delta a c^{-1/2}).\tag{2.5c}$$

For the initial value problem, $a(X, Y, 0)=1$, $c(X, Y, 0)=1$ and $\kappa(X, Y, 0)=\kappa_0$, where κ_0 is obtained from the dispersion relation when $T=0$ (e.g. $\kappa_0=3.9226$ when $\delta=1$).

3. CONSERVATION LAWS AND DERIVATION OF TRANSPORT EQUATIONS

In order to complete the description of the slowly varying modon, transport equations must be obtained for the slowly varying modon parameters A , a , c , κ , l and k . These equations are derived from the conservation of energy, potential vorticity and enstrophy. As mentioned in the introduction, the theory described in this section is a development of Grimshaw's (1970, 1971, 1979a, b and 1981) analyses for slowly varying one-dimensional solitary waves.

The required conservation laws can be expressed [in the original

(x, y, t) variables] as equations of the form (Pedlosky, 1979, Sec. 3.27)

$$E_t + \nabla \cdot \mathbf{F} = 0, \quad (3.1)$$

with

$$E = \nabla \cdot (H^{-1} \nabla \psi), \quad (3.2a)$$

$$\mathbf{F} = \mathbf{e}_3 \times \nabla \psi H^{-1} [\nabla \cdot (H^{-1} \nabla \psi) + f],$$

$$E = (2H)^{-1} \nabla \psi \cdot \nabla \psi, \quad (3.2b)$$

$$\mathbf{F} = -\psi H^{-1} \nabla \psi_t - \mathbf{e}_3 \times \nabla \psi \psi H^{-1} [\nabla \cdot (H^{-1} \nabla \psi) + f],$$

$$E = H^{-1} [\nabla \cdot (H^{-1} \nabla \psi) + f]^2, \quad (3.2c)$$

$$\mathbf{F} = \mathbf{e}_3 \times \nabla \psi H^{-2} [\nabla \cdot (H^{-1} \nabla \psi) + f]^2,$$

where $\mathbf{e}_3 = (0, 0, 1)$, corresponding to the conservation of potential vorticity, energy and potential enstrophy, respectively. When the perturbation expansion (2.2) is substituted in (3.1), expressions of the form

$$-clE_\xi^{(0)} + \nabla \cdot \mathbf{F}^{(0)} + \varepsilon E_T^{(0)} + \varepsilon \nabla_X \cdot \mathbf{F}^{(0)} - \varepsilon clE_\xi^{(1)} + \varepsilon \nabla \cdot \mathbf{F}^{(1)} + \dots = 0 \quad (3.3)$$

are obtained, where $\nabla_X = (\partial_X, \partial_Y)$ and $\nabla = (l\partial_\xi, \partial_y)$. The first-order perturbation quantity $E^{(1)}$ appearing in (3.3) will be a relatively complicated function of the form

$$E^{(1)} = E^{(1)}(\partial_X, \partial_Y, \partial_T, \partial_\xi, \partial_y, \psi^{(0)}, \psi^{(1)})$$

(with a similar expression for $\mathbf{F}^{(1)}$).

Following the development of one-dimensional theories (e.g. Grimshaw, 1979a, b; Ablowitz, 1971; Ablowitz and Kodama, 1979; Kodama and Ablowitz, 1980, 1981) it will be assumed that $\nabla \psi^{(1)} \rightarrow 0$ as $r \rightarrow \infty$. In addition $\psi^{(1)}$ will be assumed to be at least twice

continuously differentiable since $\psi^{(0)}$ is. It follows that $E^{(1)}$ and $\mathbf{F}^{(1)}$ will be continuous at $r=a$ and that both will vanish as $r \rightarrow \infty$.

The transport equation for $A(X, Y, T)$ is obtained from the conservation of potential vorticity [i.e. (3.3) with E and \mathbf{F} defined by (3.2a)]. Taking the limit $r \rightarrow \infty$, recalling $\psi^{(0)}$ and $\nabla\psi^{(1)} \rightarrow \mathbf{0}$ as $r \rightarrow \infty$, yields to $O(\varepsilon)$

$$[(AH)_X f]_y = 0$$

(from the $\nabla \cdot \mathbf{F}^{(1)}$ term), which reduces to simply

$$(AH)_X = 0. \quad (3.4)$$

As in one-dimensional theories (e.g. Grimshaw, 1979a, b), the leading term in the expansion (2.2a) (i.e. AH) satisfies the long-wave approximation to the governing Eq. (2.1). The remaining transport equations are obtained by averaging (3.3) over the fast variable domain [i.e. $-\infty < (\xi, y) < \infty$] for the energy (3.2b) and enstrophy (3.2c).

Define the averaging operator

$$\langle (*) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (*) d\xi dy.$$

Upon averaging (3.3), it follows that [to $O(\varepsilon)$]

$$\langle E^{(0)} \rangle_T + \nabla_X \cdot \langle \mathbf{F}^{(0)} \rangle = 0. \quad (3.5)$$

The operations of averaging and slow differentiation do not commute since $r=a(X, Y, T)$ is contained in the integration domain. However, all boundary integrals that result from interchanging differentiation and averaging sum to zero due to the continuity of $E^{(0)}$ and $\mathbf{F}^{(0)}$ at $r=a$. Also note that the averages of the $O(\varepsilon)$ fast variable divergences integrate to zero since $E^{(1)}$ and $\mathbf{F}^{(1)}$ are continuous at $r=a$ and vanish as $r \rightarrow \infty$.

It turns out that several terms in the leading order energy and enstrophy averages integrate to zero. (The physical significance of this fact will be discussed after the transport equations have been derived.) The component of the leading order energy and energy flux

that in the end give non-zero averages are, respectively,

$$E^{(0)} = H \nabla \psi^{(0)} \cdot \nabla \psi^{(0)} / 2 \quad (3.6a)$$

$$\mathbf{F}^{(0)} = \mathbf{e}_1 [H c l^2 \psi_{\xi\xi}^{(0)} \psi^{(0)} + \psi_y^{(0)} H \psi^{(0)} (\Delta \psi^{(0)} + \delta^2 y)], \quad (3.6b)$$

and similarly for the leading order enstrophy and enstrophy flux

$$E^{(0)} = (\Delta \psi^{(0)})^2 / H \quad (3.7a)$$

$$\mathbf{F}^{(0)} = -\mathbf{e}_1 \psi_y^{(0)} (\Delta \psi^{(0)} + \delta^2 y)^2 / H. \quad (3.7b)$$

After some algebra, the averaged energy transport equation can be put into the form

$$\partial_T (l a^2 c^2 E_1) + \partial_X (l a^2 c^3 F_1) = -l a^2 c^3 F_1 H^{-1} H_X, \quad (3.8)$$

with E_1 and F_1 nondimensional scalar functions given by

$$E_1 = \gamma^4 [4v^2 J_1^2(v)]^{-1} [J_1^2(v) - J_0(v)J_2(v)] - \gamma^2 (\gamma^2 + v^2) J_2(v) [2v^3 J_1(v)]^{-1} \\ + \gamma^2 [4K_1^2(\gamma)]^{-1} [K_1^2(\gamma) - K_0(\gamma)K_2(\gamma)], \quad (3.9a)$$

$$F_1 = -(\gamma^2 + v^2)/8 + \gamma^4 [8v^2 J_1^2(v)]^{-1} [J_1^2(v) - J_0(v)J_2(v)] \\ - 3\gamma^2 (\gamma^2 + v^2) J_2(v) [4v^3 J_1(v)]^{-1} + (\gamma^2 + v^2)^2 (8v^2)^{-1} \\ + \gamma^2 [8K_1^2(\gamma)]^{-1} [K_1^2(\gamma) - K_0(\gamma)K_2(\gamma)], \quad (3.9b)$$

respectively, where $\gamma^2 = \delta^2 a^2 / c$ and $v = \kappa a$. Similarly, the transport equation for the averaged enstrophy is found to be

$$\partial_T (l c^2 E_2) + \partial_X (l c^3 F_2) = l c^3 F_2 H^{-1} H_X, \quad (3.10)$$

where E_2 and F_2 are nondimensional functions given by, respectively,

$$E_2 = \gamma^4 [2J_1^2(v)]^{-1} [J_1^2(v) - J_0(v)J_2(v)] \\ - \gamma^4 [2K_1^2(\gamma)]^{-1} [K_1^2(\gamma) - K_0(\gamma)K_2(\gamma)], \quad (3.11a)$$

$$\begin{aligned}
F_2 = & \frac{1}{4}(v^4 - \gamma^4) + \gamma^4 [2J_1^2(v)]^{-1} [J_1^2(v) - J_0(v)J_2(v)] \\
& - 2\gamma^2(\gamma^2 + v^2)J_2(v)[vJ_1(v)]^{-1} + \frac{1}{4}(\gamma^2 + v^2)^2 \\
& + 2v^3\{\gamma^2 J_2(v)[vJ_1(v)]^{-1} - \frac{1}{4}(\gamma^2 + v^2)\} \\
& - \gamma^4 [2K_1^2(\gamma)]^{-1} [K_1^2(\gamma) - K_0(\gamma)K_2(\gamma)] \\
& - 2\gamma^3 K_2(\gamma)[K_1(\gamma)]^{-1}. \tag{3.11b}
\end{aligned}$$

The complete set of nonlinear equations, describing the slowly varying modon parameters A , l , c , a , κ and k are given by (2.3), (2.5c), (3.4), (3.8) and (3.10). Several qualitative observations can be made about these equations. Note that the leading term in the expansion (2.2a) (i.e. AH) has completely decoupled from the dynamics governing the other modon parameters. Thus, with no loss of generality in the theory developed here, we take $A \equiv 0$. However, if the free surface effect had been included in (2.1), (3.4) would have allowed the propagation of long nondispersive barotropic planetary waves. Westward-travelling modons require a free surface or deformable pycnocline in order to exist (see Flierl *et al.*, 1980), thus it seems that any extension of the present theory to westward-travelling or baroclinic modons would have to include a possible interaction between the modon and Rossby waves.

The dynamical Eqs. (3.8) and (3.10) do not contain any ∂_Y terms. Also, (2.3) implies $l_Y = c_Y = 0$. The meridional component of the energy and enstrophy fluxes to integrate to zero due to the periodicity in θ of the modon. Physically, the lack of any dependence of the parameters on meridional topographic gradients can be attributed to the scaling demand that the planetary vorticity gradient dominates topographic gradients. The essential physics retained in (3.8) and (3.10) is therefore the adjustment of the modon to zonal topographic gradients, due to the conservation of energy and relative enstrophy. Also, note that all terms proportional to the inverse Rossby number [recall $f = (r_0)^{-1} + \delta^2 y$] integrate to zero, due to the trigonometric integrations. Physically, these terms vanish because the $O(1)$ velocity field is horizontally nondivergent [see (2.1b) and (2.2a)].

In summary, the leading order description of the modon as it

propagates over slowly varying finite amplitude topography will be given by (2.5) with a , c , κ and l obtained from (2.3), (2.5c), (3.8) and (3.10) evaluated at the position of the modon center, say $X_c(T)$, given implicitly by

$$Y=0, \quad X_c(T) = \varepsilon \xi_0 + \int_0^T c(X_c, 0, T) dT. \quad (3.12a, b)$$

These nonlinear equations require, in general, a numerical solution since no exact solution could be found. However, the essential features of the slowly varying modon are described by exploiting the smallness of the topographic amplitude parameter μ (i.e. $0 < \varepsilon \ll \mu \ll 1$, which is the case in many applications) in order to obtain an analytical perturbation solution.

4. PERTURBATION SOLUTION FOR SMALL-AMPLITUDE TOPOGRAPHY

The governing equations for the leading order modon parameters are displayed here again for convenience, in the matrix form

$$\mathbf{A} \cdot \mathbf{b}_T + \mathbf{B} \cdot \mathbf{b}_X = \mathbf{G} H^{-1} H_X, \quad (4.1)$$

$$c_Y = l_Y = 0, \quad (4.2)$$

$$-\delta J_2(\kappa a) K_1(\delta a c^{-1/2}) = c^{1/2} \kappa J_1(\kappa a) K_2(\delta a c^{-1/2}), \quad (4.3)$$

where \mathbf{A} and \mathbf{B} are the 3×4 Jacobian matrices

$$\mathbf{A} = \partial(l, la^2c^2E_1, lc^2E_2)/\partial(l, a, c, \kappa),$$

$$\mathbf{B} = \partial(cl, la^2c^3F_1, lc^3F_2)/\partial(l, a, c, \kappa),$$

respectively, and \mathbf{b} and \mathbf{G} are the column vectors defined by $(l, a, c, \kappa)^t$ and $(0, -la^2c^3F_1, lc^2F_2)^t$, respectively, with $(*)^t$ the transpose of $(*)$.

When $0 < \varepsilon \ll \mu \ll 1$ (the small-amplitude topography approxi-

mation), solutions to (4.1) and (4.3) can be constructed in the form

$$a = 1 + \mu a^{(1)}(X, T) + O(\mu^2), \quad c = 1 + \mu c^{(1)}(X, T) + O(\mu^2), \quad (4.4a, b)$$

$$\kappa = \kappa_0 [1 + \mu \kappa^{(1)}(X, T) + O(\mu^2)], \quad l = 1 + \mu l^{(1)}(X, T) + O(\mu^2), \quad (4.4c, d)$$

$$H^{-1}H_X = -\mu h_X(X, 0, T) + O(\mu^2). \quad (4.4e)$$

Note that $a^{(1)}$, $c^{(1)}$, $\kappa^{(1)}$ and $l^{(1)}$ represent μ -perturbations and are not to be confused with the role played by $\psi^{(1)}$, $u^{(1)}$ and $v^{(1)}$ in the ε -perturbation expansion in (2.2).

The $O(\mu)$ terms in (4.1) and (4.3) are, respectively,

$$\mathbf{A}_0 \cdot \mathbf{b}_T^{(1)} + \mathbf{B}_0 \cdot \mathbf{b}_X^{(1)} = -\mathbf{g}h_X, \quad (4.5)$$

$$\kappa^{(1)} = N(a^{(1)} - \frac{1}{2}c^{(1)}) - \frac{1}{2}c^{(1)}, \quad (4.6)$$

where \mathbf{A}_0 and \mathbf{B}_0 are \mathbf{A} and \mathbf{B} evaluated for $a=c=l=1$ and $\kappa=\kappa_0$, respectively,

$$\mathbf{g} = [0, -(F_1)_0, (F_2)_0]^t, \quad N = -\{\delta R + (\kappa_0)^2 R/\delta\}/\{4 + \delta/R + (\kappa_0)^2 R/\delta\},$$

with $R = K_2(\delta)/K_1(\delta)$ and $\mathbf{b}^{(1)} = (l^{(1)}, a^{(1)}, c^{(1)}, \kappa^{(1)})^t$. The perturbation wavenumber $\kappa^{(1)}$ can be eliminated from (4.5) by direct substitution of (4.6) into (4.5), to give the linear dynamical system

$$\mathbf{E} \cdot \mathbf{e}_T + \mathbf{F} \cdot \mathbf{e}_X = -\mathbf{g}h_X, \quad (4.7)$$

where \mathbf{E} and \mathbf{F} are the 3×3 matrices whose components (in the standard notation) are defined by,

$$E_{11} = 1, \quad E_{1i} = 0, \quad E_{i1} = 0,$$

$$E_{i2} = (A_{i2} + NA_{i4})_0, \quad E_{i3} = [A_{i3} - \frac{1}{2}(N+1)A_{i4}]_0,$$

$$F_{11} = F_{13} = 1, \quad F_{12} = 0, \quad F_{ii} = (B_{ii} - A_{ii})_0,$$

$$F_{i2} = (B_{i2} + NB_{i4})_0, \quad F_{i3} = [B_{i3} - \frac{1}{2}(N+1)B_{i14} - A_{ii}]_0,$$

with $i=2, 3$, and $\mathbf{e}=(l^{(1)}, a^{(1)}, c^{(1)})^t$. Thus, the solution of (4.7) gives $l^{(1)}$, $a^{(1)}$ and $c^{(1)}$, and $\kappa^{(1)}$ is obtained from (4.6).

For the initial-value problem, the initial conditions on the solutions to (4.7) are taken to be

$$\mathbf{e}(T=0)=\mathbf{0}, \tag{4.8}$$

$$\mathbf{e}_T(T=0)=-\mathbf{E}^{-1} \cdot \mathbf{g}h_X, \tag{4.9}$$

$$\mathbf{e}_{TT}(T=0)=\mathbf{E}^{-1}\mathbf{F}\mathbf{E}^{-1} \cdot \mathbf{g}h_{XX}. \tag{4.10}$$

Condition (4.8) corresponds to no perturbation at $T=0$, (4.9) is obtained from (4.7) evaluated at $T=0$ and (4.10) is obtained from $(4.7)_T$ evaluated at $T=0$.

The general solution to (4.7) can be written in the form

$$\begin{aligned} \mathbf{e} = & -\mathbf{F}^{-1} \cdot \mathbf{g}h(X, 0) + \alpha_1 h(X - \sigma_1 T, 0) + \alpha_2 h(X - \sigma_2 T, 0) \\ & + \alpha_3 h(X - \sigma_3 T, 0), \end{aligned} \tag{4.11}$$

where α_1 , α_2 and α_3 are 1×3 column vectors of constants determined from the initial conditions and σ_1 , σ_2 and σ_3 are the (expected real) roots of

$$\det(\mathbf{E}\sigma - \mathbf{F})=0. \tag{4.12}$$

The initial conditions (4.8), (4.9) and (4.10) for the solution (4.11) can be put into the form

$$\mathbf{R} = \mathbf{W}\mathbf{S},$$

where \mathbf{R} , \mathbf{S} and \mathbf{W} are 3×3 matrices defined as follows. The elements of the first, second and third rows of \mathbf{R} are the elements of $\mathbf{F}^{-1} \cdot \mathbf{g}$, $\mathbf{E}^{-1} \cdot \mathbf{g}$ and $\mathbf{E}^{-1}\mathbf{F}\mathbf{E}^{-1} \cdot \mathbf{g}$, respectively. The elements of the first, second and third rows of \mathbf{S} are the elements of α_1 , α_2 and α_3 , respectively. And finally, the first, second and third rows of \mathbf{W} are the vectors $(1, 1, 1)$, $(\sigma_1, \sigma_2, \sigma_3)$ and $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$, respectively. Hence the constant vectors α_1 , α_2 and α_3 are determined from

$$\mathbf{S} = \mathbf{W}^{-1}\mathbf{R}, \tag{4.13}$$

i.e. are the rows of $\mathbf{W}^{-1}\mathbf{R}$, respectively. The position of the modon center for $\mu \ll 1$ will be given by

$$X_c(T) = \varepsilon \xi_0 + T + \mu \int_0^T c^{(1)}(X_c, T) dT + O(\mu^2), \quad (4.14)$$

which completes the analytical perturbation solution.

5. AN EXAMPLE CALCULATION WITH RIDGE-LIKE TOPOGRAPHY

In this section the qualitative properties of the solution are examined by way of an example calculation. Consider midlatitude oceanic mesoscale parameter values of $f_0 \sim 10^{-4} \text{s}^{-1}$, $\beta \sim 10^{-11} \text{m}^{-1} \text{s}^{-1}$, $a_0 \sim 100 \text{km}$ and $c_0 \sim 10 \text{cms}^{-1}$; thus $\delta = 1$ and hence $\kappa_0 = 3.9226$ [cf. (2.5c)]. In order to make the topography specific consider a ridge-like topographic configuration modelled with the Gaussian

$$H(X) = 1 - \mu \exp(-X^2), \quad (5.1a)$$

so that

$$h(X) = \exp(-X^2). \quad (5.1b)$$

The above scales yield the parameter values (see Section 4 for the relevant definitions):

$$\sigma_1 = 1.00, \quad \sigma_2 = -24.53, \quad \sigma_3 = 0.61, \quad N = -0.96,$$

$$\mathbf{F}^{-1} \cdot \mathbf{g} = (-0.69, -0.84, 0.69)^t, \quad \boldsymbol{\alpha}_1 = (0.14, -0.39, 0.00)^t,$$

$$\boldsymbol{\alpha}_2 = (-0.02, -0.24, 0.37)^t, \quad \boldsymbol{\alpha}_3 = (-0.81, -0.21, 0.32)^t.$$

Consequently, the slowly varying modon radius, translation speed and wavenumber are determined by

$$a = 1 + \mu \{ 0.84h(X_c) - 0.39h(X_c - 1.00T) - 0.24h(X_c + 24.53T) - 0.21h(X_c - 0.61T) \}, \quad (5.2)$$

$$c = 1 + \mu \{-0.69h(X_c) + 0.37h(X_c + 24.53T) + 0.32h(X_c - 0.61T)\}, \quad (5.3)$$

$$\begin{aligned} \kappa = \kappa_0 + \mu\kappa_0 \{ & -0.82h(X_c) + 0.37h(X_c - 1.00T) \\ & + 0.24h(X_c + 24.53T) + 0.21h(X_c - 0.61T)\}, \end{aligned} \quad (5.4)$$

respectively, where $X_c(T)$ will be determined by (4.14) and (5.3).

The hyperbolic waves in the solutions (5.2), (5.3) and (5.4) are transients required to adjust the modon to the sudden imposition of a variable medium at $T=0$. When the topography is isolated and flat near the initial position of the modon [i.e. $h(\varepsilon\xi_0) \sim 0$ with $\varepsilon\xi_0 \ll 0$, say], then in the neighborhood of the topography (i.e. $X \sim 0$) the *transients* make no contribution and the solutions for the modon parameters are approximately given by

$$a \simeq 1 + 0.84\mu h(X_c), \quad (5.5)$$

$$c \simeq 1 - 0.69\mu h(X_c), \quad (5.6)$$

$$\kappa \simeq \kappa_0 - 0.82\mu\kappa_0 h(X_c). \quad (5.7)$$

These “steady-state” solutions are simply the small-amplitude solutions when the phase variable ξ is constructed in the form

$$\xi = -t + \varepsilon^{-1} \int_0^X c^{-1}(x') dx',$$

with $c(X)$ the leading order spatially slowly varying translation speed. In such a solution a *zonal* wavenumber is not needed as an additional parameter since in consequence of (2.3), $l = c^{-1}$ and thus $l^{(1)} = -c^{(1)}$.

The solutions (5.5), (5.6) and (5.7) predict that as the modon propagates over a topographic feature for which $h > 0$, the modon radius increases, the translation speed decreases and the modon wavenumber decreases. If the topography corresponded to a deepening of the fluid (i.e. $h < 0$), then these properties are reversed. In the absence of any variable topography (i.e. $h = 0$) the modon parameters do not, of course, change. These qualitative features are exactly those expected on the basis of heuristic continuity and vorticity arguments

for isolated, incompressible vortex tubes [see, for example, Pedlosky (1979) Chap. 3].

Figure 1a shows the position of the modon center $X_c(T)$ vs. T for the Gaussian-ridge topographic profile given by (5.1) with $\mu=0.2$, (μ has been chosen as large as 0.2 in order to accentuate topographically-induced effects), subject to the steady-state solutions (5.5), (5.6) and (5.7) with $X_c(0)=-4.0$. The slope of the curve depicted in Figure 1a is the slowly varying translation speed $c(T)$ vs. T , which is shown in Figure 1b. The minimum translation speed is given by $1-0.69\mu$ [=0.863 for $\mu=0.2$, cf. (5.6)] occurring when the

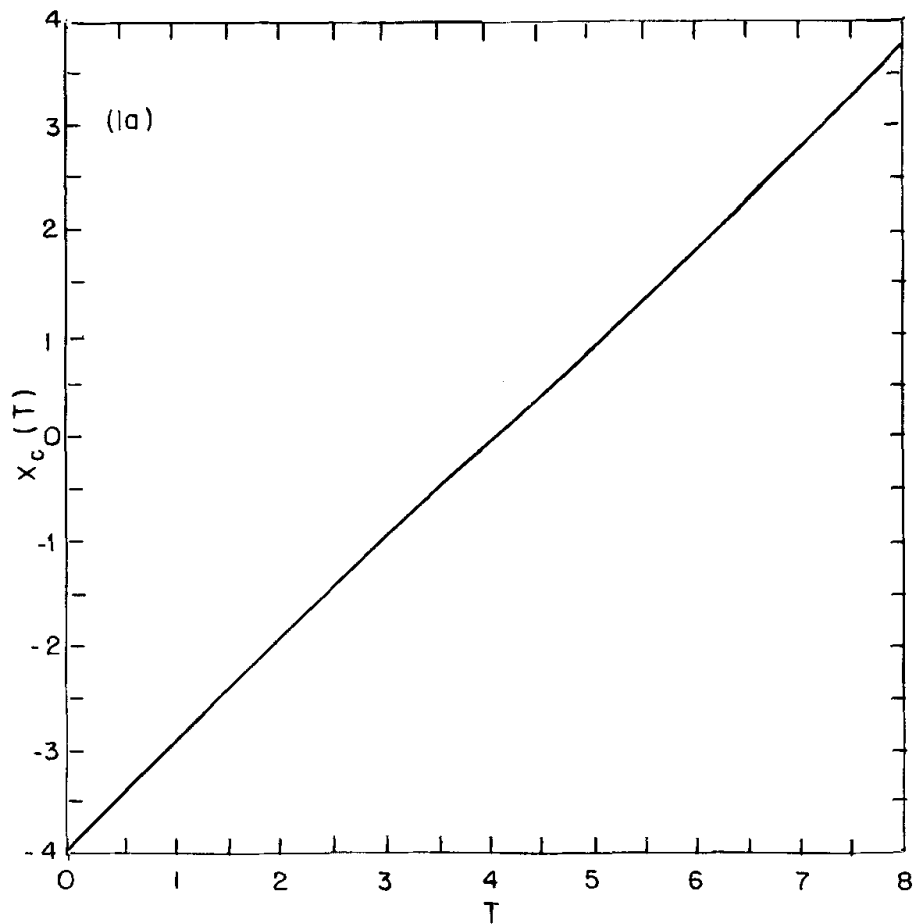


FIGURE 1a Plot of the modon center $X_c(T)$ vs. the slow time T for the “steady-state” dynamics described by (5.6) and (4.14). Parameter values used are $\mu=0.2$ and $X_c(0)=-4.0$. The topography used in the calculation is given by (5.1). The interaction between the modon and the topography leads to changes in the slope of the curve $X_c(T)$, which are confined to a relatively small region about $X \sim 0.0$.

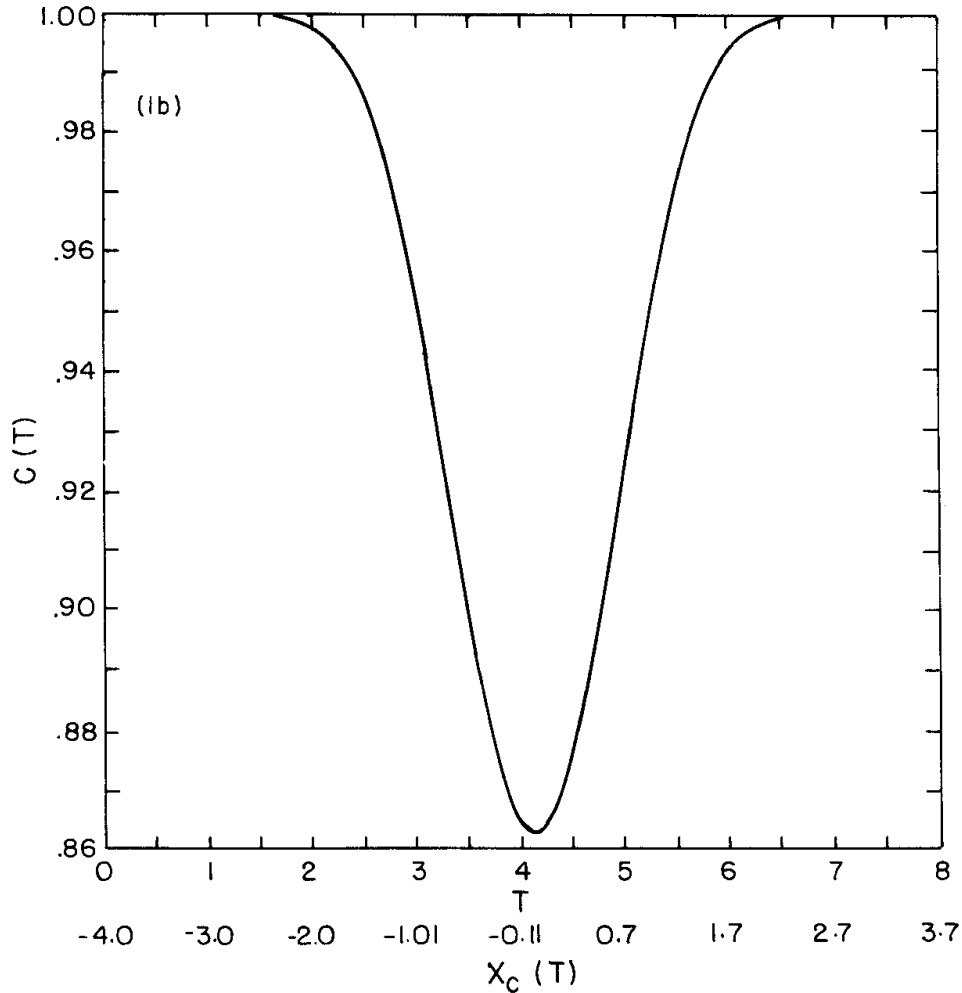


FIGURE 1b Plot of the modon translation speed $c(T)$ vs. the slow time T subject to the parameter values described in the Figure 1a caption. The upper and lower rows of numbers along the abscissa are the slow time T and the position of the modon center $X_c(T)$, respectively.

modon center is located at the point of maximum topographic amplitude (i.e. $X_c=0$ which occurs in this simulation at $T=4.13$). Figures 1c and 1d are plots of $a(T)$ and $\kappa(T)$ vs. T , respectively. Along the abscissa of Figures 1b, 1c and 1d are two rows of numbers, the upper being the slow time T and the lower set the position of the modon center $X_c(T)$ at that time.

Figures 2 and 3 are sequences of contour plots of the streaklines $\psi^{(0)} + cy$ and relative vorticity $\Delta\psi^{(0)}$, respectively, for space-time

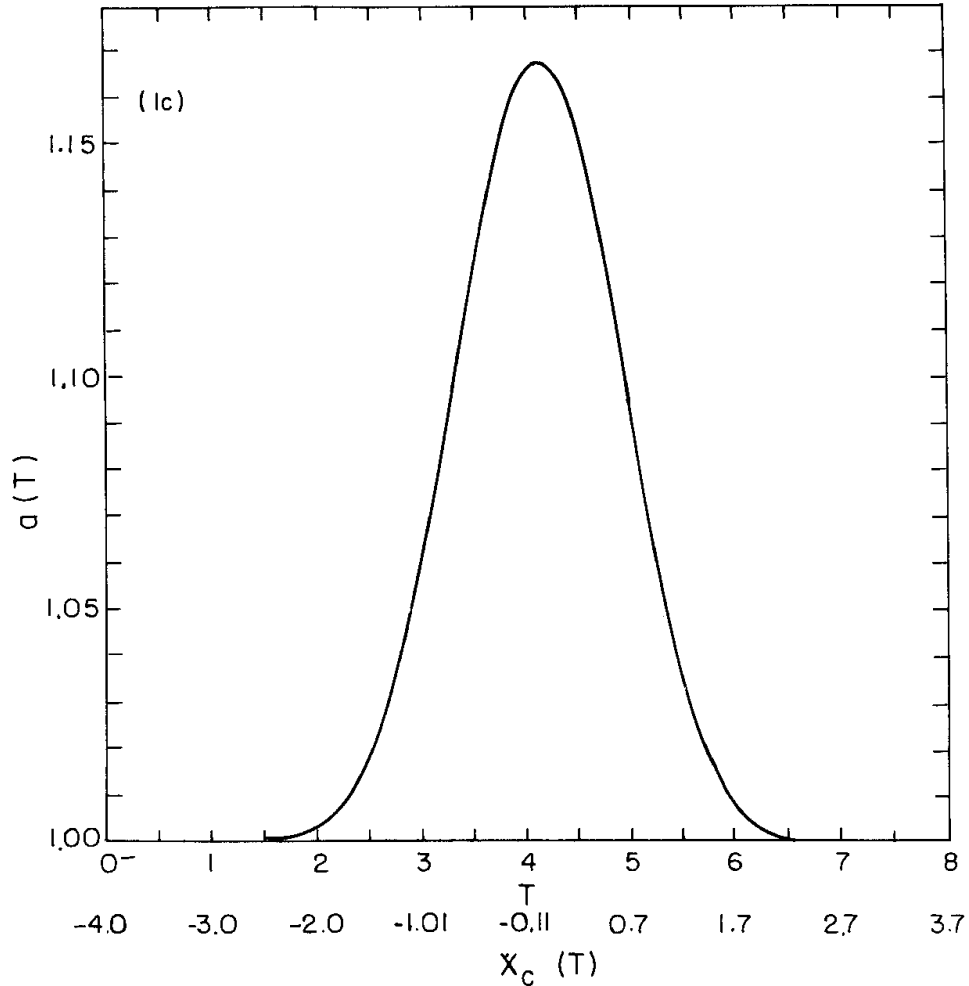


FIGURE 1c Plot of the modon radius $a(T)$ vs. the slow time T subject to the parameter values described in the caption of Figure 1a. The upper and lower rows of numbers along the abscissa are the slow time T and the position of the modon center $X_c(T)$, respectively.

coordinates $(X_c(T), T)$ given by $(-4.0, 0.0)$, $(-0.79, 3.24)$ and $(0.01, 4.14)$. These times were chosen in order to illustrate the initial condition and the modon deformation during an intermediate and near maximum state of the *steady-state* interaction with the topography.

The values of the modon parameters (a, c, κ) in Figures 2a, 3a; 2b, 3b and 2c, 3c are $(1.0, 1.0, 3.9226)$; $(1.09, 0.93, 3.59)$ and $(1.17, 0.86, 3.30)$, respectively. (Note that as in Figure 1, $\mu=0.2$.) The modon

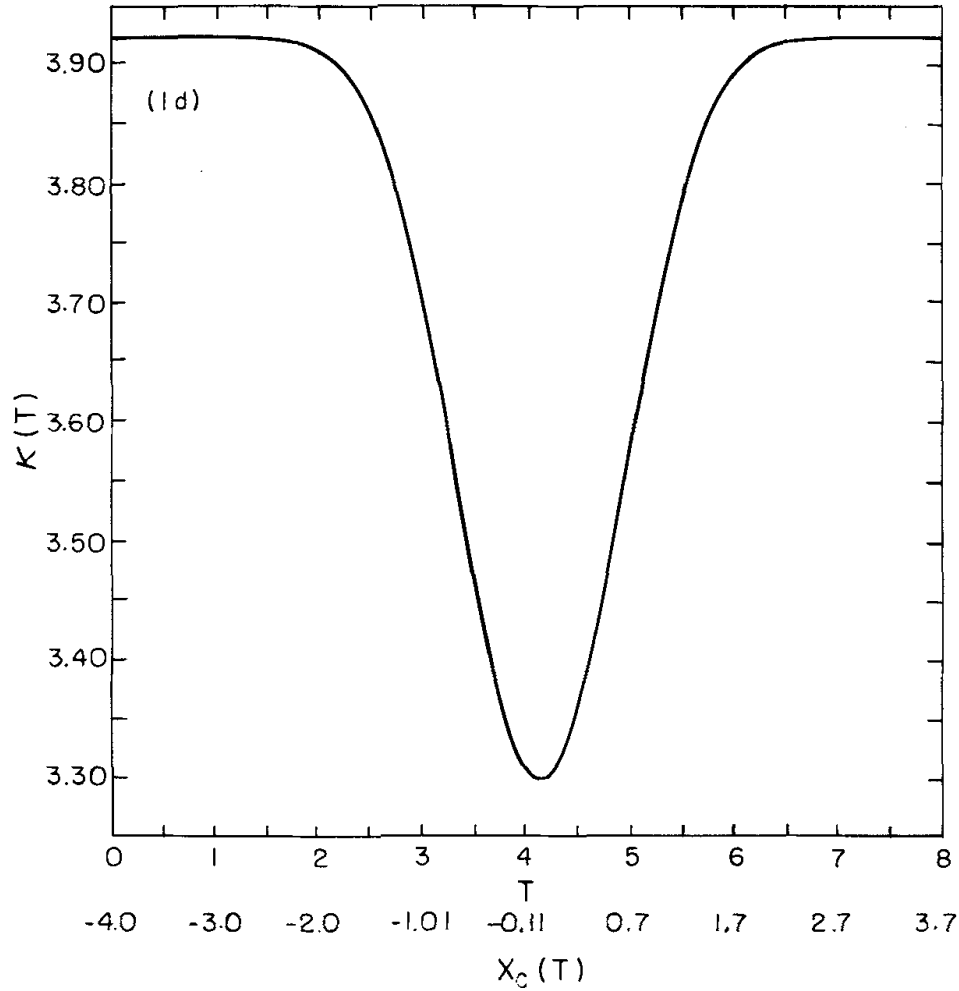


FIGURE 1d Plot of the modon wavenumber $\kappa(T)$ vs. the slow time T subject to the parameter values described in the caption of Figure 1a. The upper and lower rows of numbers along the abscissa are the slow time T and the position of the modon center $X_c(T)$, respectively.

radius is the circular 0.0-value contour Figure 2 and occurs just radially outward of the 0.0-value contour in Figure 3 [cf. (2.5)]. The contour intervals in Figures 2a and 2b are ± 0.5 and in Figure 2c is ± 2.0 . In Figures 3a and 3b the contour intervals are ± 5.0 and in Figure 3c is ± 20.0 .

There is an increase in the absolute value of the local extrema in the interior (i.e. $r < a$) streaklines and vorticity in the $X < 0$ region since the local depth is decreasing [when $X > 0$ these extrema return

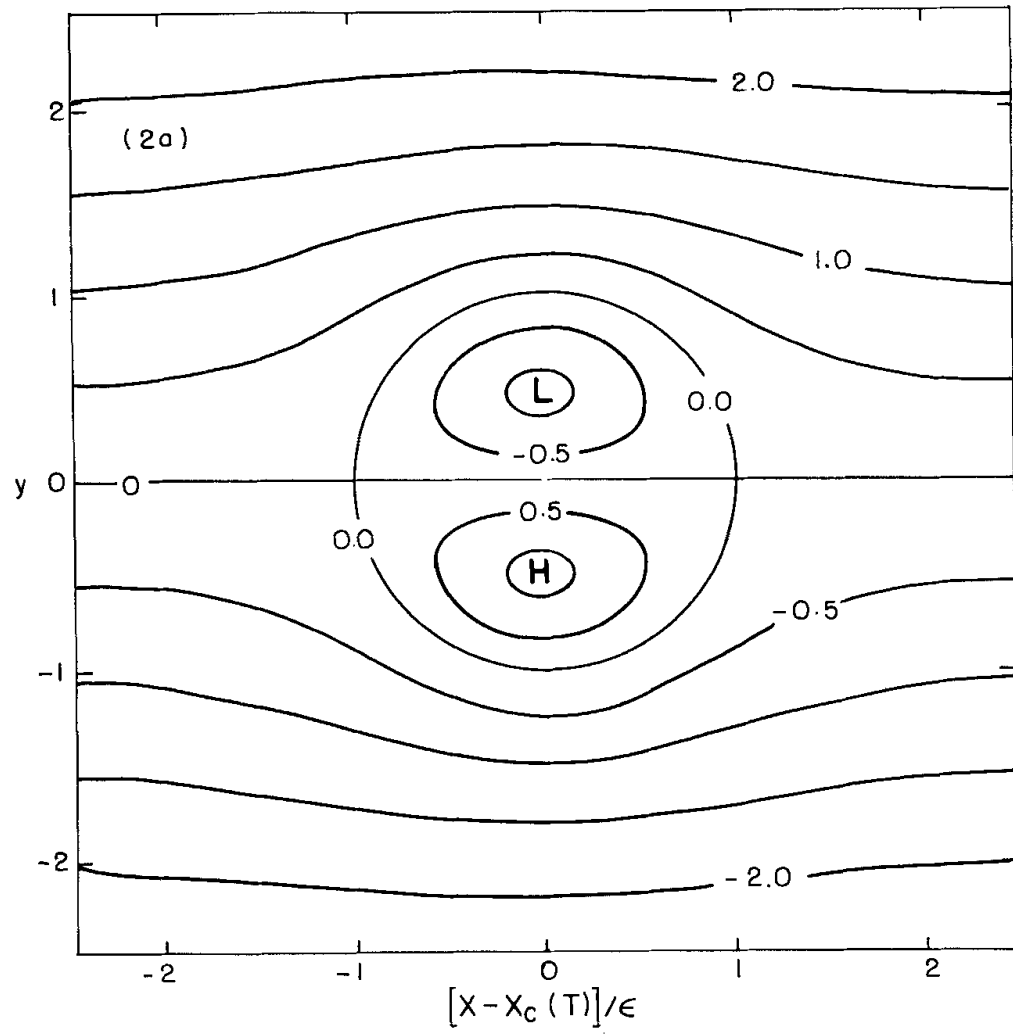


FIGURE 2a

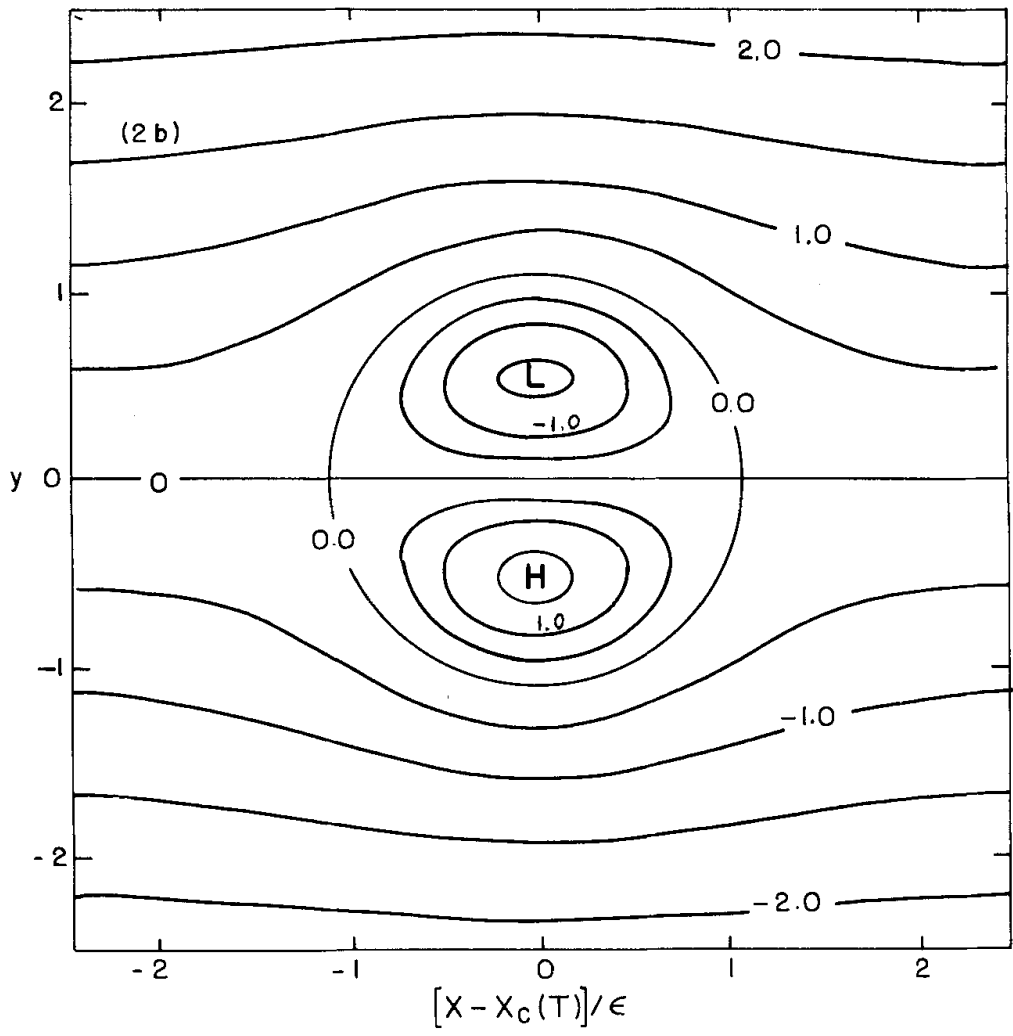


FIGURE 2b

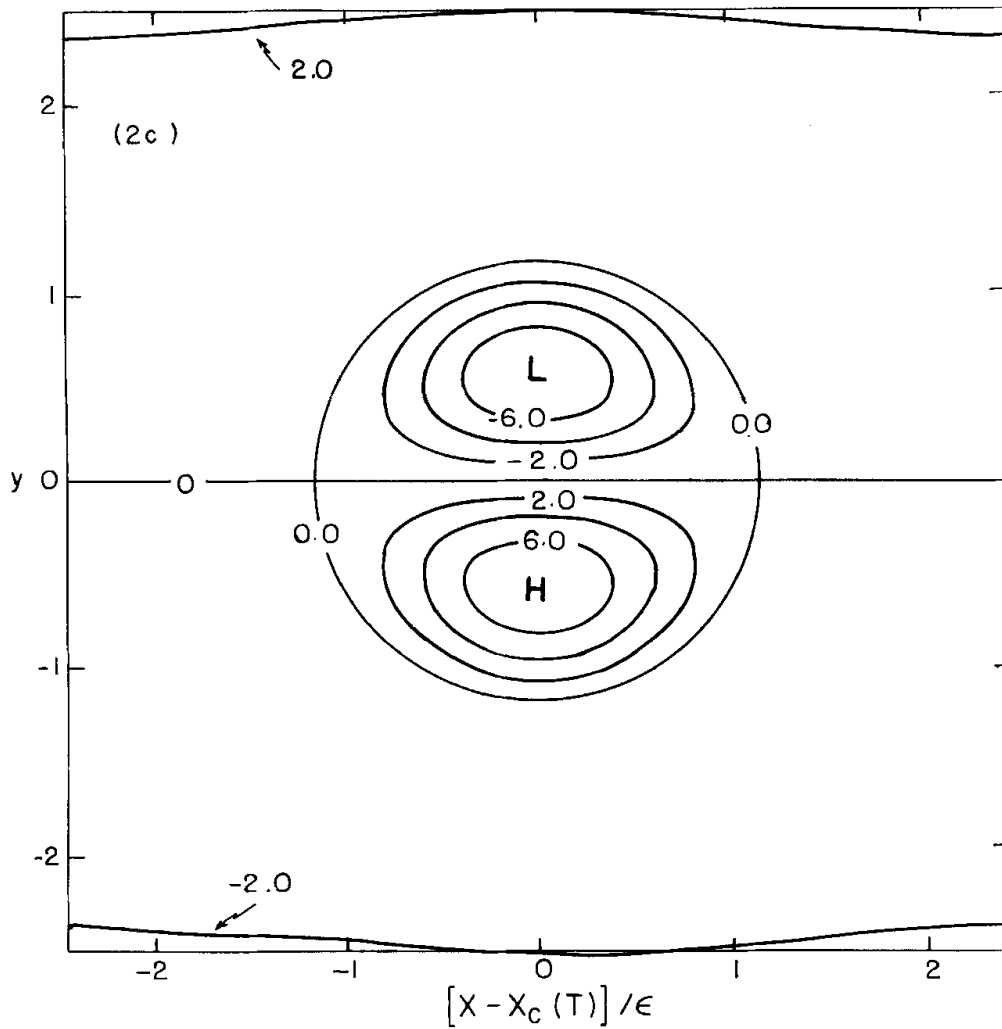


FIGURE 2c

FIGURE 2 Sequence of contour plots of the streaklines $\psi^{(0)} + cy$ subject to the parameter values described in the caption for Figure 1a. The $(X_c(T), T)$ coordinates of (a), (b) and (c) are $(-4.0, 0.0)$, $(-0.79, 3.24)$ and $(0.01, 4.14)$, respectively. The contour interval in (a) and (b) is ± 0.5 and in (c) is ± 2.0 . The position of the modon radius $a(T)$ is the circular 0.0-value contour. The L and H symbols denote the minimum and maximum values of the streaklines in the modon interior, respectively, which in (a), (b) and (c) are ± 1.08 , ± 1.61 and ± 7.83 , respectively.

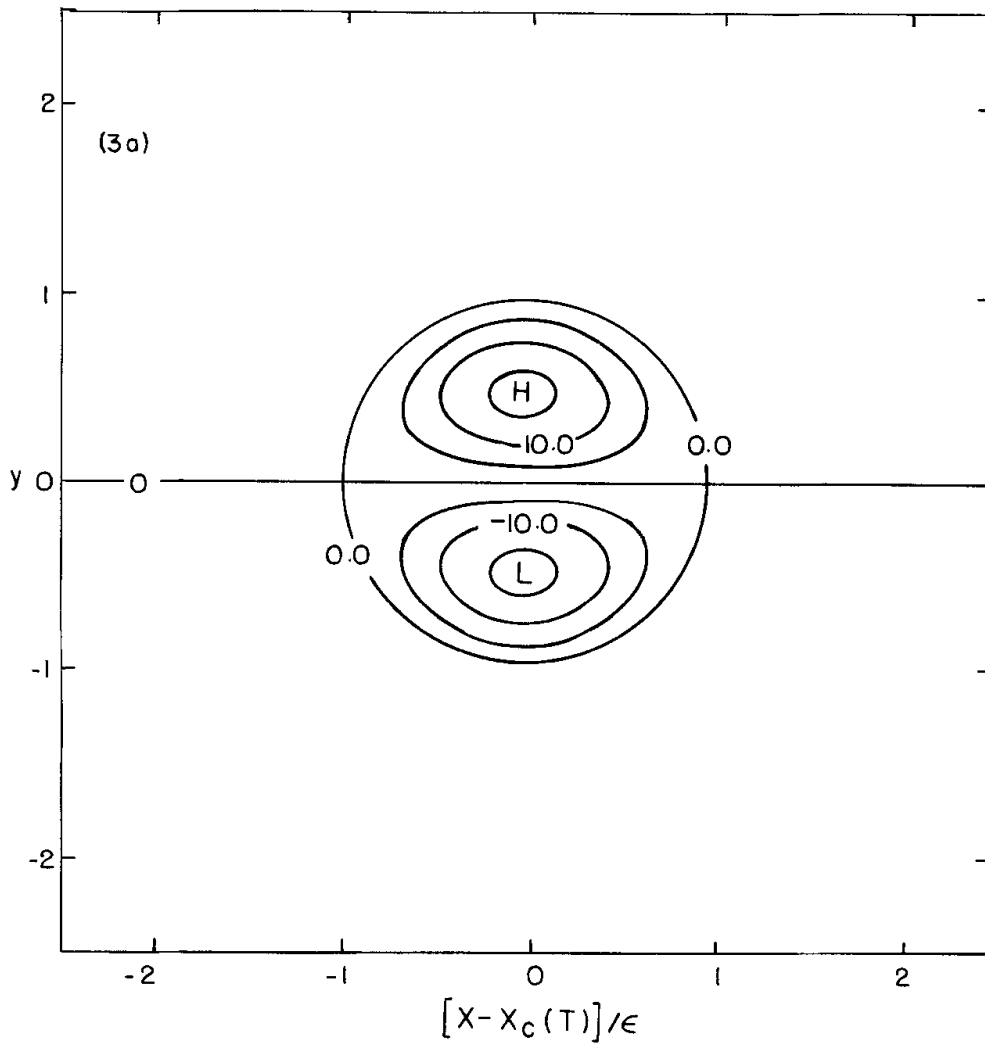


FIGURE 3a

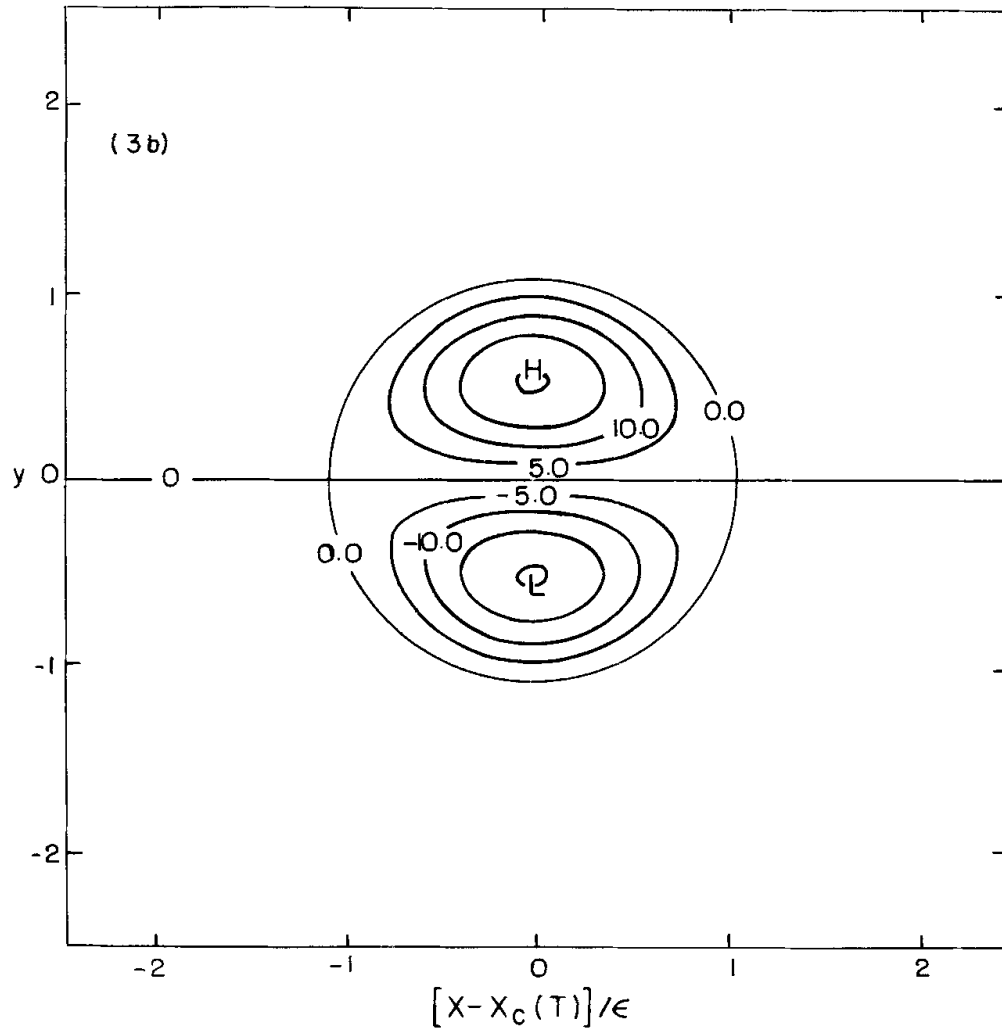


FIGURE 3b

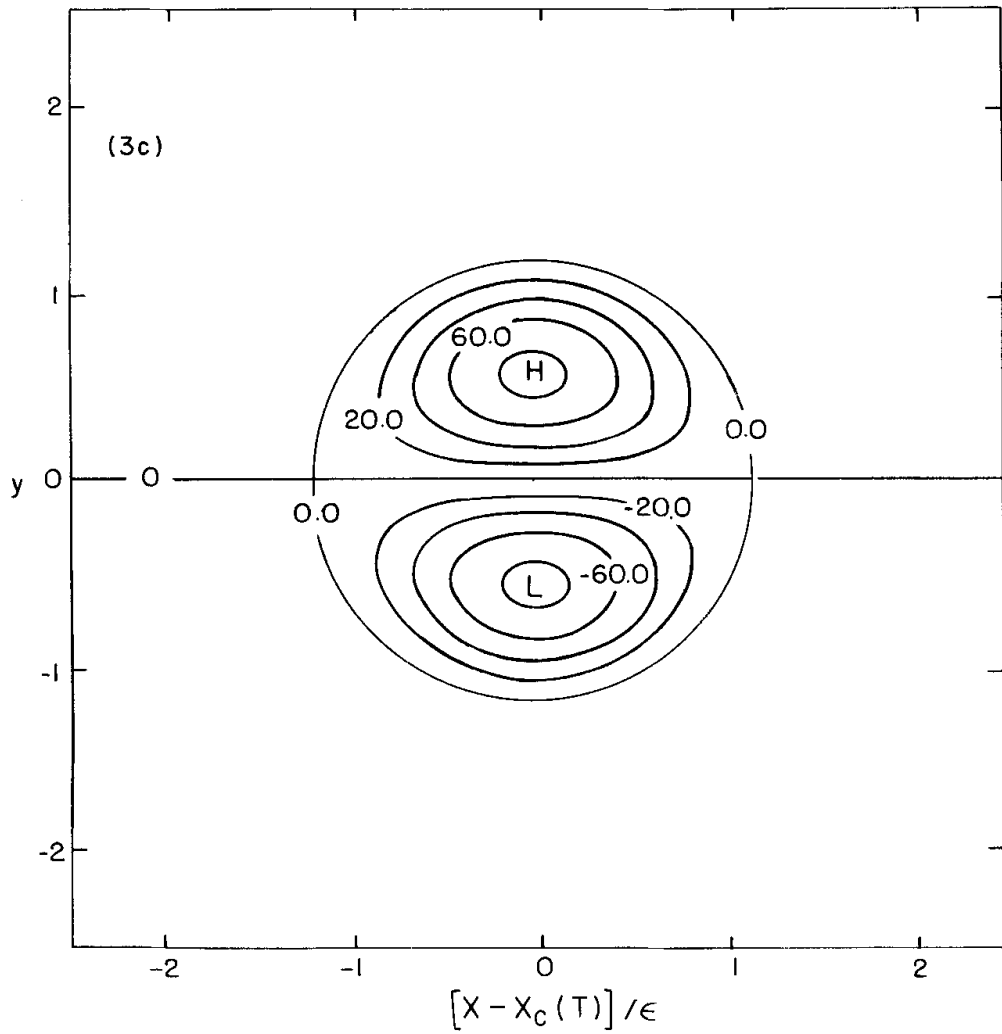


FIGURE 3c

FIGURE 3 Sequence of contour plots of the relative vorticity $\Delta\psi^{(0)}$ subject to the parameter values described in the caption for Figure 1a. The $(X_c(T), T)$ coordinates are those described in Figure 2. The contour interval in (a) and (b) is ± 5.0 and in (c) is ± 20.0 . The L and H symbols denote the minimum and maximum values of the relative vorticity in the modon interior, respectively, which in (a), (b) and (c) are ± 16.1 , ± 20.2 and ± 84.5 , respectively.

to their original values since the local depth is increasing back to one, see (5.1)]. The low and high interior streakline extrema in Figures 2a, 2b and 2c are ± 1.08 , ± 1.61 and ± 7.83 , respectively. The low and high interior vorticity extrema in Figures 3a, 3b and 3c are ± 16.1 , ± 20.2 and ± 84.6 , respectively. Consequently, there is an amplification of the modon as it propagates into a region of shallower fluid.

The qualitative features of the solution remain the same for other parameter values and topographic configurations (Swaters, 1985). The linear perturbation solutions suggests the interesting possibility that for finite-amplitude topography [i.e. the nonlinear dynamics (4.1) and (4.3)] the modon may amplify and “stall” and be unable to propagate past the topography. Thus the theory developed here may have application to the initial formation of atmospheric blocks (see also McWilliams, 1980). Warn and Brasnett (1983) have proposed a similar theory by attempting to model the onset of regional blocking as the weak topographic modulation of atmospheric K-DV solitons.

Another property that the nonlinear Eqs. (4.1) and (4.3) may possess that the linear solutions do not is the possibility that the characteristics in space-time may cross. Grimshaw (1979a) conjectured that this type of breakdown in slowly varying solitary wave problems may be associated with formation of further solitary waves. Clearly there may be interesting modon-topographic interactions which the linear analysis presented here is unable to describe.

The leading order solution $\psi^{(0)}$ is not expected to be uniformly valid as $\xi \rightarrow \pm \infty$. It can be easily shown that $\Delta\psi^{(0)}$ will be a homogeneous solution to the $O(\varepsilon)$ vorticity equation and that certain inhomogeneities in the $O(\varepsilon)$ vorticity equation are proportional to $\Delta\psi^{(0)}$ (Swaters, 1985). Thus resonance will in general occur and the expansion (2.2) is not expected to be valid for all ξ .

In one dimensional solitary waves this nonuniformity manifests itself as a “shelf” ahead and behind the solitary wave (e.g. Ko and Kuelh, 1978; Grimshaw, 1979a, b; Kodama and Ablowitz, 1980). The shelf appears in the solution as $\psi^{(1)} \rightarrow B^+$ and B^- as $\xi \rightarrow +\infty$ and $-\infty$, respectively, where B^+ and B^- are constants. In general, it is assumed that ahead of the solitary wave the fluid (to leading order) is undisturbed, thus $B^+ = 0$ (e.g. Grimshaw, 1979a, b). Physically, the shelf is identified with a slowly developing mass increment behind the solitary wave (see Ko and Kuelh, 1978; Grimshaw, 1979a). The

shelf region is removed by the introduction of an outer expansion valid for $\xi \simeq O(\varepsilon^{-n})$ (the power n is determined in the problem) (e.g. Grimshaw, 1979a, b; Kodama and Ablowitz, 1980). However, since it is $\nabla\psi^{(1)}$ that determines part of the $O(\varepsilon)$ velocity field, this constant does not contribute to $(u^{(1)}, v^{(1)})$. Consequently a nonuniformity of this type does not seem to be physically relevant here.

6. SUMMARY

A multiple-scale perturbation theory has been developed to describe modon propagation over slowly varying topography. The theory as it is developed here follows Grimshaw's analyses of slowly varying one-dimensional solitary waves, based on averaged conservation laws.

The point of origin for the theory is the rigid-lid shallow-water equations, for which the eastward-travelling barotropic modon is an exact solution in the absence of variable topography. Nonlinear transport equations for the slowly varying modon are derived based on averaged (over the fast phase variables) conservation laws for energy, enstrophy and vorticity. To leading order the slowly varying modon depends only parametrically on the meridional topographic structure since the planetary vorticity gradient has been scaled $O(1)$ relative to topographic gradients. No analytical solution for the nonlinear transport equations were found, however numerical calculations suggested they are strictly hyperbolic.

In the limit of small-amplitude topography, analytical perturbation solutions were obtained for the transport equations. The solutions were composed of hyperbolic transients acting to adjust the modon to the sudden imposition of variable topography at time zero, and a stationary component which describes the slow steady deformation of the modon as it travels over variable topography. As the modon moves into a region of deeper (shallower) fluid, the modon translation speed increases (decreases), the modon radius decreases (increases) and the modon wavenumber increases (decreases). These properties are in qualitative agreement with the known dynamics of inviscid, incompressible vortex-tubes.

The linear solutions suggest that for finite-amplitude topography, the modon may become topographically-trapped and amplified.

Thus it is possible that modon-topographic interactions may lead to the onset of blocking-like states. It was also speculated that modon-topographic interactions may lead to a fission-like process in which further solitary waves are generated.

Higher order effects in the perturbation solution were not examined. The possible creation of a slowly varying "tail" behind the modon due to a slow mass increment in the first order solutions has been ignored. Also, the possibility that meridional topographic structure may lead to higher order radiation damping was not considered. An analysis allowing for meridional motion, baroclinic dynamics and so on is also required before the full range of modon-topographic interactions is understood.

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