NOTES AND CORRESPONDENCE

Ekman Layer Dissipation in an Eastward-Traveling Modon

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ABSTRACT

A perturbation solution for an eastward-traveling modon in the presence of a bottom Ekman boundary layer is presented. The modon radius, translation speed and wavenumber are allowed to be functions of a slow time and the geostrophic pressure is expanded in the small damping coefficient $E^{1/2}/(2r_0)$, where E and r_0 are the vertical Ekman and Rossby numbers, respectively. The modon amplitude and translation speed decay exponentially and the modon wavenumber increases exponentially as the slow time increases. The resulting dissipation in the streamfunction and vorticity is qualitatively similar to the McWilliams and others numerical solution, although it is unable to describe the eventual transition to Rossby waves. For oceanic and atmospheric scales the decay takes place over a 100- and 10-day time scale respectively, with the modon traveling about 5 modon radii before complete dissipation.

1. Introduction

Studies of modon dynamics have generally focused on numerical integrations of the potential vorticity equation (McWilliams *et al.*, 1981; McWilliams and Zabusky, 1982; McWilliams, 1983; and Mied and Lindemann, 1982). In particular, McWilliams *et al.* (1981) numerically calculated the effect of (linear, Newtonian and biharmonic) vorticity dissipation on eastward-traveling barotropic modons and concluded that the decay was approximately exponential and shape preserving. The modon parameters (i.e., the radius, translation speed and wavenumber) evolved (to a first approximation) in such a manner as to preserve the modon dispersion relationship. In the final stages of the dissipation the modon degenerated into a field of westward-traveling Rossby waves. These observations suggest that the dissipation of a modon due to bottom friction can be viewed in the context of the evolution of solitary waves in slowly varying media. The main purpose of this note is to describe a theory for analytically obtaining the lead-order solution of an eastward-traveling modon in the presence of a bottom boundary layer. The solution we obtain agrees with the numerical calculation of McWilliams *et al.* (1981) for the dissipation of a modon, although it is unable to describe the transition to westward-traveling Rossby waves in the final stages of the decay.

The physical problem we examine is highly idealized. Only eastward-traveling barotropic modons forced by Ekman dissipation are considered. Other dissipation processes such as Newtonian or biharmonic damping or horizontal friction are not explicitly addressed. In addition, reduced gravity, fully baroclinic and mixed baroclinic/barotropic dynamics are not considered. While the precise analysis and conclusions presented here are limited in scope, much of what follows is qualitatively relevant for an analysis of these more realistic problems.

For typical oceanic and atmospheric modon scales (described in the next section) the effect of bottom friction is an order of magnitude smaller than the inertial and dispersive terms in the potential vorticity equation. To lead order such weak frictional effects will induce slow structural changes in the propagating modon. Whitham (1965) described the slow evolution of nonlinear waves in a dispersive medium by the slow variation of the parameters characterizing the waves. Ablowitz (1971), Kodama and Ablowitz (1981) and Ablowitz and Segur (1981) calculated the stability of various one-dimensional solitary waves forced by transverse or dissipative perturbations using analogous methods.

The dissipation of a barotropic modon when the effects of a bottom Ekman layer are included in the vorticity equation can be viewed in a similar context. The parameters which describe the modon solitary wave are allowed to be functions of the slowly varying media. Necessary constraints are exercised on the modon to derive initial value problems describing the evolution of the parameters. The method exploited in this paper can be generalized in an obvious way to describe modon propagation in a variety of slowly varying media (such as those described previously). The plan of the note is as follows. In Section 2 the

lead-order solution is obtained. In Section 3 we describe the results and in Section 4 a summary is provided.

2. Perturbation solution

The nondimensional barotropic potential vorticity equation in which the interior of the fluid is asymptotically matched to a bottom Ekman boundary layer is (Pedlosky, 1979)

$$\Delta\psi_t + J(\psi, \Delta\psi + \delta^2 y) = -\epsilon\Delta\psi \tag{2.1}$$

where ψ is the geostrophic pressure field; $J(*, \cdot)$ is the Jacobian determinant $\partial(*, \cdot)/\partial(x, y)$ with x, y and t the usual east, north and time coordinates; and where Δ is the horizontal Laplacian. The parameters $\delta^2 = \beta a_0^2/c_0$ and $\epsilon = E^{1/2}/(2r_0)$ are the planetary vorticity factor and damping coefficient respectively, with E the vertical Ekman number $2a_\nu f^{-1}H^{-1}$ where a_ν, f and H are the vertical eddy viscosity, Coriolis parameter and fluid depth respectively, and where r_0 is the Rossby number $c_0 f^{-1}(a_0)^{-1}$ with a_0 and c_0 the undamped modon radius and translation speed respectively. The space, time and velocity scalings have been chosen as a_0, a_0/c_0 and c_0 respectively. For typical oceanic (atmospheric) modon parameter values of β, a_0, c_0, a_ν, H and f of 1.6×10^{-11} m^{-1} s^{-1}, 100 (1000) km, 10^{-1} (10) m s^{-1}, 10^{-3} (10) m^2 s^{-1}, 4 (10) km and 10^{-4} s^{-1} respectively, it follows that $\epsilon \approx 10^{-2} - 10^{-1}$. Thus for modons, the rhs of (2.1) can be viewed as a weak dissipative term. The values of a_0 and c_0 were chosen to give an order unity planetary vorticity factor while satisfying quasigeostrophy. Equation (2.1) does not include a free surface since for oceanic applications the length scale a_0 is much smaller than the external deformation radius $(gH/f)^{1/2} \approx 2000$ km.

Defining fast variables

$$(\xi, y) = \left(x - \epsilon^{-1}\int_0^{\epsilon t} c(t')dt', y\right)$$

and the slow variable

$$T = \epsilon t,$$

(2.1) is rewritten

$$J(\psi + cy, \Delta\psi + \delta^2 y) = -\epsilon\Delta\psi - \epsilon\Delta\psi_T \tag{2.2}$$

where the Jacobian is taken with respect to ξ and y. The solution to (2.2) is constructed in the form

$$\psi \approx \psi^{(0)} + \epsilon\psi^{(1)} + \cdots.$$

The O(1) problem is

$$J(\psi^{(0)} + cy, \Delta\psi^{(0)} + \delta^2 y) = 0$$

the solution of which is taken to be the modon (Flierl *et al.*, 1980)

$$\psi^{(0)} = -caK_1(\delta c^{-1/2}r)\sin(\theta)/K_1(\delta a c^{-1/2})$$
$$\Delta\psi^{(0)} = -\delta^2 aK_1(\delta c^{-1/2}r)\sin(\theta)/K_1(\delta a c^{-1/2}), \quad r > a \tag{2.3}$$

$$\psi^{(0)} = \delta^2 a\kappa^{-2}J_1(\kappa r)\sin(\theta)/J_1(\kappa a) - (\delta^2 + \kappa^2 c)\kappa^{-2} r\sin(\theta)$$
$$\Delta\psi^{(0)} = -\delta^2 aJ_1(\kappa r)/J_1(\kappa a)\sin(\theta), \quad r < a \tag{2.4}$$

where J_1 and K_1 are the ordinary and modified Bessel functions of order one, $r^2 = \xi^2 + y^2$, $\tan(\theta) = y/\xi$ and the modon wavenumber κ is the first nonzero solution of the dispersion relation

$$-\delta J_2(\kappa a)K_1(\delta a c^{-1/2}) = c^{1/2}\kappa J_1(\kappa a)K_2(\delta a c^{-1/2}). \tag{2.5}$$

The modon parameters a, c and κ are allowed to be functions of the slow time (see Ablowitz, 1971; Kodama and Ablowitz, 1981; and Ablowitz and Segur, 1981 for a discussion of these methods) with the initial conditions $a(0) = 1$, $c(0) = 1$ and $\kappa(0) = \kappa_0$ where κ_0 solves the modon dispersion relation for $a = c = 1$ ($\kappa_0 = 3.9226$).

The O(ϵ) problem for $r > a$ is

$$J(\psi^{(0)} + cy, \Delta\psi^{(1)} - \delta^2 c^{-1}\psi^{(1)}) = -\Delta\psi^{(0)} - \Delta\psi_T{}^{(0)}. \tag{2.6}$$

The homogeneous adjoint equation associated with (2.6) is

$$(\Delta - \delta^2 c^{-1})J(\psi^{(0)} + cy, u) = 0$$

for which $u = \psi^{(0)}(r > a)$ is a solution. A compatibility condition on $\psi^{(0)}$ for $r > a$ is therefore (see Ablowitz, 1971; Kodama and Ablowitz, 1981; and Ablowitz and Segur, 1981)

$$\int_{-\pi}^{\pi}\int_a^{\infty}\psi^{(0)}(\Delta\psi^{(0)} + \Delta\psi_T{}^{(0)})rdrd\theta = 0. \tag{2.7}$$

The O(ϵ) problem for $r < a$ is

$$J(\psi^{(0)} + cy, \Delta\psi^{(1)} + \kappa^2\psi^{(1)}) = -\Delta\psi^{(0)} - \Delta\psi_T{}^{(0)} \tag{2.8}$$

with the related homogeneous adjoint equation

$$(\Delta + \kappa^2)J(\psi^{(0)} + cy, u) = 0$$

for which $u = \psi^{(0)}(r < a)$ is a solution. A compatibility condition on $\psi^{(0)}$ for $r < a$ is therefore

$$\int_{-\pi}^{\pi}\int_0^{a}\psi^{(0)}(\Delta\psi^{(0)} + \Delta\psi_T{}^{(0)})rdrd\theta = 0. \tag{2.9}$$

After a little algebra it can be shown that (2.7) and (2.9) imply respectively

$$Aa^{-1}a_T - [A - 1](2c)^{-1}c_T = -1 \tag{2.10}$$

$$Ba^{-1}a_T + [B - 1]\kappa^{-1}\kappa_T = -1 \tag{2.11}$$

where

$$A = \gamma K_2(\gamma)/K_1(\gamma) - 1 - K_1{}^2(\gamma)/D_1$$
$$D_1 = -\gamma^{-1}[\gamma K_1{}^2(\gamma) - 2K_0(\gamma)K_1(\gamma) - \gamma K_0{}^2(\gamma)]$$

$$B = 1 - \{kJ_0(k)D_2\delta^2/J_1^3(k) - [kJ_0(k)/J_1(k) + 2]$$
$$(\delta^2 + \kappa^2 c)J_2(k)/[kJ_1(k)] + \delta^2 + 2\kappa^2 c\}/$$
$$\{\delta^2 D_2/J_1^2(\gamma) - 2(\delta^2 + \kappa^2 c)J_2(k)/[kJ_1(k)]\}$$
$$D_2 = k^{-1}\{kJ_2^2(k) - 2J_2(k)J_1(k) + kJ_1^2(k)\}$$

where $\gamma = (\delta^2 a^2/c)^{1/2}$ and $k = \kappa a$. Differentiating (2.5) with respect to T gives

$$\kappa^{-1}\kappa_T = N[a^{-1}a_T - (2c)^{-1}c_T] - (2c)^{-1}c_T \quad (2.12)$$

where $N = -[\gamma R + \kappa^2 R/\gamma]/[4 + \gamma/R + \kappa^2 R/\gamma]$ and $R = K_2(\gamma)/K_1(\gamma)$.

The solutions to (2.10), (2.11) and (2.12) are simply (irrespective of A, B and N)

$$a_T = -a \quad c_T = -2c \quad \kappa_T = \kappa \text{ i.e.,}$$
$$a = \exp(-\epsilon t), \quad c = \exp(-2\epsilon t), \quad \kappa = \kappa_0 \exp(\epsilon t). \quad (2.13)$$

The solutions (2.13) are the principal result of our calculation. Concomitant with the intuitive expectation that the rhs of (2.3) must result in a exponential-like decay in the modon, (2.13) implies that the vorticity and streamfunction amplitudes decay as $\exp(-\epsilon t)$ and $\exp(-3\epsilon t)$ respectively.

3. Discussion

The solutions for a, c and κ satisfy $(\kappa a)_T = 0$, $(ac^{-1/2})_T = 0$ and $(\kappa c^{1/2})_T = 0$. Therefore (2.5) reduces to

$$-\delta J_2(\kappa_0)K_1(\delta) = \kappa_0 J_1(\kappa_0)K_2(\delta)$$

for all T, implying that the dispersion relationship is invariant during the decay. Thus the modon remains dynamically equivalent to its initial state, at least initially, and to O(1) as any WKB-like theory must predict. The exponential decay of the streamfunction and vorticity, and the invariance of the dispersion relation which we have obtained is in agreement with the numerical solution of (2.1) (McWilliams *et al.*, 1981) for a modon initial state. Figure 1 is a sequence of contour plots showing the decay in the vorticity field as T increases. The observer is fixed with respect to the fluid at infinity. The modon moves to the right with speed $c(T)$.

The compatibility conditions in (2.7) and (2.9) are in fact sufficient to eliminate the secularity in $\psi^{(1)}$ [note that $\Delta\psi^{(0)}$ is a homogeneous solution to (2.6) and (2.8)] since $\Delta\psi^{(0)} + \Delta\psi_T^{(0)}$ is indentically zero as a consequence of (2.13) (introduce the change of variable $r \rightarrow a(T)r$).

An upper bound on the distance over which the dissipating modon travels as a modon can be obtained from the characteristic equation

$$dx/dt = c(T)$$

which integrates to

$$x(t) = x_0 + [1 - \exp(-2\epsilon t)]/(2\epsilon)$$

so that the modon travels a maximum distance $(2\epsilon)^{-1}$ (about 5 modon radii) before breaking up into a field of Rossby waves.

McWilliams *et al.* (1981) estimate that for $t < 15$ the modon decays as a modon (based on similar parameter values for δ^2 and r) and when $t \approx 15$ a modon Rossby wave transition occurs. Based on our scaling this transition takes place when $T \approx 1.5$–15. At this stage the amplitudes of $\psi^{(0)}$ and $\Delta\psi^{(0)}$ are very small (see Fig. 1) and thus the above solution, we believe, qualitatively describes the principal decay mechanism. For oceanic scales of a_0 and c_0 of 100 km and 0.1 m s^{-1}, respectively, the above perturbation solution will be asymptotically valid for a time scale of 100–1000 days, whereas for atmospheric scales for a_0 and c_0 of 1000 km and 10 m s^{-1} respectively, (McWilliams, 1980) the perturbation solution will be asymptotically valid on a time scale of ten days.

4. Summary

A perturbation solution for the propagation of an eastward-traveling modon with a bottom boundary layer has been obtained. The geostrophic pressure has been expanded in the damping coefficient $\epsilon = E^{1/2}/(2r_0) \approx 10^{-1}$ with E the vertical Ekman number and r_0 the Rossby number.

The modon radius (a), translation speed (c) and wavenumber (κ) are allowed to be functions of the slow time $T = \epsilon t$. The O(ϵ) equations require a necessary compatibility condition on the O(1) solutions (taken to an eastward-traveling modon) resulting in nonlinear initial-value problems for the modon parameters.

The solutions $a = \exp(-T)$, $c = \exp(-2T)$ and $\kappa = \kappa_0 \exp(T)$ leave the modon dispersion relationship invariant during the decay. The amplitude of the modon streamfunction and vorticity decays like $\exp(-3T)$ and $\exp(-T)$, respectively. The maximum distance over which the modon travels before complete dissipation is about 5 modon (initial) radii. Based on a comparison with a numerical solution (McWilliams *et al.*, 1981) for the frictional dissipation of an eastward-traveling modon, the asymptotic solution obtained here describes the decay over a 100–1000 day time scale for oceanic parameters and a ten day time scale for atmospheric parameters.

The physical problem considered here is simple. The spatial homogeneity of the dissipative perturbation enabled the introduction of a single slow variable with a relatively straight forward solution. However, the solution technique is easily extended to other, and perhaps more interesting, physical problems. Modon interactions with slowly varying topography

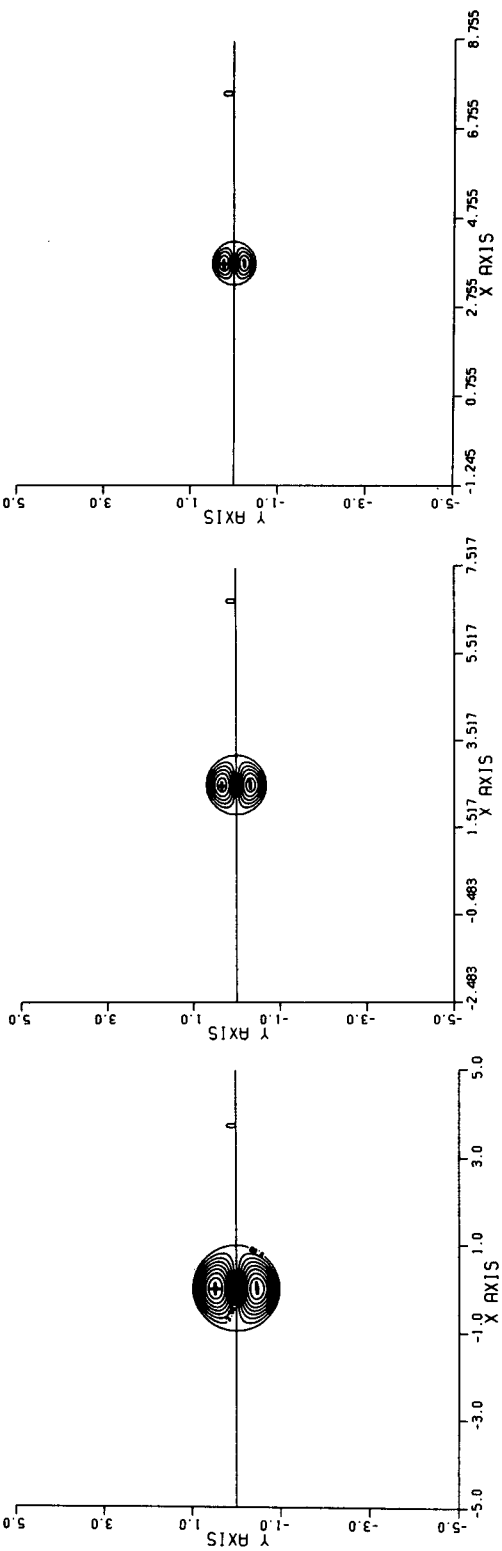

FIG. 1. Sequence of contour plots of the vorticity $\Delta\psi^{(0)}$. The observer is fixed with respect to the fluid at infinity. The contour intervals are ± 2. The zero contour is marked with a 0. The values of a, c and κ at each slow time T are listed in the upper left-hand corner.

and currents can be studied with similar perturbation methods and will be described in subsequent papers.

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