

Spectral Properties in Modon Stability Theory

By Gordon E. Swaters

Spectral properties of an invariant functional, denoted by H , for the linear stability equation associated with the *modon*, or solitary drift vortex, solutions of the quasi-geostrophic equivalent barotropic potential vorticity, or Charney–Hasegawa–Mima (CHM), equation are investigated. It is shown that H , which is the only known quadratic invariant in modon stability theory, is identical in form to the second variation of a “Benjamin-like” variational principle for solitary vortices. However, such a principle does not exist for the modon. The discrete spectrum of the “form operator” in H contains two simple negative eigenvalues and the simple zero eigenvalue. For the leftward-traveling solution there are only a finite number of positive eigenvalues. For the rightward-traveling solution, there are a countable infinity of positive eigenvalues. A sharp lower bound on the spectrum, for both the rightward- and leftward-traveling solutions, and a sharp upper bound for the leftward traveling solution, is determined. For the leftward-traveling solutions, the eigenfunctions span a finite-dimensional vector space and are orthogonal with respect to an inner product which is valid for all of L_2 . For the rightward-traveling solutions, the eigenfunctions span an infinite-dimensional Hilbert space, but are orthogonal with respect to an inner product, which is not valid for all of L_2 .

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1. Introduction

Modons are isolated steadily traveling dipole vortex solutions of the Charney–Hasegawa–Mima (or CHM) equation which, in nondimensional form, can be written as

$$(\Delta - 1)\psi_t + \psi_x + J(\psi, \Delta\psi) = 0, \quad (1)$$

where $J(A, B) \equiv A_x B_y - A_y B_x$ and $\Delta \equiv \partial_{xx} + \partial_{yy}$, where alphabetical subscripts imply partial differentiation (unless otherwise noted), and where the stream function $\psi(x, y, t)$ is related to the velocity field $\mathbf{u}(x, y, t)$ via

$$\mathbf{u} = (u, v) = \mathbf{e}_3 \times \nabla\psi = (-\psi_y, \psi_x).$$

By “isolated” we mean, firstly, that the finite energy and relative enstrophy constraints

$$\left. \begin{aligned} \iint_{\mathbb{R}^2} \nabla\psi \cdot \nabla\psi + \psi^2 \, dx \, dy < \infty, \\ \iint_{\mathbb{R}^2} (\Delta\psi - \psi)^2 \, dx \, dy < \infty, \end{aligned} \right\} \quad (2)$$

are satisfied and, additionally, that there exists a bounded region in \mathbb{R}^2 in which there are closed isolines of the streak function, i.e., the stream function in the co-moving frame of reference.

In the context of geophysical fluid dynamics, (1) is the quasi-geostrophic potential vorticity equation for the baroclinic dynamics of the ocean or atmosphere in a reduced gravity approximation [1]. In this situation, ψ is the leading order dynamic pressure field in the fluid. In the plasma context, the CHM equation can be derived in a formal asymptotic analysis of, for example, the dynamics of a cold ion fluid in an electrostatic field [2]. In this situation, ψ is the leading order electrostatic potential. In many respects, the CHM equation is the canonical $(2 + 1)$ -dimensional model for inviscid vortex dynamics in which baroclinic stretching (the ψ_t term) and a constant background vorticity gradient (the ψ_x term) are present and aligned with each other.

Stern [3] obtained the first steadily traveling dipole vortex solution of the CHM equation in the rigid lid limit where $\Delta \gg 1$ in (1). The solution found by Stern had the undesirable property that the relative vorticity, i.e., $\Delta\psi = \mathbf{e}_3 \cdot \nabla \times \mathbf{u}$, was not continuous $\forall (x, y) \in \mathbb{R}^2$. Larichev and Reznik [4] independently found a generalization of the Stern modon solution to (1) which had a vorticity field continuous $\forall (x, y) \in \mathbb{R}^2$.

It was originally conjectured that modons might correspond to a genuine $(2 + 1)$ -dimensional soliton. This point of view was further supported by early numerical simulations, which suggested that modons possessed soliton-like stability and interaction characteristics [5, 6].

The CHM equation does not, however, possess an infinite number of independent conservation laws so it is unlikely that there is a multidimensional

inverse scattering procedure for constructing solutions to it. Consequently, modons are *not* true multidimensional analogs of one-dimensional solitons. Nevertheless, modon-like solutions have been found for a large number of fluid and plasma models [7] and are thought to be important for describing aspects of the dynamics of coherent eddy-like features in turbulent fluid and plasma flows. Kloeden [8] has presented an argument for the assertion that the modon is the *unique* steadily traveling isolated solution to the CHM equation on the domain \mathbb{R}^2 .

There have been many attempts to establish the stability of modons [9–14]. All of these analyses are wrong [15–17]. The entire problem of *rigorously* proving the stability or instability of modons is an open mathematical problem.

For the leftward-traveling solution (i.e., traveling in the negative x -direction), it has been argued that stability is out of the question because of the so-called *tilt instability* [17, 18]. This instability, which is observed in numerical simulations, corresponds to perturbing the direction of propagation of the leftward-traveling modon so that initially there is a small component in the y -direction. No matter how small this initial perturbation, if it is nonzero, the modon follows a very complicated cycloid-like trajectory, which can only be identified as a significant deviation from the trajectory of an unperturbed leftward-traveling modon.

While it has been suggested that leftward-traveling modons are unstable, the situation for the rightward-traveling solutions (i.e., traveling in the positive x direction) is not so clear. Numerical experiments [5–7] seem to suggest that the rightward-traveling solution is stable for a rather large class of perturbations.

The establishment of a mathematical theory for the stability or instability of modons is complicated by a number of factors. As a steadily traveling dipole vortex solution to the CHM equation, modons, obviously, do not correspond to parallel shear flows (even in the co-moving frame of reference). Thus, the spatial part of the perturbation field in the linear stability problem cannot be separated into an along flow propagating normal mode and a transverse structure function that must satisfy a Rayleigh-like stability equation. The linear stability equation for the modon is, so far as we are aware, analytically intractable. This suggests that a mathematical stability or instability theory for modons likely must be built on a more general approach to the problem.

It is well known that the CHM equation is an infinite-dimensional noncanonical Hamiltonian dynamical system [19–21]. Benjamin [22] showed that, for a certain class of solitary steadily traveling solutions to the CHM equation, there exists a variational principle. This fact opens up the possibility that it might be possible to construct a stability theory for modons based on the energy-Casimir stability algorithm [19–21]. Unfortunately, as we show here, the modon falls outside the class of solutions for which the Benjamin [22] variational principle is applicable. Moreover, even if the modon was within the class of solutions for which the Benjamin variational principle was applicable,

Andrews's theorem [23] rules out the possibility that the modon can be proved stable by the energy-Casimir algorithm.

Swaters [11] and Laedke and Spatschek [12] independently found an invariant quadratic functional, which we henceforth denote by H , for the linear stability equation for the modon. Perhaps surprisingly, and as we show here, H is identical in form to what the second variation of the “Benjamin variational functional” evaluated at the modon solution would look like if such a “Benjamin variational principle” existed as articulated in [22] (which it does not).

Regardless of the origins of the functional H , the fact that it is quadratic with respect to the perturbation field and invariant with respect to time (for the linear stability problem) implies that if it were possible to find conditions on the modon solution so that H could be proved to be positive or negative definite for all perturbations, then linear stability in the sense of Liapunov could be established. However, and as we show here, H cannot be bounded away from zero since $H \equiv 0$ when evaluated for the translational mode (this is the dynamical implication of *Andrews's theorem* [23]). The translational mode is the variation or perturbation associated with an incremental spatial translation of the steadily traveling solution parallel to the direction of propagation.

This problem is not necessarily insurmountable. A similar situation arises in the stability theory for $(1 + 1)$ -dimensional solitons [24–26]. In its initial stages, the stability theory for $(1 + 1)$ -dimensional solitons is similar to the energy-Casimir algorithm. Both exploit the underlying Hamiltonian structure of the model equation and both begin with establishing a variational principle for the solution one wants to examine the stability of. The invariant functional in the variational principle is the sum of the Hamiltonian, a linear impulse functional (with a “Lagrange multiplier” given by the translation velocity) and an appropriately chosen Casimir.

The *linear* stability argument in the energy-Casimir procedure involves (see, [19–23] and references therein) determining simple conditions on the Casimir integrand or density function, which are sufficient to ensure the definiteness of the second variation of the invariant functional used in the variational principle (evaluated at the soliton solution). In soliton stability theory the mathematical analysis is much deeper since it must take into account the fact that the second variation of the invariant functional used in the variational principle evaluated at the steady solution is not definite.

The same problem also holds in the *nonlinear* stability argument, but in this situation one must work with the so-called pseudo-variational functional (the pseudo-variational functional is the invariant functional given by the difference between the functional used in the variational principle evaluated at the perturbed steady solution and the functional used in the variational principle evaluated at the unperturbed steady solution). Because of the indefiniteness of the second variation, the straightforward approach of the energy-Casimir method cannot succeed in soliton stability theory.

The integrand of the second variation of the invariant functional used in the soliton variational principle is a quadratic form in which the form matrix (in the linear algebra context) is replaced, in the present context, with a linear operator, henceforth denoted by \mathcal{L} . The stability theory for the KdV soliton as developed by Benjamin [24] and Bona [25] systematically exploited the properties associated with the discrete spectrum of \mathcal{L} , henceforth denoted by $\text{spec}(\mathcal{L})$, and the use of a novel sliding metric to factor out the translational mode.

The properties associated with $\text{spec}(\mathcal{L})$ that are sufficient to establish the stability of soliton solutions to many $(1 + 1)$ -dimensional models have been succinctly summarized by Albert et al. [26]. They are (see, also, Chapter 6 in [21]):

1. The eigenvalue 0 is simple (the eigenfunction associated with the zero eigenvalue is, of course, proportional to the translational mode).
2. There is only a single, simple negative eigenvalue.
3. A certain technical inequality holds involving the single, simple negative eigenvalue, its corresponding normalized eigenfunction, and the minimum positive eigenvalue.

The principal purpose of the present paper is give a systematic description of the discrete spectrum and corresponding eigenfunctions of the operator \mathcal{L} , which arises in the integrand of the invariant quadratic functional found by Swaters [11] and Laedke and Spatschek [12]. We do not give a stability theorem for modons. Our goal, rather, is to describe important mathematical properties of what is, so far as we are aware, the only known invariant quadratic functional in modon stability theory.

The outline of this paper is as follows. In Section 2, we briefly describe the modon solution to (1) and formulate the stability problem. We show that the modon solution does not fit into the variational principle framework developed by Benjamin [22] for steadily traveling solutions to the CHM equation. Notwithstanding this fact, we also show that the invariant functional (for the linear stability problem) found by Swaters [11] and Laedke and Spatschek [12] has exactly the form one would expect of the second variation of the “Benjamin variational functional” evaluated at the modon solution if such a “Benjamin variational principle” existed as articulated in [22].

In Section 3, we describe the spectral properties. We begin by establishing a number of preliminary general results such as the fact that the eigenvalues are real. We also show that for the eigenfunctions to satisfy the finite energy and enstrophy constraints (2), it follows, for the leftward-traveling modon that the spectrum is bounded above *and* below, but for the rightward-traveling modon, the spectrum is bounded only below. In both the leftward- and rightward-traveling modon, the lower bound is strictly negative. The upper bound on the spectrum for the leftward-traveling modon is positive. Our computational work suggests the theoretically determined bounds are sharp.

It is possible to obtain an implicit relation from which the eigenvalues can be determined numerically. For both the rightward- and leftward-traveling modon, we prove that there will always exist two negative simple eigenvalues and the single, simple zero eigenvalue. The translational mode is, of course, the eigenfunction (up to a multiplicative constant) associated with the zero eigenvalue. It is tempting to speculate that the number of negative eigenvalues obtained in this kind of “second variation” spectral analysis in solitary wave/vortex stability theory will be equal to the number of spatial dimensions in the solitary wave/vortex model.

For the rightward-traveling modon, we prove that there are an infinite number of discrete positive eigenvalues that increase without bound. For the leftward-traveling modon, there is at most a finite number of discrete positive eigenvalues. We have found leftward-traveling modon parameter values for which there are no positive eigenvalues.

For the leftward-traveling modon, the vector space spanned by the eigenfunctions is, of course, only finite dimensional. For the rightward-traveling modon, the vector space spanned by the eigenfunctions *is* a Hilbert space, but it is a subspace which is properly contained in $L_2(\mathbb{R}^2)$. In fact, we show by direct example, that it *cannot* be $L_2(\mathbb{R}^2)$.

It is possible to explicitly determine the eigenfunctions. For both the leftward- and rightward-traveling modon, the eigenfunctions can be described as oscillatory in the “interior” region of the modon (where fluid parcels are actually transported by the dipole) and exponentially decaying toward zero in the “exterior” region of the modon. Some concluding remarks are made in Section 4.

2. Problem formulation

The modon is a twice continuously differentiable solution to (1) in the form

$$\psi(x, y, t) = \varphi_s(x - ct, y). \quad (3)$$

Substitution of (3) into (1) yields

$$J(\varphi_s + cy, \Delta\varphi_s - \varphi_s + y) = 0, \quad (4)$$

where it is understood that $J(A, B) \equiv A_\xi B_y - A_y B_\xi$ and $\Delta \equiv \partial_{\xi\xi} + \partial_{yy}$ where $\xi = x - ct$ so that we may write $\varphi_s = \varphi_s(\xi, y)$.

We can immediately integrate (4) to yield

$$\varphi_s + cy = \Phi(q_s), \quad (5)$$

where $\Phi(\cdot)$ is a yet to be determined function of its argument and

$$q_s \equiv \Delta\varphi_s - \varphi_s + y.$$

The streak function is $\varphi_s + cy$ and q_s is the potential vorticity (which is the sum of the relative vorticity $\Delta\varphi_s$ term, the baroclinic stretching $-\varphi_s$ term, and the background vorticity gradient $+y$ term). It is important to note that (3) explicitly excludes solutions, which have a component of propagation in the y -direction. It is well known [27] that there are no *isolated* steadily traveling solutions to (1) with a component of propagation in the y -direction.

For those streak lines, i.e., the isolines of the streak function, which extend to infinity, it follows from (5) that

$$\Phi(q_s) = cq_s, \tag{6}$$

which when substituted into (5) implies that φ_s satisfies

$$\Delta\varphi_s - \left(1 + \frac{1}{c}\right)\varphi_s = 0. \tag{7}$$

For those streak lines that do not extend to infinity there is no boundary condition to *a priori* determine the form of $\Phi(q_s)$. The modon solution is obtained by invoking the *ansatz* that

$$\Phi(q_s) = -\frac{q_s}{1 + \kappa^2}, \tag{8}$$

for all those streak lines which do not extend to infinity. The parameter κ is called the *modon wave number*, and it is determined by differentiability conditions. If (8) is substituted into (5), it follows that all those streak lines which do not extend to infinity satisfy

$$\Delta\varphi_s + \kappa^2\varphi_s = -(1 + c + c\kappa^2)y. \tag{9}$$

The boundary between the region containing streak lines which extend to infinity and the region containing streak lines which do not extend to infinity is assumed to be the circle $r \equiv \sqrt{\xi^2 + y^2} = a$, where the parameter a is called the *modon radius*. Thus one may write

$$\Phi(q) = \begin{cases} cq, & \text{for } r > a, \\ -q/(\kappa^2 + 1), & \text{for } r < a. \end{cases} \tag{10}$$

The region $r > a$ will be referred to as the exterior region and the region $r < a$ will be referred to as the interior region.

Since the solution is continuous on the modon radius, it follows that

$$\lim_{r \downarrow a} \varphi_s + cy = \lim_{r \uparrow a} \varphi_s + cy,$$

which together with (5), (6), and (8) implies that

$$[c + (\kappa^2 + 1)^{-1}](\Delta\varphi_s - \varphi_s + y)|_{r=a} = 0.$$

Thus either $c = -(\kappa^2 + 1)^{-1}$ or $q_s|_{r=a} = 0$. If $c = -(\kappa^2 + 1)^{-1}$, then $1 + c^{-1} = -\kappa^2 < 0$ and the solutions to (7) will not satisfy (2) and thus must be excluded. Hence $q_s|_{r=a} = 0$, which implies (see (5) and (10))

$$\varphi_s + cy = 0 \quad \text{on } r = a. \tag{11}$$

This condition is not sufficient to ensure that q_s is continuous at $r = a$. In addition, it is required that

$$\lim_{r \downarrow a} \nabla \varphi_s = \lim_{r \uparrow a} \nabla \varphi_s. \tag{12}$$

The modon wave number is determined from (12).

There is a slight generalization of (8) which allows for the superposition of an additional radially symmetric field on top of the modon solution just derived, i.e., the so-called ‘‘rider’’ solutions [27]. However, the rider solutions introduce a finite step discontinuity in the vorticity field across the modon boundary and are barotropically unstable [28]. We do not consider them further here.

The condition (2) imposes further restrictions on the allowed values for the translation velocity c . If $1 + c^{-1} \leq 0$, there are no solutions to (9) which satisfy (2). Thus only $1 + c^{-1} > 0$ is possible, which can be rearranged to imply that

$$c < -1 \quad \text{or} \quad c > 0. \tag{13}$$

The set of allowed values for the modon translation velocity is therefore disjoint from the set of allowed values for the x -direction phase velocity of the linear dispersive Rossby wave solutions to (1), which are given by $c_{\text{Rossby}} \in (-1, 0)$. The leftward- and rightward-traveling solutions correspond to $c < -1$ and $c > 0$, respectively.

The solution to (7), satisfying (2) and (11), is given by

$$\varphi_s(r, \theta) = -\frac{acK_1(\gamma r/a) \sin(\theta)}{K_1(\gamma)}, \tag{14}$$

where $\gamma \equiv a\sqrt{1 + c^{-1}}$, and where $K_1(\cdot)$ is the modified Bessel function of the first kind of order one.

The bounded solution to (9), satisfying (11), is given by

$$\varphi_s(r, \theta) = \frac{a(1 + c)J_1(\nu r/a) \sin(\theta)}{\kappa^2 J_1(\nu)} - \frac{(1 + c + c\kappa^2)r \sin(\theta)}{\kappa^2}, \tag{15}$$

where $\nu \equiv \kappa a$, and where $J_1(\cdot)$ is the ordinary Bessel function of the first kind of order one.

The remaining constraint is (12). It is straightforward to verify that $\partial \varphi_s / \partial \theta$ is continuous on $r = a$. The condition

$$\lim_{r \downarrow a} \frac{\partial \varphi_s}{\partial r} = \lim_{r \uparrow a} \frac{\partial \varphi_s}{\partial r} \iff \frac{\gamma K_1(\gamma)}{K_2(\gamma)} = -\frac{\nu J_1(\nu)}{J_2(\nu)}. \tag{16}$$

The latter relation is called the *modon dispersion relationship*. It is usual to consider that (16) defines $\kappa = \kappa(a, c)$. There are a countable infinity of $\kappa > 0$ solutions for each (a, c) . The smallest nontrivial solution is called the ground state wave number and the corresponding modon the ground state modon.

Figures 1(a) and (b) are contour plots of a rightward- and leftward-traveling (ground state) modon, respectively. In Figure 1(a), $c = a = 1.0$ which implies that $\kappa \simeq 3.98$. In Figure 1(b), $c = -4.0$ and $a = 1.0$ which implies that

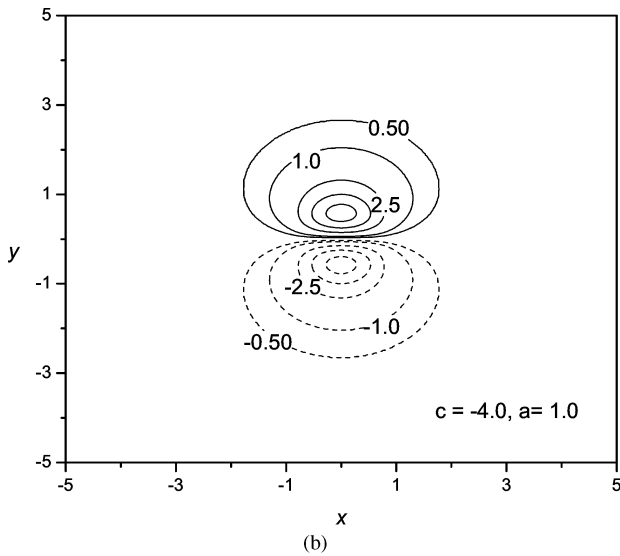
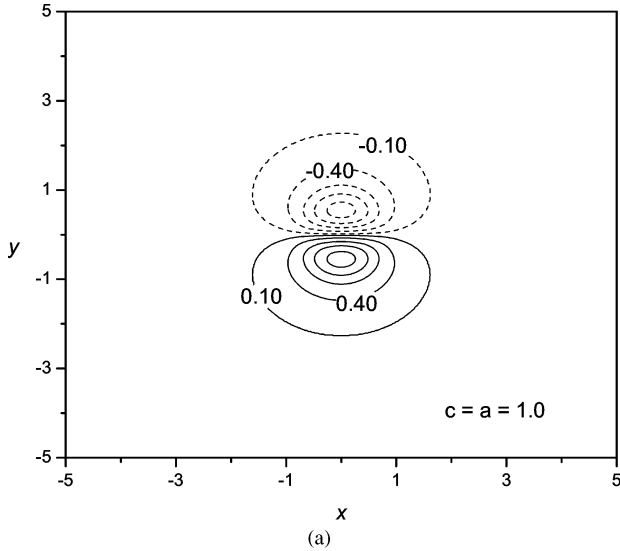


Figure 1. (a) φ_s for the rightward-traveling modon $c = a = 1.0$. (b) φ_s for the leftward-traveling modon $c = -4.0$ and $a = 1.0$.

$\kappa \simeq 3.90$. Within the interior region, $r < a$, fluid parcels are trapped and are thus transported in the appropriate x -direction. In the exterior region, $r > a$, the stream function decays exponentially to zero. The modon is a strongly localized dipole, which steadily travels at a speed “proportional” to its maximum amplitude. The ground state modon possesses a nodal line in the stream function only along the x -axis. The higher states have progressively more nodal lines (in the interior region).

2.1. Stability problem and invariant functionals

Benjamin [22] showed that *general* steadily traveling solutions to the CHM equation of the form

$$\psi_s(\xi, y) = F(q_s(\xi, y)), \quad q_s = y + (\Delta - 1)\psi_s, \quad \xi = x - ct, \quad (17)$$

where $F(q)$ is a sufficiently smooth function of q , satisfy the first-order necessary conditions for an extrema to the functional

$$\mathcal{H}(q) = H(q) - c M(q) + \iint_{\mathbb{R}^2} \left\{ \int_y^q F(\zeta) d\zeta \right\} d\xi dy, \quad (18)$$

where

$$H(q) = \frac{1}{2} \iint_{\mathbb{R}^2} |\nabla\psi|^2 + \psi^2 d\xi dy = \frac{1}{2} \iint_{\mathbb{R}^2} (y - q)\psi d\xi dy,$$

$$M(q) = \iint_{\mathbb{R}^2} y(\Delta\psi - \psi) d\xi dy = \iint_{\mathbb{R}^2} y(q - y) d\xi dy,$$

where it is formally understood that $\psi = (\Delta - 1)^{-1}(q - y)$.

In terms of the well-developed noncanonical Hamiltonian structure for the CHM equation [21–23], H , M and the third functional in \mathcal{H} are, respectively, the Hamiltonian (and the energy), x -direction linear impulse, and a Casimir. All three functionals are individually conserved and from the viewpoint of Noether’s Theorem [29], H and M arise due to the translational invariance associated with t and x , respectively, and the Casimirs arise due to the particle relabeling symmetry. Clearly, $\mathcal{H}(q)$ is an invariant of the CHM equation since it is the sum of three individually invariant functionals.

The first variation $\delta\mathcal{H}(q)$ is given by

$$\delta\mathcal{H}(q) = \iint_{\mathbb{R}^2} [F(q) - \psi - cy] \delta q d\xi dy.$$

Thus, $\delta\mathcal{H}(q_s) = 0 \forall \delta q = (\Delta - 1)\delta\psi$ (in the appropriate Hilbert space) because $\psi_s + cy = F(q_s)$.

The second variation of $\mathcal{H}(q)$ evaluated at the steadily traveling solution (17) can be written in the form

$$\delta^2\mathcal{H}(q_s) = \iint_{\mathbb{R}^2} |\nabla\delta\psi|^2 + (\delta\psi)^2 + F'_s(\delta q)^2 d\xi dy, F'_s \equiv \left. \frac{dF(q)}{dq} \right|_{q=q_s}. \quad (19)$$

The energy-Casimir stability algorithm [19–21, 30] exploits the fact that $\delta^2\mathcal{H}(q_s)$ is quadratic in the perturbation field δq and is always an invariant, for sufficiently smooth $F(q)$, of the linear stability equation associated with the solution (ψ_s, q_s) . Should it be the case that $F'_s \geq 0 \forall q_s$, then *linear* stability, and if $dF(q)/dq \geq 0 \forall q \in \mathbb{R}$ then *nonlinear* stability in the sense of Liapunov for (ψ_s, q_s) can be proved [19–21, 30]. This stability argument is sometimes called *Arnol'd's first stability theorem* and reduces to *Fjortoft's theorem* for a parallel shear flow [30, 31]. We note that although, in principle, stability is not ruled out if $F'_s < 0$, the energy-Casimir stability argument requires a Poincaré inequality [19–21, 30, 32] which does not exist for the domain \mathbb{R}^2 . And we point out that the energy-Casimir stability algorithm provides sufficient, and not necessary, stability conditions.

The energy-Casimir argument cannot be used to examine the stability of the modon for a number of reasons. First, if one substitutes $F = \Phi(q)$, as given by (10), into (18) the resulting $\mathcal{H}(q)$ is no longer an invariant of (1) for general q because $F(q)$ is not an analytic function of its argument. This means that $\mathcal{H}(q)$ cannot be used to examine the nonlinear stability of modons. Second, the above derivations do not allow for variations in the modon boundary. Presumably, for general q , there is no reason to suppose that the modon boundary (even in the co-moving frame) cannot be a function of (θ, t) . Third, $\delta\mathcal{H}(q_s) \neq 0$ for all possible variations. Fourth, even if the first three issues just raised did not exist, if F'_s is computed assuming $F = \Phi(q)$, F'_s is not nonnegative $\forall (\xi, y) \in \mathbb{R}^2$ and thus the energy-Casimir stability argument is not available.

Nevertheless, and perhaps surprisingly, if F'_s is computed assuming $F = \Phi(q)$ and substituted into the right hand side of $\delta^2\mathcal{H}(q_s)$ in (19), it follows that the resulting functional, denoted by H (so as to not suggest that it is $\delta^2\mathcal{H}(q_s)$), and written in the form

$$H = \iint_{\mathbb{R}^2} |\nabla\varphi|^2 + \varphi^2 + \Phi'_s(\Delta\varphi - \varphi)^2 d\xi dy, \quad (20)$$

$$\Phi'_s = \begin{cases} c, & \text{for } r \geq a, \\ -(\kappa^2 + 1)^{-1} & \text{for } r < a. \end{cases} \quad (21)$$

is an invariant of the linear stability equation for the modon [11, 12], given by

$$(\Delta - 1)\varphi_t + J(q_s, \Phi'_s\Delta\varphi - [1 + \Phi'_s]\varphi) = 0, \quad (22)$$

or, equivalently,

$$(\Delta - 1)\varphi_t + J(\psi_s + cy, \Delta\varphi - [1 + 1/\Phi'_s]\varphi) = 0,$$

written in the co-moving frame of reference, where $\varphi(\xi, y, t)$ is the perturbation, i.e., $\psi \simeq \varphi_s + \varphi$. Even though $\partial\Phi'_s/\partial r$ possesses a delta function along $r = a$, (22) remains valid (at least in the sense of distributions) on the modon boundary since $\partial q_s(a, \theta)/\partial\theta = 0$ and we insist that, at least, $\varphi \in C^{(1)}(\mathbb{R}^2)$ and, additionally, that φ satisfies (2) with its second derivatives continuous, except possibly across the modon boundary (which is a set of measure zero in \mathbb{R}^2).

The fact that Φ'_s is piecewise constant leads one to hope that H might be an invariant of the nonlinear stability equation. Of course H is, as one might expect, *not* an invariant of the *nonlinear* stability equation given by

$$(\Delta - 1)\varphi_t + J(q_s, \Phi'_s\Delta\varphi - [1 + \Phi'_s]\varphi) + J(\varphi, \Delta\varphi) = 0.$$

In fact, one can show, assuming the perturbation φ evolves fully nonlinearly, that

$$\frac{dH}{dt} = - \iint_{\mathbb{R}^2} \Phi'_s J(\varphi, (\Delta\varphi - \varphi)^2) d\xi dy.$$

Thus, $dH/dt = 0$ for the fully nonlinear problem, only if either φ or $\Delta\varphi - \varphi$ is assumed constant on $r = a$. Neither of these assumptions is acceptable. (We note that the possibility that $c = -(\kappa^2 + 1)^{-1}$ has already been ruled out.)

3. Spectral properties

We noted above that since Φ'_s is not nonnegative $\forall (\xi, y) \in \mathbb{R}^2$ it is not possible to establish, as would be required in energy-Casimir stability theory, that H is positive definite for all φ (in the appropriate Hilbert space). For example, substituting $\varphi = \partial\varphi_s/\partial x = \partial\varphi_s/\partial\xi$, i.e., the translational mode, into (20) results in $H = 0$.

This latter fact is easily seen as a consequence of observing that

$$H|_{\varphi=\partial\varphi_s/\partial\xi} = 0 \iff \iint_{r < a} \Phi'_s \frac{\partial(7)}{\partial\xi} \frac{\partial q_s}{\partial\xi} d\xi dy + \iint_{r > a} \Phi'_s \frac{\partial(9)}{\partial\xi} \frac{\partial q_s}{\partial\xi} d\xi dy = 0, \quad (23)$$

and integrating the right-hand side of the equivalence statement by parts once, exploiting the fact that $\varphi_s \in C^{(1)}(\mathbb{R}^2)$, $q_s \in C(\mathbb{R}^2)$ and that (2) holds for $\psi = \partial\varphi_s/\partial\xi$. In the context of the energy-Casimir stability theory for the CHM equation, (23) is understood as *Andrews's theorem* [21, 23, 33, 34] Clearly, (23) rules out the possibility that H is positive definite for all appropriate φ . Indeed, there are, as we will show, φ for which H is negative.

Regardless, (23) does not constitute a mathematical proof that modons are unstable. It implies only that any mathematical stability theory for modons must be more sophisticated than the energy-Casimir algorithm. Indeed, this is already well known in the stability theory for (1 + 1)-dimensional solitary waves. As shown by Benjamin [24] and Bona [25] for the KdV soliton (although the result holds much more generally), the translational mode will always be a member of the kernel of the integrand of the second variation of the constrained Hamiltonian (for which the solitary wave satisfies the first-order necessary conditions for an extrema). As articulated by Albert et al. [26], and first shown by Benjamin [24], it is, in part, the spectral properties of the linear operator associated with the integrand of the second variation of the constrained Hamiltonian, which are critical in developing a rigorous theory for the stability of solitary waves (see, also, Bona and Soyeur [35]).

The functional H can be re-written as

$$H = \iint_{\mathbb{R}^2} (\Delta\varphi - \varphi) \mathcal{L}(\Delta\varphi - \varphi) d\xi dy, \quad \mathcal{L} = -(\Delta - 1)^{-1} + \Phi'_s. \quad (24)$$

In this representation, the integrand in H takes the form of a quadratic form with the dependent variable given by the *perturbation* vorticity $(\Delta - 1)\varphi$ and the “form operator” \mathcal{L} . We note that $\mathcal{L}(\Delta\varphi - \varphi) = -\varphi + \Phi'_s(\Delta\varphi - \varphi)$. This form for H is motivated by the structure of the integrand of the functional examined in the stability theory for (1 + 1)-dimensional solitons [24–26]. We remind ourselves that in the stability theory for (1 + 1)-dimensional solitons, the quadratic part of the functional examined *is* the second variation of the constrained Hamiltonian for which the soliton satisfies the first-order necessary conditions for an extremal.

3.1. The eigenvalue problem

The eigenvalue problem we examine is given by

$$\mathcal{L}[(\Delta - 1)\phi] = \lambda\phi, \quad (\xi, y) \in \mathbb{R}^2,$$

or, equivalently,

$$\Delta\phi - \left(\frac{\Phi'_s + \lambda + 1}{\Phi'_s} \right) \phi = 0, \quad (\xi, y) \in \mathbb{R}^2, \quad (25)$$

where $\phi \in C^{(1)}(\mathbb{R}^2)$ and it is required that (2) holds for $\psi = \phi$. In the exterior and interior regions, (25) takes the form, respectively,

$$\Delta\phi - \left(\frac{\lambda + 1 + c}{c} \right) \phi = 0 \quad \text{for } r > a, \quad (26)$$

$$\Delta\phi + [\kappa^2 + (1 + \kappa^2)\lambda]\phi = 0 \quad \text{for } r < a, \quad (27)$$

where it is understood that $1 + c^{-1} > 0$ and that $\kappa = \kappa(a, c)$ is obtained from (16).

LEMMA 1. $\lambda \in \mathbb{R}$.

Proof: The fact that the eigenvalues are necessarily real is not entirely trivial because of the discontinuity and possible sign change in Φ'_s across $r = a$. If (25) is multiplied by the complex conjugate of ϕ and the result integrated over \mathbb{R}^2 , it follows that

$$\iint_{\mathbb{R}^2} |\nabla\phi|^2 + \left(\frac{\Phi'_s + \lambda + 1}{\Phi'_s} \right) |\phi|^2 d\xi dy = 0. \tag{28}$$

If (28) is subtracted from the complex conjugate of itself, it follows that

$$(\lambda^* - \lambda) \iint_{\mathbb{R}^2} \frac{|\phi|^2}{\Phi'_s} d\xi dy = 0, \tag{29}$$

where λ^* is the complex conjugate of λ . For the $c < -1$ modon solutions, $\Phi'_s < 0 \forall (\xi, y) \in \mathbb{R}^2$ so, necessarily, $\lambda \in \mathbb{R}$. However, the $c > 0$ modon solutions, Φ'_s changes sign across $r = a$, so an alternate argument is needed.

If $\lambda \neq \lambda^*$, then the integral in (29) is necessarily zero. This fact, together with (28), implies

$$\iint_{\mathbb{R}^2} |\nabla\phi|^2 + |\phi|^2 d\xi dy = 0,$$

so that only the trivial solution $\phi = 0$ is possible. ■

3.2. The eigenvalues

The discrete doubly indexed continuously differentiable orthonormal eigenfunctions can be written in the form

$$\phi_{nm} = A_{nm} [\cos(n\theta), \sin(n\theta)] \begin{cases} K_n(\gamma_{nm}r/a)/K_n(\gamma_{nm}) & \text{for } r \geq a, \\ J_n(v_{nm}r/a)/J_n(v_{nm}) & \text{for } r < a, \end{cases} \tag{30}$$

with $n \in \{0, 1, 2, \dots\}$, $m \in \{1, 2, \dots\}$,

$$A_{nm} \equiv \left[\frac{(2 - \delta_{n0})(\kappa^2 + 1 + 1/c)^{-1} v_{nm}^2 K_n^2(\gamma_{nm})}{\pi a^2 [v_{nm}^2 K_n^2(\gamma_{nm}) + a^2 K_{n-1}(\gamma_{nm}) K_{n+1}(\gamma_{nm})]} \right]^{1/2},$$

$$\gamma_{nm} \equiv a\sqrt{1 + (1 + \lambda_{nm})/c}, \quad v_{nm} \equiv a\sqrt{\kappa^2 + (1 + \kappa^2)\lambda_{nm}},$$

and where λ_{nm} is the m th ordered root, for a given n , of

$$\frac{\gamma_{nm} K_{n+1}(\gamma_{nm})}{K_n(\gamma_{nm})} = \frac{v_{nm} J_{n+1}(v_{nm})}{J_n(v_{nm})}. \tag{31}$$

We will discuss the sense in which the $\phi_{nm}(r, \theta)$ are orthonormal after we describe the λ_{nm} solutions to (31).

THEOREM 1. *The eigenvalues satisfy the bounds*

$$\lambda_{nm} > \lambda_{\text{lower}} \equiv -\kappa^2/(1 + \kappa^2) \quad \text{if } c > 0, \tag{32}$$

$$\lambda_{\text{lower}} < \lambda_{nm} < \lambda_{\text{upper}} \equiv -c(1 + c^{-1}) \quad \text{if } c < -1. \tag{33}$$

Proof: For solutions of the form (30) to satisfy (2), it is required that $\lambda_{nm}/c > -(1 + c^{-1})$ which, in turn, implies that $\lambda_{nm} > -c(1 + c^{-1})$ if $c > 0$ and $\lambda_{nm} < -c(1 + c^{-1})$ if $c < -1$, respectively. Moreover, $\lambda_{nm} > -\kappa^2/(1 + \kappa^2)$ since, if not, then (31) cannot be satisfied since if $\lambda_{nm} < -\kappa^2/(1 + \kappa^2)$, the right-hand side of (31) is given by

$$-|v_{nm}|I_{n+1}(|v_{nm}|)/I_n(|v_{nm}|) < 0 \quad \forall |v_{nm}| \neq 0,$$

while the left-hand side is strictly positive. In addition, because

$$\kappa^2/(1 + \kappa^2) < c(1 + c^{-1}) \quad \forall \kappa \in \mathbb{R} \text{ and } c > 0,$$

the theorem follows. ■

Our computational work suggests the theoretically determined bounds are sharp. Thus, for the $c > 0$ modon solutions, the discrete spectrum of \mathcal{L} , denoted $\text{spec}(\mathcal{L})$, is bounded below but not above.

LEMMA 2. *For $c > 0$, the eigenvalues are countably infinite and increase without limit.*

Proof: We argue directly from (31). It follows from the properties of Bessel functions that the left-hand side of (31) is finite and positive for all real γ_{nm} and that the zeros of J_{n+1} and J_n separate each other and increase without bound. Thus, thought of as a function of v_{nm} , the range of the right-hand side of (31) is \mathbb{R} for the open interval between *each* consecutive pair of zeros of J_n . By the Intermediate Value Theorem, there exists v_{nm} which will satisfy (31) for *any* γ_{nm} between *each* consecutive pair of zeros of J_n . Therefore, if $c > 0$, there exists for each value of n , a countable infinity of λ_{nm} for which $\lambda_{nm} < \lambda_{n(m+1)}$ and $\lambda_{nm} \rightarrow \infty$ as $m \rightarrow \infty$ for fixed n . Similarly, for a given value of m , there exists, if $c > 0$, a countable infinity of λ_{nm} for which $\lambda_{nm} < \lambda_{(n+1)m}$ and $\lambda_{nm} \rightarrow \infty$ as $n \rightarrow \infty$ for fixed m . ■

For the $c < -1$ modon solutions, $\text{spec}(\mathcal{L})$ is bounded below *and* above (and, as it turns out, contains only a finite number of eigenvalues). The $\{\lambda_{nm}\}$

Table 1
 λ_{nm} for $n = 0, \dots, 4$ and $m = 1, \dots, 5$ for $c = a = 1.0$.

m	n				
	0	1	2	3	4
1	-0.81	-0.51	0	0.68	1.53
2	0.14	1.02	2.10	3.37	4.91
3	2.26	3.73	5.39	7.24	9.26
4	5.56	7.61	9.85	12.28	14.89
5	10.03	12.66	15.48	18.49	21.68

are not necessarily of definite sign for either the $c > 0$ or $c < -1$ modon solutions, respectively. And, of course, it is already known that $\lambda_{2m} = 0$ for some m with the corresponding eigenfunction being proportional to $\partial\varphi_s/\partial\xi$.

Table 1 lists the eigenvalues λ_{nm} for the range $n = 0, \dots, 4$ and $m = 1, \dots, 5$ for the rightward-traveling modon $c = a = 1.0$ ($\implies \kappa \simeq 3.98$ and $\lambda_{\text{lower}} \simeq -0.94$). One can see that there are two negative eigenvalues (λ_{01} and λ_{11}) and that the zero eigenvalue is λ_{21} .

THEOREM 2. *There are exactly two negative eigenvalues (λ_{01} and λ_{11}) $\forall a > 0$ and $c > 0$ or $c < -1$ and the zero eigenvalue is always λ_{21} . All the remaining eigenvalues, should they exist, are positive.*

Proof: It follows from the properties of Bessel functions and (31) that $0 < \nu_{01} < j_{0,1}$ where $j_{n,m}$ is the m th nontrivial ordered zero of J_n , i.e., $J_n(j_{n,m}) = 0$ and $0 < j_{n,m} < j_{n,m+1} \forall n$. Substituting in for ν_{01} implies that

$$-\frac{\kappa^2}{1 + \kappa^2} < \lambda_{01} < \frac{j_{0,1}^2 - \nu^2}{a^2(1 + \kappa^2)} < 0, \tag{34}$$

since it follows from (16) that (for the ground state modon) $j_{1,1} < \nu < j_{2,1}$ and obviously $j_{1,1} > j_{0,1}$.

Similarly, it follows from the properties of Bessel functions and (31) that $j_{0,1} < \nu_{11} < j_{1,1}$, which in turn implies that

$$\frac{j_{0,1}^2 - \nu^2}{a^2(1 + \kappa^2)} < \lambda_{11} < \frac{j_{1,1}^2 - \nu^2}{a^2(1 + \kappa^2)} < 0, \tag{35}$$

since $j_{0,1} < j_{1,1} < \nu < j_{2,1}$. Thus, we have proved that $\lambda_{01} < \lambda_{11} < 0 \forall a$ and $c > 0$ or $c < -1$.

To see that $\lambda_{21} = 0$, we first note that for $n = 2$ and $m = 1$, (31) can be re-written, using recursion relations for Bessel functions, as

$$\frac{\gamma_{21}K_1(\gamma_{21})}{K_2(\gamma_{21})} = -\frac{\nu_{21}J_1(\nu_{21})}{J_2(\nu_{21})}, \tag{36}$$

i.e., (16), from which it follows that $\lambda_{21} = 0$ uniquely.

We now show that, provided they exist, all the remaining eigenvalues are positive. As we have seen, if $c > 0$, there are an infinite number of additional eigenvalues. However, $c < -1$, positive eigenvalues may not exist and, in any event, there will be only a finite number of them. In the proof we now present for the positiveness of the remaining eigenvalues, it is important to keep in mind that the argument is only valid if, in fact, the eigenvalue actually exists, i.e., a real solution for λ_{nm} actually exists to (31). We will show, later in the section, that there exist values of a , $c < -1$ and κ for which there are no positive eigenvalues.

For $n > 2$ and $m \geq 1$, we begin by noting that (31) can be re-written in the form

$$\frac{\gamma_{nm}K_{n-1}(\gamma_{nm})}{K_n(\gamma_{nm})} = -\frac{\nu_{nm}J_{n-1}(\nu_{nm})}{J_n(\nu_{nm})}, \tag{37}$$

from which it follows that, necessarily, $j_{n-1,1} < \nu_{n1} < j_{n,1}$ (irrespective of whether or not there actually exists λ_{n1} actually satisfying (37)). That is, for any positive real value of γ_{n1} , there will exist $\nu_{n1} \in (j_{n-1,1}, j_{n,1})$ such that the right-hand side equals the left-hand side in (37). Since the $j_{n,1}$ are discrete and increasing and $\partial \nu_{n1} / \partial \lambda_{n1} > 0$ with $\lambda_{21} = 0$, it follows, provided they exist, that $\lambda_{n1} > 0$ for $n > 2$. We have already shown that the λ_{nm} are increasing with respect to m for a given n . Thus we have established that, should the λ_{nm} exist, they must satisfy $\lambda_{nm} > 0 \forall n > 2$ and $m \geq 1$.

All that remains to be argued is that $\lambda_{nm} > 0$ for $n = 0, 1$ and $m \geq 2$. It is sufficient to show that $\lambda_{02} > 0$ since, if this is true, then by the fact that the λ_{nm} are increasing with respect to m (for fixed n) and with respect to n (for fixed m), the rest follows.

Let $f(\lambda)$ be the function given by

$$f(\lambda) = \frac{K_0(\gamma_{02})}{\gamma_{02}K_1(\gamma_{02})} - \frac{J_0(\nu_{02})}{\nu_{02}J_1(\nu_{02})}, \tag{38}$$

or, equivalently,

$$f(\lambda) = \frac{K_2(\gamma_{02})}{\gamma_{02}K_1(\gamma_{02})} + \frac{J_2(\nu_{02})}{\nu_{02}J_1(\nu_{02})} - 2 \left(\frac{1}{\gamma_{02}^2} + \frac{1}{\nu_{02}^2} \right), \tag{39}$$

where we write, for the moment,

$$\gamma_{02}(\lambda) = a\sqrt{1 + (1 + \lambda)/c}, \quad \nu_{02}(\lambda) \equiv a\sqrt{\kappa^2 + (1 + \kappa^2)\lambda}.$$

Here, we are considering f as a function of the continuous real variable λ . Comparing (38) with (31) or (37) we see that λ_{02} satisfies $f(\lambda_{02}) = 0$. We

begin by noting that based on the properties of Bessel functions it follows from (31) that λ_{02} satisfies $j_{1,1} < \nu_{02}(\lambda_{02}) < j_{0,2}$. Thus, $f(\lambda)$ is a continuously differentiable function on the interval $j_{1,1} < \nu_{02}(\lambda) < j_{0,2}$ satisfying

$$\lim_{\nu_{02} \downarrow j_{1,1}} f(\lambda) = -\infty \quad \text{and} \quad f(\lambda)|_{\nu_{02}=j_{0,2}} > 0,$$

regardless of the finite positive value of γ_{02} . Since $f(\lambda)$ is continuous it clearly has a zero, regardless of the finite positive value of γ_{02} , for some value of ν_{02} in the interval $j_{1,1} < \nu_{02} < j_{0,2}$. Again, we emphasize that this argument does not imply that λ_{02} necessarily exists; that depends on whether or not the value of λ_{02} associated with the value of the ν_{02} for which $f = 0$, also satisfies (33).

In addition, it follows from (39) and the modon dispersion relation (16) that

$$f(0) = -2 \left(\frac{1}{\gamma^2} + \frac{1}{\nu^2} \right) < 0,$$

so that $\lambda_{02} \neq 0$.

Finally, we show $df/d\lambda > 0$. After a little algebra, it follows from (38) that

$$\frac{df}{d\lambda} = \frac{a^2}{2c\gamma_{02}^2} \left(\frac{K_0^2(\gamma_{02})}{K_1^2(\gamma_{02})} - 1 \right) + \frac{a^2(1 + \kappa^2)}{2\nu_{02}^2} \left(1 + \frac{J_0^2(\nu_{02})}{J_1^2(\nu_{02})} \right).$$

If $c < -1 < 0$, $df/d\lambda > 0$ since $0 < K_0(\gamma_{02})/K_1(\gamma_{02}) < 1$. If $c > 0$, $df/d\lambda > 0$ since

$$-\frac{1}{c\gamma_{02}^2} + \frac{(1 + \kappa^2)}{\nu_{02}^2} = \frac{a^2(1 + 1/c + \kappa^2)}{\gamma_{02}^2 \nu_{02}^2} > 0.$$

Hence, we have shown that $f(\lambda)$ is a monotonically increasing continuously differentiable function over the interval $j_{1,1} < \nu_{02}(\lambda) < j_{0,2}$, which possesses a zero somewhere in this interval, i.e., $f(\lambda_{02}) = 0$, and that $f(0) < 0$. Hence, by continuity, $\lambda_{02} > 0$. Because the eigenvalues λ_{nm} increase (provided they exist) with respect to m for fixed n , and with respect to n for fixed m , it follows that $\lambda_{nm} > 0$ for $n = 0, 1$ and $m \geq 2$. ■

Table 2 lists *all* the eigenvalues λ_{nm} for the leftward-traveling modon $c = -4.0$ and $a = 1.0$ ($\implies \kappa \simeq 3.90$, $\lambda_{\text{lower}} \simeq -0.94$ and $\lambda_{\text{upper}} = 3.0$). We remark that the fact that λ_{01} , λ_{11} , and λ_{lower} appear to be the same in Tables 1 and 2 is a consequence only of the fact that the numbers we report are rounded-off to only two decimal places. They are not, in fact, identical.

The first thing to note is that, in contrast to the rightward-traveling modon, for the leftward-traveling modon there are only a finite number of eigenvalues. Because of the existence of a finite λ_{upper} (see (33)) the eigenvalues cannot increase without bound and there is no finite accumulation point (which would allow the possibility of an infinite number of λ_{nm}).

Table 2
 λ_{nm} for $c = -4.0$ and $a = 1.0$.

m	n					
	0	1	2	3	4	5
1	-0.81	-0.51	0	0.70	1.58	2.61
2	0.12	0.99	2.10	$\bar{\neq}$	$\bar{\neq}$	$\bar{\neq}$
3	2.19	$\bar{\neq}$	$\bar{\neq}$	$\bar{\neq}$	$\bar{\neq}$	$\bar{\neq}$

For the leftward-traveling modon, there will always be a minimum of three eigenvalues (λ_{01} , λ_{11} , and λ_{21}). That is, two negative eigenvalues and the zero eigenvalue but that there need not necessarily exist a strictly positive eigenvalue. For example, for $a = 1.0$ and $c = -1.025$ ($\implies \kappa \simeq 3.83$, $\lambda_{\text{lower}} \simeq -0.94$, and $\lambda_{\text{upper}} = 0.025$) we find that there are just the three eigenvalues $\lambda_{01} \simeq -0.81$, $\lambda_{11} \simeq -0.51$, and $\lambda_{21} = 0$ and, thus, there is no positive eigenvalue. We have not been able to determine simple sufficient conditions for the nonexistence of positive eigenvalues when $c < -1$. Nevertheless, we believe the following to be true.

CONJECTURE 1. *For the leftward-traveling modon, there will exist $c^* < -1$ (where c^* depends on a) such that $\forall c$ in the open interval $c^* < c < -1$, $\text{spec}(\mathcal{L})$ contains only two negative eigenvalues and the zero eigenvalue and there are no positive eigenvalues.*

3.3. The eigenfunctions

We now turn to discussing the sense in which the ϕ_{nm} are orthonormal. It follows from (26) and (27) that

$$(\lambda_{nm} - \lambda_{\tilde{n}\tilde{m}})\{\phi_{nm}, \phi_{\tilde{n}\tilde{m}}\} = 0, \tag{40}$$

and by *direct calculation* that

$$\{\phi_{nm}, \phi_{nm}\} = 1, \tag{41}$$

$\forall n, m, \tilde{n}, \tilde{m}, a$, and c , where the symmetric bilinear functional $\{f, g\}$ is given by

$$\{f, g\} \equiv \iint_{\mathbb{R}^2} \frac{fg}{\Phi'_s} d\xi dy = (\kappa^2 + 1) \iint_{r < a} fg d\xi dy - \frac{1}{c} \iint_{r > a} fg d\xi dy, \tag{42}$$

where f and g are square integrable.

The A_{nm} coefficients in (30) were determined by substituting (30) into (41) and explicitly computing the integrals. Perhaps surprisingly, and this

is an important observation, regardless of the (nonzero) choice of the A_{nm} coefficients, direct calculation establishes the following Lemma.

LEMMA 3. $\{\phi_{nm}, \phi_{nm}\} > 0 \forall n, m, a$ and c (irrespective of whether $c > 0$ or $c < -1$ or the nonzero choice of A_{nm}).

We also remark that although (40) follows from (26) and (27), it is possible to directly verify the following lemma.

LEMMA 4. $\{\phi_{nm}, \phi_{\tilde{n}\tilde{m}}\} = 0$ if $n \neq \tilde{n}$ or (perhaps less obviously) if $m \neq \tilde{m} \forall a$ and c (again, irrespective of whether $c > 0$ or $c < -1$ or the nonzero choice of A_{nm}).

For the leftward-traveling modon, $c < -1 \implies \{f, g\}$ is a properly defined inner product for the entire Hilbert space $L_2(\mathbb{R}^2)$. That is, in addition to being symmetric and bilinear, $\{f, f\} \geq 0 \forall f \in L_2(\mathbb{R}^2)$ with $\{f, f\} = 0 \implies f = 0$ (almost everywhere). Since there are only a finite number of ϕ_{nm} if $c < -1$, $\text{span}\{\phi_{nm} | c < -1\}$ only forms a finite-dimensional subspace of $L_2(\mathbb{R}^2)$.

For the rightward-traveling modon $c > 0$ and $\{f, f\}$ is not sign definite $\forall f \in L_2(\mathbb{R}^2)$. For example, consider a compactly supported continuous $f \geq 0$ where the support is in the exterior region $r > a$. Clearly, $f \in L_2(\mathbb{R}^2)$ and $\{f, f\} < 0$. On the other hand, for a compactly supported continuous $f \geq 0$ where the support is in the interior region $r < a$, again $f \in L_2(\mathbb{R}^2)$, but in this case $\{f, f\} > 0$. Thus, for the rightward-traveling modon where $c > 0$, $\{f, g\}$ is *not* a properly defined inner product for the entire space $L_2(\mathbb{R}^2)$. Nevertheless, in the case $c > 0$, (40) and (41) can be rewritten as $\{\phi_{nm}, \phi_{\tilde{n}\tilde{m}}\} = \delta_{n\tilde{n}}\delta_{m\tilde{m}} \forall n, m, \tilde{n}$, and \tilde{m} . The $\text{span}\{\phi_{nm} | c > 0\}$ with respect to the norm $\sqrt{\{\cdot, \cdot\}}$, denoted as Λ , is a Hilbert space with $\{f, g\}$ as its inner product.

It is important to point out that Λ is *not* the set of $L_2(\mathbb{R}^2)$ functions for which $\{\cdot, \cdot\}$ is positive (although, obviously, Λ is a subset of this set). Consider two functions $f, g \in L_2(\mathbb{R}^2)$ where it is assumed that $\{f, f\} > 0$ and we write f and g in the form

$$f = \begin{cases} f_{>} & \text{if } r \geq a, \\ f_{<} & \text{if } r < a, \end{cases} \quad g = \begin{cases} 0 & \text{if } r \geq a, \\ -f_{<} & \text{if } r < a. \end{cases}$$

Obviously, $\{g, g\} > 0$. No matter the particular choice of f , it follows that

$$\{f + g, f + g\} = -\frac{1}{c} \iint_{r>a} (f_{>})^2 d\xi dy < 0,$$

if $c > 0$, which implies that the set of $L_2(\mathbb{R}^2)$ functions for which $\{\cdot, \cdot\}$ is positive is *not* a vector space whereas Λ is. However, we suspect that the following conjecture is true.

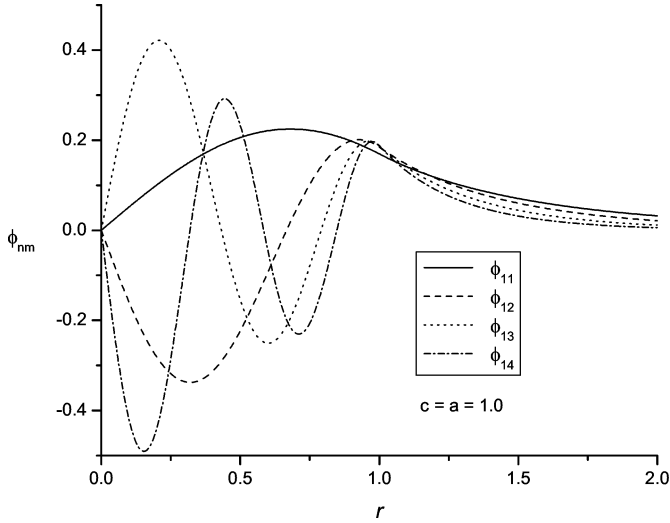


Figure 2. Radial part of ϕ_{1m} for $m = 1, \dots, 4$ if $c = a = 1.0$.

CONJECTURE 2. Λ is the “largest” subspace contained in $L_2(\mathbb{R}^2)$ for which $\{\cdot, \cdot\}$ is positive.

Figure 2 is a plot of the radial part of the first four ϕ_{nm} for $n = 1$ ($m = 1, \dots, 4$) for the rightward-traveling modon $c = a = 1.0$, i.e., (40) without the θ dependence. One can see that there are $m - 1$ nodal points associated with ϕ_{1m} (this holds $\forall n$) and that, of course, the nodal points are restricted to the interior region $r < a$. Examining Figure 2 we see that the value of the ϕ_{nm} near the modon boundary $r = a$ are all very similar. This is a straightforward consequence of the property that $A_{1m} \rightarrow \sqrt{2/\pi}(1 + 1/c + \kappa^2)/a \simeq 0.19$, which is independent of m , as the λ_{1m} increase (rather quickly).

Figure 3 is the corresponding plot for the radial part of ϕ_{nm} for $n = 1$ and $m = 1$ and 2 for the leftward-traveling modon $c = -4.0$ and $a = 1.0$. The qualitative structure of the ϕ_{nm} in Figures 2 and 3 are similar. The most important difference is that, of course, for the rightward-traveling solution there are an infinite number of orthogonal eigenfunctions ϕ_{1m} whereas for the leftward-traveling modon with $c = -4.0$ and $a = 1.0$ there are only two orthogonal eigenfunctions.

4. Discussion

Given the piecewise linear relationship between the potential vorticity and the streak function for the modon, it seems that it should have been all but trivial to rigorously establish the stability or instability of modons. This would

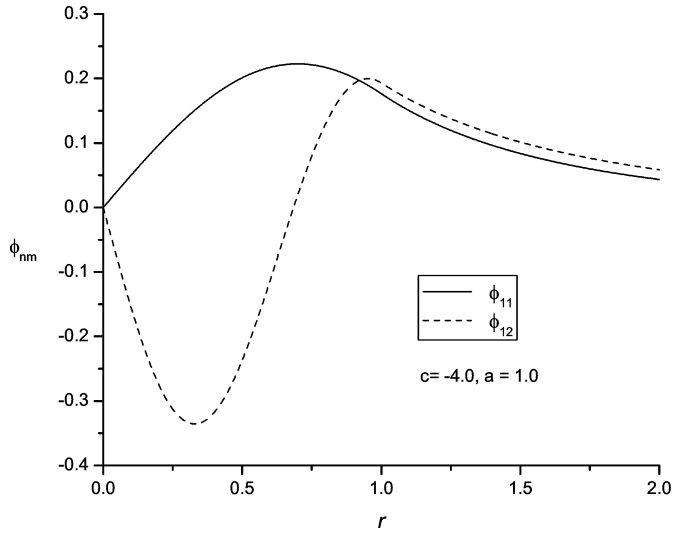


Figure 3. Radial part of ϕ_{1m} for $m = 1, 2$ if $c = -4.0$ and $a = 1.0$.

be an important result given the ubiquitous occurrence of robust propagating dipole vortices in quasi-two-dimensional turbulence in laboratory experiments and numerical simulations in both fluid and plasma dynamics. Yet, all such attempts have failed.

The purpose of the present contribution was to begin to systematically understand the mathematical structure of the only known quadratic invariant functional in linear modon stability theory. This invariant, which we labelled as H , has precisely the same mathematical form as the principal part would have for the disturbance field (or, equivalently, the second variation) in a “Benjamin-like” variational principle for solitary vortices. It is known, however, that such a variational principle does not exist for the modon.

The properties of the discrete spectrum associated with the “form operator” in the second variation of the Benjamin variational functional is critical in $(1 + 1)$ -dimensional soliton stability theory. Our goal here was to determine the discrete spectrum and the structure of the eigenspaces associated with the form operator in H . In particular, it was shown that

1. There are exactly two simple negative eigenvalues and a simple zero eigenvalue for both the leftward- and rightward-traveling modon.
2. The zero eigenvalue corresponds, of course, to the translational mode.
3. For the leftward-traveling modon, there are at most only a finite number of positive eigenvalues. Indeed, there are modon parameter values for which there are no positive eigenvalues. The eigenfunctions span only a finite-dimensional vector space and are orthogonal with respect to an inner product valid for all of L_2 .

4. For the rightward-traveling modon, there are always a countable infinity of positive eigenvalues. The eigenfunctions span a Hilbert space, which we denote by Λ , but are orthogonal with respect to an inner product which is not valid for all of L_2 . That is, $\Lambda \subset L_2$.

These results are encouraging. In $(1 + 1)$ -dimensional soliton stability theory, even though the form operator possesses negative eigenvalues, it is still possible to show that the pseudo-variational functional (see Section 1), associated with the Benjamin variational principle, is positive definite and thus stability can be established. The fact that we have found negative eigenvalues for the form operator in H does not imply instability. However, any such stability argument, presumably, will necessarily have to be based on a pseudo-variational functional which would have H as its principal part. Such a pseudo-variational functional is unknown at present.

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