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Derivation and analysis of a McPhee-like damping term for inertially oscillating ice drift

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Summary

It is shown that the empirical McPhee [8] damping term for inertial oscillations in time-dependent ice-ocean motion can be derived as a first-order correction when the air stress is quadratic in air relative to ice velocity. Analytical expressions are derived for the leading term transient ice and surface oceanic boundary-layer velocities and the mass transport function of a perturbation expansion in a small parameter $\epsilon[O(10^{-1})-O(10^{-2})]$, the ratio of scale ice speed to air speed.

1. Introduction

Consider a large ice sheet and underlying ocean in unaccelerated motion driven by a steady wind stress, which is subsequently subjected to a different steady wind stress. As the ice-ocean system readjusts to the new wind stress, weakly decaying inertial oscillations in the ice velocity field are known to occur. For example, Hunkins [4] has observed these oscillations in the drift record of ice island *T*-3 in the Arctic Ocean. McPhee [8] and Khandekar [5] have modelled inertial oscillations of drifting sea ice observed in the Beaufort Sea.

A difficulty of most models of inertially-oscillating ice drift (see Kheysin and Yvchenko [6], Colony and Thorndike [3], McPhee [8] and Lewis and Denner [7], among others) is that an empirical transport damping term is required in order that the ice-water transport converges to a steady-state value. Without some damping in the transport equation, the above models predict that the transient ice-water transport will oscillate without attenuation, in contrast to the observed behaviour (see McPhee [8] and Khandekar [5]).

McPhee [8] showed that observations of freely-drifting inertially-oscillating sea ice are essentially described by a balance between inertial, Coriolis, wind and water stresses. Further, McPhee [8] also showed that the observed decay in the inertial oscillations of the ice-water transport were accurately modelled by an empirical damping term that was proportional to the component of the ice-water transport antiparallel to the air stress. The main purpose of this paper is to show that if the air stress depends on the air relative to ice velocity, the leading-order damping terms have the same direction and magnitude as the empirical McPhee damping term.

Traditionally, the ice velocity is neglected in the air stress formula since the ice speed is approximately 2% of the ten-meter wind speed. This approximation is shown to be equivalent to obtaining the leading term of a perturbation expansion of the ice and surface oceanic boundary-layer (OBL) velocity fields in the small parameter ϵ which is the ratio of scale ice speed to the air speed. The scale ice speed is determined by requiring the $O(1)$ balance in the sea ice equation to be between the air and the water stress terms. However this perturbation expansion is shown to be secular. Moreover, this secularity is independent of the form of the water stress or OBL.

As in previous studies of inertially-oscillating ice drift (e.g. Kheysin and Yvchenko [6], McPhee [8] and Khandekar [5]), we take the OBL to be an infinitely deep time-dependent Ekman layer and the (ice-) water stress as proportional to the vertical gradient of the OBL velocity field evaluated at the ice-water interface. The air (-ice) stress is quadratic in air speed, but formulated with air relative to ice velocity.

The problem is formulated in Section 2. In Section 3, perturbation solutions (in ϵ) are obtained for the ice and OBL velocities and the mass transport. It is shown that for $t < \epsilon^{-2} f^{-1}$s (f is the Coriolis parameter $2\Omega \sin(\theta)$ where Ω is the magnitude of the Earth's angular velocity vector and θ the latitude), neglecting the ice velocity terms in the air stress approximates the nonlinear solutions to $O(\epsilon)$. When $t \simeq \epsilon^2 f^{-1}$s (i.e., approximately 5 to 10 days) the perturbation solutions are no longer valid, with higher-order terms becoming unbounded as $t \to \infty$. For $t > \epsilon^{-2} f^{-1}$s, the $O(\epsilon)$ terms in the ice equation are essential to describe the damping of the inertial oscillations of the $O(1)$ transients. Correct perturbation solutions for $t > \epsilon^{-2} f^{-1}$s are given. In addition, it is shown that the ice, OBL and transport decay about their respective equilibrium values as $O(t^{-3/2})$, $O(t^{-3/2})$ and $O(t^{-1/2})$ respectively, as $t \to \infty$. These decay rates are faster than the $O(t^{-1/2})$ for the ice and OBL velocities and the absence of any decay in the transport that are predicted by models in which the ice velocity is neglected in the air stress (Kheysin and Yvchenko [6]). The results are summarized in Section 4.

2. Formulation of governing equations

The nondimensional equations of motion for unsteady wind-driven ice motion can be written as (McPhee [8])

$$\mu(\partial_t + \mathrm{i})V = |A - \epsilon V|(A - \epsilon V) - \partial_z W(z = 0, t), \tag{2.1}$$

$$(\partial_t + \mathrm{i})W = \partial_{zz} W, \quad z < 0, \tag{2.2}$$

where $\mu = \rho h[\rho_w(\nu/f)^{1/2}]^{-1}$ and $\epsilon = |A^*|\rho_a c_a[\rho_w(f\nu)^{1/2}]^{-1}$ with ρ, h, f, ν, ρ_w, i, c_a and ρ_a the ice density, ice thickness, Coriolis parameter, vertical OBL eddy viscosity, OBL density, $(-1)^{1/2}$, air drag coefficient and air density, respectively. Positive z points vertically upward and $z = 0$ corresponds to the ocean surface, just below the ice sheet. The complex ice, OBL and air velocities are given by V, W and A, respectively, with the real part positive east and the imaginary part positive north. The air velocity is taken to be constant and is typically the wind at a height of ten meters.

The nondimensional variables are related to the dimensional (asterisked) variables by

$$z^* = z(\nu/f)^{1/2}, \quad t^* = f^{-1}t, \quad A^* = |A^*|A,$$

$$\{V^*(t^*), W^*(z^*, t^*)\} = \{V(t), W(z, t)\}\rho_a c_a |A^*|^2 \left[\rho_w (f\nu)^{1/2}\right]^{-1}.$$

The scalings for z^* and t^* are the e-folding depth in an Ekman boundary layer and the inverse inertial frequency, respectively. The air velocity is scaled by $|A^*|$, so that $|A| = 1$. The OBL and ice velocities are scaled by demanding that the air stress balance the water stress. The magnitude of the air stress is $\rho_a c_a |A^*|^2$ and the magnitude of the water stress is $\rho_w (\nu f)^{1/2} V^*$. Equating the two implies that the OBL and ice velocity fields are scaled by $\rho_a c_a |A^*|^2 [\rho_w (\nu f)^{1/2}]^{-1}$.

The nondimensional numbers μ and ϵ measure inertial terms in the ice cover spinup relative to the inertial terms in the OBL and the ratio of scale ice speed to air speed, respectively. Data for typical sea ice covers and OBLs (Campbell [2], Hunkins [4], McPhee [8], Pollard [9]) suggest $\rho h \simeq 3 \cdot 10^3$ kgm^{-3}, $f \simeq 1.4 \cdot 10^{-4}$ s^{-1}, $\rho_w \simeq 10^3$ kgm^{-1}, $\nu \simeq 10^{-3} - 10^{-1}$ m^{2}s^{-1}, $|A^*| \simeq 10 - 20$ ms^{-1} and $c_a \simeq 2.7 \cdot 10^{-3}$. It follows that $\mu \simeq O(10^{-1}) - O(1)$ and $\epsilon \simeq O(10^{-2}) - O(10^{-1})$. It will be subsequently assumed $\mu \simeq O(1)$ and $\epsilon \ll 1$.

The ice sheet is assumed initially to be in equilibrium with respect to a constant ten-meter wind $B \neq A$. This assumption allows the OBL to contain a nontrivial vertical shear while retaining analytical simplicity. Deep in the ocean the wind-induced currents vanish thus $W(z, t) \to 0$ as $z \to -\infty$. At the ice-water interface the no-slip condition holds, i.e. $W(z = 0, t) = V(t)$. The nondimensional initial condition becomes $W(z, t = 0) = W_0(z) = V_0 \exp(i^{1/2} z)$ with V_0 satisfying

$$|B - \epsilon V_0|(B - \epsilon V_0) - (i^{1/2} + i\mu) V_0 = 0. \tag{2.3}$$

The equilibrium solution of (2.1) and (2.2) is given by $W_\infty(z) = V_\infty \exp(i^{1/2} z)$, with V_∞ satisfying

$$|A - \epsilon V_\infty|(A - \epsilon V_\infty) - (i^{1/2} + i\mu) V_\infty = 0. \tag{2.4}$$

The inertial oscillations in the ice velocity $V(t)$ that result from numerically integrating (2.1) and (2.2) (with a fourth-order Runge-Kutta and an explicit finite-difference scheme, respectively) are shown in Figs. 1 and 2.

The transient OBL and ice velocities, OBL transport and ice-water transport, defined by $\Psi(z, t) = W(z, t) - W_\infty(z)$, $\Phi(t) = V(t) - V_\infty$, $\Pi(t) = \int_{-\infty}^{0} \Psi(z, t)\, dz$ and $\Pi(t) +$

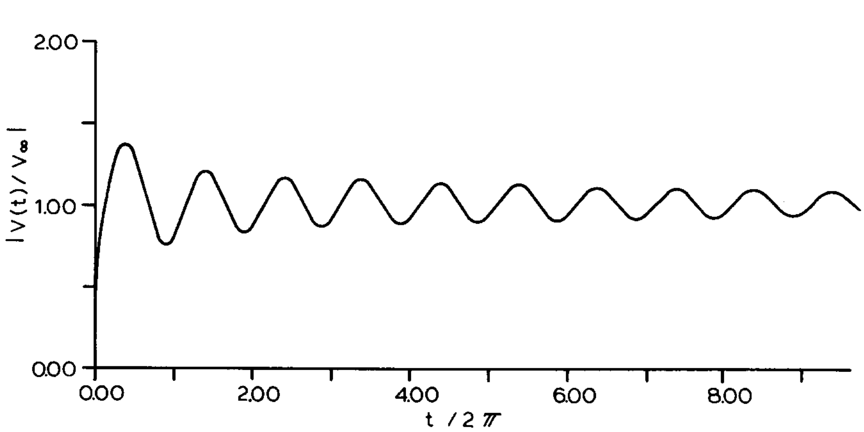


Figure 1. Magnitude of the nondimensional ice velocity $|V(t)/V_\infty|$ vs. $t(2\pi)^{-1}$ where $V(t)$ and V_∞ are numerical solutions to (2.1), (2.2) and (2.4). Parameter values used are: $A^* = 10$ ms^{-1}, $B^* = 0$ ms^{-1}, $\mu = 0.1$ and $\epsilon = 0.01$.

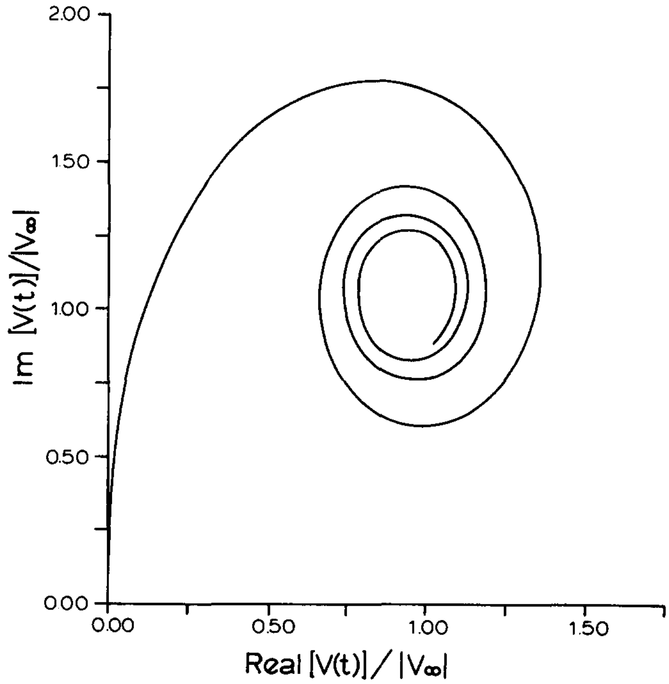


Figure 2. Hodograph of the nondimensional ice velocity $V(t)/V_\infty$, obtained by a numerical integration of (2.1) and (2.2). Parameter values used are: $A^* = 10\ \mathrm{ms}^{-1}$, $B^* = 0\ \mathrm{ms}^{-1}$, $\mu = 0.1$ and $\epsilon = 0.01$.

$\mu\Phi(t)$ respectively, satisfy

$$\mu(\partial_t + \mathrm{i})\phi = |A - \epsilon\Phi - \epsilon V_\infty|(A - \epsilon\Phi - \epsilon V_\infty) - |A - \epsilon V_\infty|(A - \epsilon V_\infty) - \partial_z\Psi(z = 0, t), \tag{2.5}$$

$$(\partial_t + \mathrm{i})\Psi = \partial_{zz}\Psi, \quad z < 0, \tag{2.6}$$

$$(\partial_t + \mathrm{i})[\Pi + \mu\Phi] = |A - \epsilon\Phi - \epsilon V_\infty|(A - \epsilon\Phi - \epsilon V_\infty) - |A - \epsilon V_\infty|(A - \epsilon V_\infty) \tag{2.7}$$

with $\Psi(z = 0, t) = \Phi(t)$ and $\Psi(z, t) \to 0$ as $z \to -\infty$. Equation (2.7) is obtained by $\int_{-\infty}^{0}$ (2.6) $\mathrm{d}z$ and applying (2.5).

Neglecting the $O(\epsilon)$ terms in (2.7) will result in its right-hand side being identically zero. Consequently, the transient ice-water transport will oscillate with the inertial frequency, without attenuation. This conclusion is true for all time-dependent ice-OBL models which 1) take the air stress to depend on the air velocity unreferenced to the ice velocity, 2) have an OBL with vertical shear stress continuous at the ice-water interface, and 3) have negligible shear stress at the bottom of the OBL. It can be shown that models which satisfy these three conditions will predict that

$$\Pi(t) + \mu\Phi(t) = (\Pi_0 + \mu\Phi_0)\exp(-\mathrm{i}t) \tag{2.8}$$

for all $t > 0$, where $\Pi_0 = \Pi(t=0)$ and $\Phi_0 = \Phi(t=0)$. It is shown in the next section that the $O(\epsilon)$ terms in (2.1) provide the necessary damping so that nondimensional OBL transport, say $M(t)$, satisfies

$$M(t) = \int_{-\infty}^{0} W(z, t)\, \mathrm{d}z \to M_\infty = \int_{-\infty}^{0} W_\infty(z)\, \mathrm{d}z = \mathrm{i}^{-1/2} V_\infty \tag{2.9}$$

(i.e. $\Pi(t) = M(t) - M_\infty \to 0$) as $t \to \infty$.

3. Asymptotic analysis for $\epsilon \to 0$

Solutions to (2.5)–(2.7) are constructed by expanding Ψ, Φ, Π, V_∞, V_0, W_∞, W_0 in a straightforward power series in ϵ. To $O(\epsilon)$ the solutions for V_0 and V_∞ are

$$V_0 = V_0^{(0)} - \epsilon\left[\beta V_0^{(0)} + \beta^{-1} B\left(B \cdot V_0^{(0)}\right)\right]\left(\mu \mathrm{i} + \mathrm{i}^{1/2}\right)^{-1} + O(\epsilon^2),$$

$$V_\infty = V_\infty^{(0)} - \epsilon\left[V_\infty^{(0)} + A\left(A \cdot V_\infty^{(0)}\right)\right]\left(\mu \mathrm{i} + \mathrm{i}^{1/2}\right)^{-1} + O(\epsilon^2),$$

where $V_\infty^{(0)} = A(\mu \mathrm{i} + \mathrm{i}^{1/2})^{-1}$, $V_0^{(0)} = \beta B(\mu \mathrm{i} + \mathrm{i}^{1/2})^{-1}$, $M \cdot N$ is defined $\mathrm{Re}(M)\,\mathrm{Re}(N) + \mathrm{Im}(M)\,\mathrm{Im}(N)$ and $\beta = |B^*/A^*|$.

Equations (2.5) and (2.7) are to $O(\epsilon)$:

$$\mu(\partial_t + \mathrm{i})\Phi = -\epsilon\Phi - \epsilon A(A \cdot \Phi) - \partial_z \Psi(z=0, t) + O(\epsilon^2), \tag{3.1}$$

$$(\partial_t + \mathrm{i})[\Pi + \mu\Phi] = -\epsilon\Phi - \epsilon A(A \cdot \Phi) + O(\epsilon^2). \tag{3.2}$$

The quadratic air stress has resulted in two terms which will damp the transient ice-water transport (see (3.2)). The first of these, $-\epsilon\Phi$, is proportional and antiparallel to the ice transport $\mu\Phi$. The second damping term, $-\epsilon A(A \cdot \Phi)$, is proportional to the component of the transient ice transport antiparallel to the air stress.

In dimensional form the $O(\epsilon)$ transport damping terms in (3.2) are

$$-\left(c_\mathrm{a}\rho_\mathrm{a}(\rho h)^{-1}|A^*|\right)\left[\rho h\Phi^* + A^*(A^* \cdot \rho h\Phi^*)\,|A^*|^{-2}\right],$$

which can be rewritten as

$$-\left(c_\mathrm{a}\rho_\mathrm{a}(\rho h)^{-1}|A^*|\right)\left[\rho h\Phi^* + \Upsilon_\mathrm{a}^*(\Upsilon_\mathrm{a}^* \cdot \rho h\Phi^*)\,|\Upsilon_\mathrm{a}^*|^{-2}\right], \tag{3.3}$$

where Υ_a^* is the $O(1)$ dimensional air stress $c_\mathrm{a}\rho_\mathrm{a}|A^*|A^*$ in (2.1). In our notation the McPhee damping term (McPhee [8]) is

$$-d_0 f(2\pi)^{-1}\Upsilon_\mathrm{a}^*\left[\Upsilon_\mathrm{a}^* \cdot (\Pi^* + \rho h\Phi^*)\right]|\Upsilon_\mathrm{a}^*|^{-2}$$

where d_0 is nondimensional damping parameter. This empirical damping term is oriented in the same direction as the second of the two derived transport-damping terms in (3.3). The damping coefficients, while appearing quite different, are in fact the same order of magnitude. Applications of the McPhee damping term (McPhee [8] and Khandekar [5]) use $d_0 \simeq 0.5-1.0$, so that $d_0 f(2\pi)^{-1} \simeq 10^{-6}-10^{-5}\mathrm{s}^{-1}$. Typical ten-meter air speeds and drag coefficients (given earlier) imply that the order of magnitude of the theoretically derived damping coefficient $c_a \rho_a (\rho h)^{-1} |A^*|$ is $10^{-6}-10^{-5}\mathrm{s}^{-1}$ (based on h between 2 and 20 meters), which is identical with the range for the empirically determined McPhee damping coefficient.

The effect of the $O(\epsilon)$ terms in (3.1) and (3.2) in determining the leading-order behaviour of the solutions for large time is most easily seen in $\Psi^T(z, s)$, $\Phi^T(s)$ and $\Pi^T(s) = \int_{-\infty}^{0} \Psi^T(z, s)\, \mathrm{d}z$, the Laplace transforms of $\Psi(z, t)$, $\Phi(t)$ and $\Pi(t)$, respectively. The solution of the transformed (2.6) and (3.1) is

$$\Psi^T(z, s) = \left(\Phi^T - \Phi_0^{-1}\right) \exp\left[(s+\mathrm{i})^{1/2}\right] - \Phi^T s^{-1} \exp[\mathrm{i}^{1/2} z],$$

$$\Pi^T(s) = \Pi_0 + (s+i)^{-1/2}\left(\Phi^T - \Phi_0 s^{-1}\right),$$

with $\Phi_0 = V_0 - V_\infty$, $\Pi_0 = \Phi_0 \mathrm{i}^{-1/2}$, and $\Phi^T(s)$ determined from

$$\begin{aligned} &\left\{\mu(s+\mathrm{i}) + (s+\mathrm{i})^{1/2} + \epsilon[1 + A(A\cdot)]\right\}\Phi^T \\ &\quad = \left[\mu(s+\mathrm{i}) + (s+\mathrm{i})^{1/2}\right]\Phi_0 s^{-1} - (\mu\mathrm{i} + \mathrm{i}^{1/2})\Phi_0 s^{-1} + O(\epsilon^2). \end{aligned} \tag{3.4}$$

A power-series solution of (3.4) in the form

$$\Phi^T(s) \doteq \Phi^{(0)T}(s) + \epsilon\Phi^{(1)T}(s) + \dots. \tag{3.5}$$

results in the leading-order term $\Phi^{(0)T}(s) = [s(\mu(s+i) + (s+i)^{1/2})]^{-1}$, which after a little algebra gives the leading-order solutions

$$\begin{aligned} \Psi^{(0)}(z, t) = {} & (\beta B - A) \exp[-\mathrm{i}t]\pi^{-1} t^{3/2} \int_0^\infty \exp(-r)\left[(r + \mathrm{i}t)(t + \mu^2 r)\right]^{-1} \\ & \times \left[r^{-1/2}\cos\left\{z(r/t)^{1/2}\right\} + \mu t^{-1/2}\sin\left\{z(r/t)^{1/2}\right\}\right]\mathrm{d}r, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Phi^{(0)}(t) = {} & (\beta B - A) \exp[-\mathrm{i}t]\pi^{-1} t^{3/2} \\ & \times \int_0^\infty \exp(-r)\left[r^{1/2}(r + \mathrm{i}t)(\mu^2 r + t)\right]^{-1} \mathrm{d}r, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \Pi^{(0)}(t) = {} & (\beta B - A)\mathrm{i}^{-1/2}\exp(-\mathrm{i}t) - (\beta B - A)\exp[-\mathrm{i}t]\mu\pi^{-1} t^{3/2} \\ & \times \int_0^\infty \exp(-r)\left[r^{1/2}(r + \mathrm{i}t)(t + \mu^2 r))\right]^{-1} \mathrm{d}r. \end{aligned} \tag{3.8}$$

As $t \to \infty$, $\Psi^{(0)}(z, t)$, $\Phi^{(0)}(t)$ and $\Pi^{(0)}(t)$ are $O(t^{-1/2}\exp(-\mathrm{i}t)]$, $O(t^{-1/2}\exp(-\mathrm{i}t)]$ and $O[\exp(-\mathrm{i}t)]$, respectively. Therefore a solution of the form (3.5) will imply that the leading-term transient transport will not decay to zero as $t \to \infty$. These leading-order solutions are, of course, exactly the solutions obtained by neglecting the ice velocity terms within the air stress and are equivalent to those derived in Kheysin and Yvchenko [6].

The expansion (3.5) is not, however, uniformily valid in t. It is straightforward to show that $\Psi^{(1)}(z, t)$ and $\Phi^{(1)}(t)$ are $O[\exp(-\mathrm{i}t)]$ and $\Pi^{(1)}(t)$ is $O[t^{1/2}\exp(-\mathrm{i}t)]$ as $t \to \infty$. Hence $|\Pi^{(1)}(t)| \to \infty$, and $\Psi^{(1)}(z, t)$ and $\Phi^{(1)}(t)$ will not decay to zero as $t \to \infty$. The breakdown of the expansion (3.5) can be viewed in the context of a boundary layer in the integrands of the Laplace inversion integrals. When $\mathrm{Re}(s + \mathrm{i}) = O(\epsilon^2)$ in (3.4) the expansion (3.5) fails to describe Φ^T, and it is within this region that the structure of the solutions for $t > O(\epsilon^{-2})$ is determined.

The correct leading term for $\Psi(z, t)$, $\Phi(t)$ and $\Pi(t)$ for $t > O(\epsilon^{-2})$ is obtained by introducing into (3.4) the boundary layer variable $s + \mathrm{i} = S\epsilon^2$ (Bender and Orszag [1]) and an expansion for $\Phi^T(S)$ in the form

$$\epsilon^{-1}(\Phi^{(0)T}(S) + \epsilon\Phi^{(1)T}(S) + \dots).$$

The $O(1)$ boundary-layer problem associated with (3.4) is

$$[S^{1/2} + 1 + A(A\,\cdot)]\Phi^{(0)T}(S) = \mathrm{i}(A - \beta B). \tag{3.9}$$

The $S^{1/2}$ coefficient of $\Phi^{(0)T}(S)$ corresponds to the momentum transfer from the ice cover to the OBL via the water stress. The $1 + A(A\,\cdot)$ coefficient of $\Phi^{(0)T}$ is derived from the $O(\epsilon)$ terms in the ice equation. Hence the $O(\epsilon)$ damping terms play an equivalent role as the stress in describing the structure of the transients for $t = O(\epsilon^{-2})$.

The solution for $\Phi^{(0)T}(S)$ is

$$\Phi^{(0)T} = \mathrm{i}[(A - \beta B)(\bar{S}^{1/2} + 3/2) + A^2(\bar{A} - \beta\bar{B})/2] \times [(\bar{S}^{1/2} + 3/2)(S^{1/2} + 3/2) - 1/4]^{-1} \tag{3.10}$$

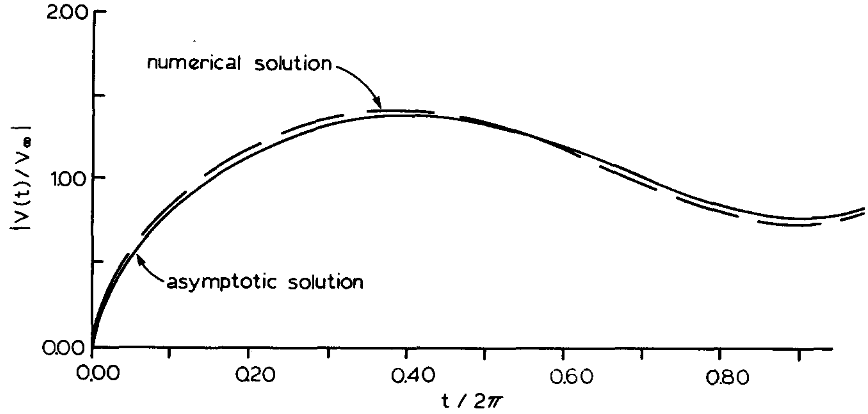
where $(\bar{S}, \bar{A}, \bar{B})$ is the complex conjugate of (S, A, B). The leading terms for $t > O(\epsilon^{-2})$ are given by

$$\Psi^{(0)}(z, t) = (A - \beta B)\exp[-\mathrm{i}t + \mathrm{i}\pi/2]\pi^{-1}t^{-1/2} \times \int_0^\infty r^{1/2}(r + 2\epsilon^2 t)^{-1}\exp(-r)\cos\{z(r/t)^{1/2}\}\mathrm{d}r, \tag{3.11}$$

$$\Phi^{(0)}(t) = (A - \beta B)\exp(-\mathrm{i}t + \mathrm{i}\pi/2)\pi^{-1}t^{-1/2} \times \int_0^\infty r^{1/2}(r + 2\epsilon^2 t)^{-1}\exp(-r)\mathrm{d}r, \tag{3.12}$$

$$\Pi^{(0)}(t) = \epsilon[(3/2)(A - \beta B) + (1/2)A^2(\bar{A} - \beta\bar{B})]\exp[-\mathrm{i}t + \mathrm{i}\pi/2]\pi^{-1}t^{1/2} \times \int_0^\infty \exp(-r)[r^{1/2}(r + 2\epsilon^2 t)]^{-1}\mathrm{d}r. \tag{3.13}$$

thus as $t \to \infty$, $\Psi(z, t)$ and $\Phi(t)$ are $O[t^{-3/2}\exp(-\mathrm{i}t)]$ and $\Pi(t)$ is $O[t^{-1/2}\exp(-\mathrm{i}t)]$. The decay rate of the inertial oscillations in the ice-water transport $\Pi(t) + \mu\Phi(t)$ is therefore $O(t^{-1/2})$.



Figur 3. Comparison between $|V(t)/V_\infty|$ (obtained by numerically integrating (2.1) and (2.2)) and $|V^{(0)}(t)/V_\infty^{(0)}| = |1+\Phi^{(0)}(t)/V_\infty^{(0)}|$ (given by (3.7)+(3.12)-(overlap term)) for $t<2\pi$ (one inertial period). Parameter values used are: $A^* = 10$ ms^{-1}, $B^* = 0$ ms^{-1} ($V_0 = 0$), $\mu = 0.1$ and $\epsilon = 0.01$.

Figure 3 compares the numerical solution for $|V(t)/V_\infty|$ obtained from (2.1) and (2.2) with $|V^{(0)}(t)/V_\infty^{(0)}| = |1+\Phi^{(0)}(t)/V_\infty^{(0)}|$ for $t<2\pi$ (i.e. one inertial period). The perturbation solution is obtained as (3.7) + (3.12) minus a term describing the overlap region [$t \simeq O(\epsilon^{-2})$], obtained in the usual manner by substituting the boundary-layer variable $s = S\epsilon^2$ into $I^{(0)}(s)$ of (3.4) and retaining the leading term as $\epsilon \to 0$ (Bender and Orszag [1]). The parameter values are listed in the figure caption. The figure depicts the similarity in the dynamical structure of the inertially-oscillating transients in agreement with the leading-term analysis developed here. The intersection at about $t \simeq \pi$ of the two solutions can be attributed to the nonlinearity of the ice velocity in the quadratic air stress which will tend to displace the period of the oscillation from the inertial period.

4. Summary and conclusions

On the basis of our theoretical study, the following conclusions can be made. The two $O(\epsilon)$ terms that occur in the ice equation when the air stress is expanded in a power series in ϵ are essential to describe the evolution of the $O(1)$ solutions for times greater than $\epsilon^{-2}f^{-1}$s (i.e., 5–10 days). Without these terms the transient inertial oscillations in the ice and OBL velocities decay like $O(t^{-1/2})$ and the transient ice-water transport oscillates about zero with the inertial frequency without attentuation. The $O(\epsilon)$ terms increase the decay in the transients of the ice and OBL velocities to $O(t^{-3/2})$ and in the transient ice-water transport to $O(t^{-1/2})$. The decay in the transient ice-water transport completely depends on the existence of these terms.

The first $O(\epsilon)$ damping term is antiparallel to the ice transport. The second $O(\epsilon)$ damping term is proportional to the component of the ice transport antiparallel to the air stress. The theoretically-derived damping coefficient associated with these terms has a magnitude range of $10^{-6} - 10^{-5}$ s^{-1}, which is identical to the magnitude range of the empirically postulated McPhee damping coefficient (McPhee [8] and Khandekar [5]). Moreover, the form of the second theoretically-derived damping term is nearly identical to the McPhee damping term, differing only in that the empirical damping term is formulated with the ice-water transport while the derived damping term is formulated with ice transport (or velocity).

The damping mechanism developed here suggests that the decay in the inertial oscillations in the ice-water transport is determined by the decay in the transient ice velocity. The decay of the transient ice velocity is determined by a balance between the $O(\epsilon)$ air stress terms and the water stress.

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References

[1] Bender, C.M. and Orszag, S.A., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York (1978).

[2] Campbell, W.J., On the steady-state drift of sea ice, Ph.D. Thesis, University of Washington, Seattle, Wash., (1964).

[3] Colony, R. and Thorndike, A.S., The horizontal coherency of the motion of summer Arctic sea ice, *J. Phys. Oceanog.* 10 (1980) 1281–1289.

[4] Hunkins, K., Inertial oscillations of Fletcher's ice island (T-3), *J. Geophys. Res.* 72 (1967) 1165–174.

[5] Khandekar, M.L., Inertial oscillations in floe motion over the Beaufort Sea: observations and analysis, *Atmosphere-Ocean* 18 (1980) 1–14.

[6] Kheysin D.E. and Yvchenko, V.O., Determination of the tangential stresses on the lower surface of ice drift in the presence of pure wind drift, *Tr. AANII* 332 (1976) 51–59.

[7] Lewis, J.K. and Denner, W.W., A theoretical response of a wind-driven oceanic surface layer and its application to the analysis of Lagrangian measurements, *J. Geophys. Res.* 89 (1984) 2101–2103.

[8] McPhee, M.G., A simulation of inertial oscillations in drifting pack ice. *Dyn. Atmos. Oceans* 2 (1978) 311–337.

[9] Pollard, R.T., Observations and models of the structure of the upper ocean, in: *Modelling and prediction of the upper layers of the ocean*, Ed. E.B. Krauss, Pergamon, Toronto (1977) 102–117.

[10] Thorndike, A.S. and Colony, R., Ice motion in response to geostrophic winds, *J. Geophys. Res.* 87 (1982) 5845–5852.