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★**Introduction to Hamiltonian fluid dynamics and stability theory. (English summary)**
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The subject of this book is the stability analysis, both linear and nonlinear, of equilibrium solutions in dynamical systems with Hamiltonian structure. Here, an equilibrium solution z_e of a dynamical system $\dot{z} = f(z)$, describing the evolution of points in the phase space of a physical system, is called stable in the sense of Lyapunov if there is a norm $\|\cdot\|$ on the phase space such that for every $\varepsilon > 0$ there is a $\delta > 0$ with the property that a trajectory $z(t)$ of the dynamical system with $\|z(0) - z_e\| < \delta$ satisfies $\|z(t) - z_e\| < \varepsilon$ for all times t. The equilibrium point z_e is called linearly stable if 0 is a stable equilibrium of the linearization of the dynamical system about z_e. The author first studies the case in which the dynamical system at hand is a canonical Hamiltonian system, i.e. one has $f(z) = J\nabla H(z)$, where $z = (q,p) \in \mathbf{R}^{2n}$, J is the canonical symplectic matrix and $H \in C^\infty(\mathbf{R}^{2n}, \mathbf{R})$ is called the Hamiltonian of the system. The author proves Dirichlet's criterion, that $z_e = (q_e, p_e)$ is a stable equilibrium point if $DH(z_e) = 0$ and $D^2H(z_e)$ is definite. As an example, the author analyses the stability of equilibrium solutions for the nonlinear planar pendulum. The proof of Dirichlet's criterion is based on the fact that H is constant along trajectories of the corresponding dynamical system and the fact that the unit sphere in a finite-dimensional vector space is compact. In fluid dynamical systems, which are the main concern of this book, the stability analysis is more subtle, since phase spaces are usually infinite-dimensional. Also the norm used in the stability analysis has to be chosen carefully.

With a view towards Hamiltonian systems in fluid mechanics, which are often modelled with the help of Poisson brackets, the author gives another characterization of canonical Hamiltonian systems with the help of canonical Poisson brackets on the phase space $\mathbf{R}^{2n}$. Here, one associates to two functions $F, G \in C^\infty(\mathbf{R}^{2n}, \mathbf{R})$ the function

$$\{F,G\}(q,p) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} \right).$$

The canonical Hamiltonian equations then are equivalent to $(d/dt)F(q_t,p_t) = \{F,H\}(q_t,p_t)$ for all $F \in C^\infty(\mathbf{R}^{2n}, \mathbf{R})$. The canonical Poisson bracket is bilinear, skew-symmetric and satisfies Jacobi's identity as well as the Leibniz identity (which the author calls the associative property). This means by definition that the bracket $\{\cdot,\cdot\}$ defines a Poisson structure on the phase space $\mathbf{R}^{2n}$. The author shows that many dynamical systems in fluid mechanics can be written in Hamiltonian form using *non-canonical* Poisson brackets, defined on a certain class of functionals on the usually infinite-dimensional phase space of the fluid. This has immediate consequences for the dynamics of the fluid. For example, the Hamiltonian is constant along trajectories, since $\{H,H\} = 0$, by skew-symmetry of the Poisson bracket. Aside from canonical brackets one can have many non-constant Casimir functionals, i.e. functionals for which the Poisson bracket with each other functional vanishes.

The author uses the so-called energy-Casimir method [D. D. Holm et al., Phys. Rep. **123** (1985), no. 1-2, 116 pp.; MR0794110 (86i:76027)] to study the stability of fluid equilibria z_e. A constant of motion $\mathcal{H}$ is determined and a norm $\|\cdot\|$ on phase space, such that $D\mathcal{H}(z_e)=0$ and $\mu_1\|z-z_e\|\leq \mathcal{H}(z)-\mathcal{H}(z_e)\leq\mu_2\|z-z_e\|$ for appropriately chosen $\mu_1,\mu_2>0$. The fact that $\mathcal{H}$ is constant along trajectories then shows that z_e is stable in the sense of Lyapunov with respect to the norm $\|\cdot\|$. The functional often has the form $\mathcal{H}=H+C$ where H is the Hamiltonian of the system and C is a Casimir functional. Other conserved quantities arise from symmetries of the system via Noether's theorem.

The method is applied to several models in ideal fluid mechanics such as the Euler equations for two-dimensional homogeneous ideal fluid flow, and the Charney-Hasegawa-Mima equations that model quasi-two-dimensional flow in the presence of a background vorticity gradient in geophysical fluid dynamics. In particular, a nonlinear generalization of Rayleigh's criterion for the stability of plane parallel shear flow is presented. Finally the stability of soliton solutions of the Korteweg-de Vries equations is studied. The fact that solitons are travelling wave solutions leads to the incorporation of the linear momentum in the conserved quantity used in the stability analysis. Also the choice of the distance function in the stability analysis is subtle, since a "perturbed soliton" can drift away from the equilibrium solution.

The author carefully explains how to derive the Charney-Hasegawa-Mima and the Korteweg-de Vries equations. That the axioms for a Poisson structure are satisfied for these systems is checked by explicit calculation. Also, proofs for the stability results stated are given in great detail.

A travelling wave solution is a relative equilibrium in the sense that its dynamical orbit is contained in its group orbit under the action of the translation group. For a treatment of Hamiltonian systems from a more abstract point of view that stresses even more than the book at hand the importance of symmetries one should also see [J. E. Marsden, *Lectures on mechanics*, Cambridge Univ. Press, Cambridge, 1992; MR1171218 (93f:58078)]. Here also, the recently developed energy momentum method for the stability analysis of relative equilibria in Hamiltonian systems with symmetries is described.

Reviewed by *Hans-Peter Kruse*