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A perturbation theory for the solitary-drift-vortex solutions of the Hasegawa–Mima equation

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A multiple-scales adiabatic perturbation theory is presented describing the adiabatic dissipation of the solitary vortex-pair solutions of the Hasegawa–Mima equation. The vortex parameter transport equations are derived as solvability conditions for the asymptotic expansion and are identical with the transport equations previously derived by Aburdzhaniya *et al.* (1987) using an energy- and enstrophy-conservation balance procedure. The theoretical results are compared with high-resolution numerical simulations. Global properties such as the decay in the enstrophy and energy are accurately reproduced. Local properties such as the position of the centre of the vortex pair, decay of the extrema in the vorticity and stream-function fields, and the dilation of the vortex dipole are also in good agreement. In addition, time series of vorticity–stream-function scatter diagrams for the numerical simulations are presented to verify the adiabatic ansatz.

1. Introduction

In the last few years a number of steadily travelling nonlinear solitary-vortex solutions have been obtained for a number of models in plasma and fluid dynamics (see e.g. Larichev & Reznik 1976; Meiss & Horton 1983; Shukla 1985; Yu, Shukla & Varma 1985; Horton *et al.* 1986; Nycander, Pavlenko & Stenflo 1987; Nycander 1988). It is thought that these solutions may be useful in modelling certain aspects of the convective motion associated with observed anomalous heat transports in fusion-containment devices. However, very little is known, in general, about the stability of these solutions to externally imposed fields and how the propagation characteristics of these modes might evolve if subjected to weak but non-negligible forcing. The principal purpose of this paper is to present a simple multiple-scales asymptotic theory that will describe the adiabatic dissipation of the solitary vortex solutions to the nonlinear Hasegawa–Mima equation (Hasegawa & Mima 1978; Hasegawa, Maclennan & Kodama 1979). We also present a detailed comparison between the predictions of our theory and a numerical solution of the governing equation, assuming a drift-vortex initial condition. The agreement between the two is very good.

It is our belief that the analysis presented here will be substantially applicable to other nonlinear solitary-vortex models that include the perturbative effects of other fields. A principal conclusion of our analysis is that these propagating vortex solutions are robust and that, at least for the problem considered here, undergo an adiabatic distortion in the sense that the non-analytic vorticity–stream-function relationship and dispersion relation that define these solitary dipoles are being continuously maintained. To support our claim, a time series of vorticity–stream-function scatter plots obtained from the numerical simulations is presented.

In a series of papers, Swaters, (1985, 1986*b*) and Swaters & Flierl (1988) developed a perturbation theory based on globally averaged energy and enstrophy balances to describe the weakly perturbed evolution of the drift vortex or modon solutions of the shallow-water equations on an infinite β-plane. A similar theory was presented by Aburdzhaniya *et al.* (1987) to describe the dissipation of the drift-vortex solution of the Hasegawa–Mima equation. We shall show that the transport equations derived by Aburdzhaniya *et al.* can be obtained as properly formulated solvability conditions for an asymptotic expansion, assuming a relatively small damping coefficient. The theory suggests, and the numerical computations confirm, that there is very little excitation of the continuous spectrum (i.e. linear drift waves) throughout most of the dissipation. However, towards the very final stages of the decay, the drift vortex does appear to 'break up' into a superposition of decaying linear waves. The asymptotic theory that we present is a generalization of the direct perturbation analyses that have been developed to study perturbed $(1+1)$-dimensional solitary-wave models (see e.g. Karpman 1977; Kaup & Newell 1978; Grimshaw 1979*a*, *b*; Kodama & Ablowitz 1981).

The plan of the paper is as follows. In § 2.1 the non-dimensional problem is formulated. In §2.2 the transport equations for the drift-vortex parameters are derived. In §3 we compare the predictions of our theory with a numerical simulation. The paper is summarized and some concluding remarks are made in §4.

2. Formulation of the problem and derivation of the transport equations

2.1. *Governing equations*

We write the frictionally perturbed nondimensional Hasegawa–Mima equation in the form

$$(\Delta-1)\varphi_t+\varphi_x+J(\varphi,\Delta\varphi) = -\epsilon\Delta\varphi, \tag{2.1}$$

where φ is the stream function, which is also the electrostatic potential (with corresponding ion velocity field $u = -\varphi_y$ and $v = \varphi_x$), the Jacobian is given by $J(A,B) \equiv A_x B_y - B_x A_y$, with subscripts indicating partial differentiation, ϵ is the damping parameter (we shall assume that $0 < \epsilon \ll 1$). The model (2.1) can be derived via a formal asymptotic expansion for a cold ion fluid in a electrostatic field (similar to the derivation in Hasegawa *et al.* 1979) that includes a Rayleigh dissipation term in the ion momentum equations.

The Hasegawa–Mima equation (2.1) is also a lowest-order model for baroclinic quasi-geostrophic dynamics in a planetary atmosphere or ocean (Pedlosky 1987). In that context φ plays the role of the geostrophic pressure and the drift-vortex solutions are called modons (Stern 1975). When $\Delta \gg 1$ in (2.1) the model corresponds to the barotropic rigid-lid shallow-water equations for a differentially rotating fluid. Swaters (1986*b*) has examined the propagation of barotropic modons over variable topography. Swaters & Flierl (1988) have developed a theory to describe the Ekman decay of barotropic modons.

To describe the near field of the dissipating Hasegawa–Mima drift vortex, the solution will be assumed to depend on the fast phase variables (Swaters 1985, 1986*b*; Swaters & Flierl 1988)

$$\xi = x - \epsilon^{-1}\int_0^{\epsilon t} c(t')\,dt', \quad y = y \tag{2.2a, b}$$

and the slow time variable $T = \epsilon t.$ (2.2c)

Consequently, derivatives will be rewritten:

$$\partial_x \rightarrow \partial_\xi, \quad \partial_y \rightarrow \partial_y, \quad \partial_t \rightarrow -c(T)\,\partial_\xi + \epsilon\,\partial_T. \tag{2.3a, b, c}$$

Substitution of (2.3) into (2.1) gives

$$J[\varphi + cy, \Delta\varphi - \varphi + y] = -\epsilon\Delta\varphi - \epsilon(\Delta - 1)\,\varphi_T, \tag{2.4}$$

where the derivatives in the Jacobian are now understood to be with respect to ξ and y.

The solution to (2.4) can be constructed in a straightforward asymptotic expansion of the form

$$\varphi \sim \varphi^{(0)}(\xi, y; T) + \epsilon\varphi^{(1)}(\xi, y; T) + \dots \tag{2.5}$$

Substitution of (2.5) into (2.4) yields the $O(1)$ problem

$$J(\varphi^{(0)} + cy, \Delta\varphi^{(0)} - \varphi^{(0)} + y) = 0, \tag{2.6}$$

whose solution is taken to be the drift vortex given by (for details see Larichev & Reznik 1976)

$$\left.\begin{aligned} \varphi^{(0)} &= -\frac{acK_1(\gamma r/a)\sin\theta}{K_1(\gamma)}, && (2.7a)\\ \Delta\varphi^{(0)} &= -\frac{a(1+c)\,K_1(\gamma r/a)\sin\theta}{K_1(\gamma)} && (2.7b)\end{aligned}\right\} \quad (r > a),$$

$$\left.\begin{aligned} \varphi^{(0)} &= \frac{a(1+c)\,J_1(\kappa r)\sin\theta}{\kappa^2 J_1(\nu)} - \kappa^{-2}(c\kappa^2 + c + 1)\,r\sin\theta, && (2.8a)\\ \Delta\varphi^{(0)} &= -\frac{a(1+c)\,J_1(\kappa r)\sin\theta}{J_1(\nu)} && (2.8b)\end{aligned}\right\} \quad (r < a),$$

with the dispersion relationship (obtained by requiring the continuity of $\nabla\varphi^{(0)}$ on $r = a$)

$$\gamma K_1(\gamma)\,J_2(\nu) + \nu J_1(\nu)\,K_2(\gamma) = 0, \tag{2.9}$$

where $\gamma^2 \equiv a^2(1 + 1/c)$, $\nu \equiv \kappa a$ and the co-moving polar co-ordinates $r^2 \equiv \xi^2 + y^2$ and $\tan\theta \equiv y/\xi$ have been introduced (see figure 1). Note that (2.7) and the definition of γ imply that the translation speed satisfies $1 + 1/c > 0$, which can be interpreted as implying that, to leading order, the exterior region ($r > a$) cannot contain any stationary linear drift waves. It will be assumed that throughout the dissipation process $c = c(T)$, $a = a(T)$ and $\kappa = \kappa(T)$ such that the dispersion relationship (2.9) remains continuously satisfied. The dispersion

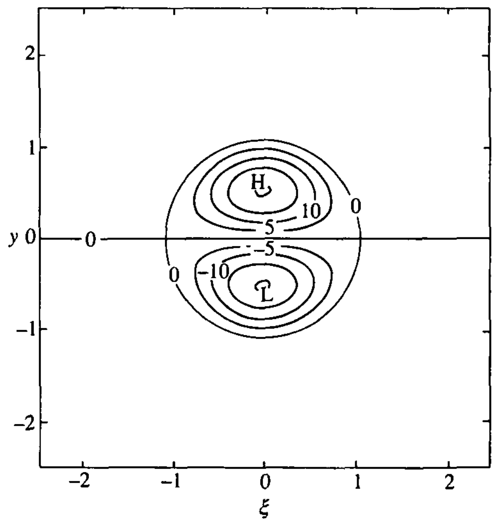

FIGURE 1. Contour plot of the vorticity field associated with a rightward-travelling drift-vortex. The contour intervals are ± 5.

relationship forms a single constraint on the evolution of a, c and κ. Two additional transport equations are required in order to uniquely determine the evolution of a, c and κ and – consequently – the leading-order dissipating drift vortex.

Linear Liapunov stability for the leftward-travelling solution (i.e. $c < -1$) has been proved by Laedke & Spatschek (1986) using a variational argument closely related to the energy–Casimir methods of Holm *et al.* (1985). For the rightward-travelling solution (i.e. $c > 0$) there is still no analytical proof of linear stability. However, Swaters (1986*a*) and Flierl (1987) have established sufficient conditions that characterize neutral modes for the linear normal-mode stability problem. There is no known proof of nonlinear stability for either the leftward- or rightward-travelling solution.† Numerical experiments by McWilliams *et al.* (1981) demonstrate, however, that the rightward-travelling solution is stable for a large class of finite-amplitude perturbations.

2.2. *Derivation of the transport equations*

Aburdzhaniya *et al.* (1987) obtained the transport equations describing the slow time evolution of the Hasegawa–Mima drift vortex from averaged conservation laws. An alternate viewpoint, described by Kodama & Ablowitz (1981) and Ablowitz & Segur (1981, §3.8), is to obtain the transport equations for slowly varying solitary waves as the consequence of orthogonality conditions derived from examining the kernel of the adjoint operator associated with the first-order equations in a perturbation theory with an asymptotically small damping coefficient. Both of these approaches must be, of course, equivalent. The asymptotic methods developed here are two-dimensional extensions of the theory that has been developed for perturbed one-dimensional solitary wave equations (see e.g. Grimshaw 1979*a*, *b*, 1981).

Unfortunately, the transport equations cannot be derived from an averaged

† We have recently become aware of a proof of 'formal' stability by Sakuma and Ghil in an article submitted to the *J. Fluid Mech.* for the stationary Stern modon based on a gauge-variable formalism.

Lagrangian since a variational principle for (2.1) in Eulerian variables is unknown. For the inviscid linear version of (2.1) a variational principle is well known (see e.g. Seliger & Whitham 1968). For the inviscid nonlinear version of (2.1) Virasoro (1981) found a variational principle, but unfortunately the action density is written in a form that makes it not particularly convenient to use in problems of the kind discussed here. In addition, it is not known whether the unperturbed equation (2.1) (i.e. with $\epsilon \equiv 0$) is integrable in the sense of a multi-dimensional inverse scattering theory (IST). If it were then the powerful machinery developed over the last decade by, among others, Karpman (1977), Karpman & Maslov (1978), Kaup & Newell (1978) and Knickerbocker & Newell (1980) for perturbed soliton equations based on the IST formalism would be available. Thus at present it seems that any analytical progress that can be made in describing the evolution of perturbed drift vortices must be based on a direct singular perturbation theory.

The $O(\epsilon)$ problem associated with (2.4) is given by

$$J[\varphi^{(0)}+cy, \Delta\varphi^{(1)}-\varphi^{(1)}]+J[\varphi^{(1)}, \Delta\varphi^{(0)}-\varphi^{(0)}+y] = -\Delta\varphi^{(0)}-(\Delta\varphi^{(0)}-\varphi^{(0)})_T. \tag{2.10}$$

The left-hand side of this equation can be simplified if we recall that the drift-vortex solution satisfies the vorticity–stream-function relationships

$$\Delta\varphi^{(0)}-\varphi^{(0)}+y = \begin{cases} c^{-1}(\varphi^{(0)}+cy) & (r>a), & (2.11a)\\ -(\kappa^2+1)(\varphi^{(0)}+cy) & (r<a). & (2.11b)\end{cases}$$

Note that the drift vortex satisfies $\Delta\varphi^{(0)}-\varphi^{(0)}+y \equiv \varphi^{(0)}+cy \equiv 0$ on $r=a$. Substitution of (2.11*a*) or (2.11*b*) into (2.10) and rearranging terms (recalling that the Jacobian is a skew-symmetric bilinear form) enables the $O(\epsilon)$ problem to be put into the form

$$J(\varphi^{(0)}+cy, \Delta\varphi^{(1)}+\lambda\varphi^{(1)}) = -\Delta\varphi^{(0)}-(\Delta\varphi^{(0)}-\varphi^{(0)})_T, \tag{2.12}$$

where $\lambda = -(1+1/c)$ in $r>a$ and $\lambda = +\kappa^2$ in $r<a$. It is easy to show that the homogeneous adjoint problem associated with (2.12) can be put into the form

$$(\Delta+\lambda)\, J(\varphi^{(0)}+cy, q) = 0, \tag{2.13}$$

for which there are two immediately obvious independent solutions:

$$q = \varphi^{(0)}, \quad q = \Delta\varphi^{(0)}. \tag{2.14a, b}$$

Therefore the right-hand-side of (2.12) must satisfy the orthogonality conditions

$$\partial_T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nabla\varphi^{(0)}.\nabla\varphi^{(0)}+\varphi^{(0)2}\, d\xi\, dy = -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nabla\varphi^{(0)}.\nabla\varphi^{(0)}\, d\xi\, dy, \tag{2.15a}$$

$$\partial_T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\nabla\varphi^{(0)})^2+\nabla\varphi^{(0)}.\nabla\varphi^{(0)}\, d\xi\, dy = -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\nabla\varphi^{(0)})^2\, d\xi\, dy. \tag{2.15b}$$

The transport equations (2.15*a, b*) are respectively the globally averaged leading-order energy and enstrophy balance equations for the drift vortex. Calculation of (2.15*a*) and (2.15*b*) yields respectively

$$(a^2c^2E_{11}+a^4c^2E_{12})_T = -2a^2c^2E_{11}, \tag{2.16a}$$

$$(c^2E_{21}+a^2c^2E_{11})_T = -2c^2E_{21}, \tag{2.16b}$$

where

$$E_{11} = \frac{\gamma^2(\gamma^2+\nu^2)}{2\nu^2}\left(1+\frac{4R}{\gamma}\right), \tag{2.16c}$$

$$E_{12} = \frac{(\gamma^2+\nu^2)(3\gamma^2-\nu^2)}{4\nu^4}\left(1+\frac{4R}{\gamma}\right)+\frac{(\gamma^2+\nu^2)R^2}{2\nu^2}, \tag{2.16d}$$

$$E_{21} = \tfrac{1}{2}\gamma^2(\gamma^2+\nu^2)R^2, \tag{2.16e}$$

where $R \equiv K_2(\gamma)/K_1(\gamma)$.

The transport equations (2.16a, b) were first derived by Aburdzhaniya *et al.* (1987) in the context of the perturbed Hasegawa–Mima drift-vortex problem. Earlier versions of these transport equations appeared in Swaters (1985) and Swaters (1986b) in the context of the perturbed β-plane modon problem. The transport equations (2.16a, b) and the dispersion relation (2.9) will uniquely determine the adiabatic dissipation of the drift vortex. In the rigid-lid shallow-water theory developed by Swaters & Flierl (1988) it turns out that the E_{12} term in (2.16a) and E_{11} term in (2.16b) are not present, and thus the transport equations can immediately be integrated to give two nonlinear algebraic expressions that depend only parametrically on the slow time.

The study by Aburdzhaniya *et al.* (1987) did not fully examine the solutions of the transport equations, focusing instead on obtaining simple analytically tractable approximations. Here we shall numerically solve the transport equations (2.16a, b) and then compare the results with a numerical solution of (2.1), assuming a drift-vortex initial condition.

The formal procedure that we adopt is to eliminate the κ_T derivatives in (2.16) (recall that $\nu = \kappa a$) in favour of a_T and c_T by exploiting the dispersion relationship (2.9). It follows from (2.9) that

$$\frac{\kappa_T}{\kappa} = \frac{Na_T}{a} - \frac{(N+1)c_T}{2c(1+c)}, \tag{2.17}$$

where

$$N \equiv -\left(\frac{\nu^2 R}{\gamma}+\gamma R\right)\Big/\left(4+\frac{\nu^2 R}{\gamma}+\frac{\gamma}{R}\right).$$

Substitution of (2.17) into (2.16) yields the 2×2 dynamical system

$$a_T = 2a(E_{21}A_{12}-E_{11}A_{22})/D, \tag{2.18a}$$

$$c_T = 2c(1+c)(A_{21}E_{11}-A_{11}E_{21})/D, \tag{2.18b}$$

with

$$A_{11} = 2E_{11}+\gamma E_{11\gamma}+(N+1)\nu E_{11\nu}+4a^2E_{12}+a^2\gamma E_{12\gamma}+(N+1)a^2\nu E_{12\nu}, \tag{2.18c}$$

$$A_{12} = 2E_{11}(1+c)-\tfrac{1}{2}\gamma E_{11\gamma}-\tfrac{1}{2}(N+1)\nu E_{11\nu} +2(1+c)a^2E_{12}-\tfrac{1}{2}a^2\gamma E_{12\gamma}-\tfrac{1}{2}(N+1)a^2E_{12\nu}, \tag{2.18d}$$

$$A_{21} = \gamma E_{21\gamma}+(N+1)\nu E_{21\nu}+2a^2E_{11}+a^2\gamma E_{11\gamma}+(N+1)a^2\nu E_{11\nu}, \tag{2.18e}$$

$$A_{22} = 2(1+c)E_{21}-\tfrac{1}{2}\gamma E_{21\gamma}-\tfrac{1}{2}(N+1)\nu E_{21\nu} +2(1+c)a^2E_{11}-\tfrac{1}{2}a^2\gamma E_{11\gamma}-\tfrac{1}{2}(N+1)a^2\nu E_{11\nu}, \tag{2.18f}$$

$X_c(t) \equiv \epsilon^{-1}\int_0^{\epsilon t} c(\tau)\, d\tau$	position of the centre of the dissipating drift vortex
$\dfrac{\mathrm{En}\,(t)}{\mathrm{En}\,(0)} \equiv \dfrac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\nabla\varphi^{(0)}.\nabla\varphi^{(0)}+\varphi^{(0)2}](\xi,y;\epsilon t)\,d\xi\,dt}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\nabla\varphi^{(0)}.\nabla\varphi^{(0)}+\varphi^{(0)2}](\xi,y;0)\,d\xi\,dy}$	time-dependent area-integrated energy normalized by its initial value
$\dfrac{\mathrm{Es}\,(t)}{\mathrm{Es}\,(0)} \equiv \dfrac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\Delta\varphi^{(0)})^2+\nabla\varphi^{(0)}.\nabla\varphi^{(0)}](\xi,y;\epsilon t)\,d\xi\,dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\Delta\varphi^{(0)})^2+\nabla\varphi^{(0)}.\nabla\varphi^{(0)}](\xi,y;0)\,d\xi\,dy}$	time-dependent area-integrated enstrophy normalized by its initial value
$\mathrm{vml}\,(t)$	radial co-ordinate of the maximum in the vorticity $\lvert\Delta\varphi^{(0)}(r,\theta;\epsilon t)\rvert$ with $\theta=\frac{1}{2}\pi$
$\dfrac{\mathrm{vm}\,(t)}{\mathrm{vm}\,(0)} \equiv \dfrac{\max[\lvert\Delta\varphi^{(0)}(\mathrm{vml},\frac{1}{2}\pi;\epsilon t)\rvert]}{\max[\lvert\Delta\varphi^{(0)}(\mathrm{vml}\,(t=0),\frac{1}{2}\pi;0)\rvert]}$	time-dependent magnitude of the maximum vorticity normalized by its initial value
$\mathrm{sml}\,(t)$	radial co-ordinate of the maximum in the stream function $\lvert\varphi^{(0)}(r,\theta;\epsilon t)\rvert$ with $\theta=\frac{1}{2}\pi$
$\dfrac{\mathrm{sm}\,(t)}{\mathrm{sm}\,(0)} \equiv \dfrac{\max[\lvert\varphi^{(0)}(\mathrm{sml},\frac{1}{2}\pi;\epsilon t)\rvert]}{\max[\lvert\varphi^{(0)}(\mathrm{sml}\,(t=0),\frac{1}{2}\pi;0)\rvert]}$	time-dependent magnitude of the maximum in the stream function normalized by its initial value

TABLE 1. Definition of symbols used in the diagnostic calculations

where $D \equiv A_{11}A_{22}-A_{12}A_{21}$, subscripts γ and ν indicate partial derivatives with respect to these variables, and the wavenumber $\kappa(T)$ is understood to be an implicit function of the radius $a(T)$ and translation speed $c(T)$ via the dispersion relationship (2.9).

3. Comparison between theory and a numerical simulation

In this section a comparison is presented between the results of the leading-order asymptotic theory developed in §2 and a direct numerical solution of (2.1) assuming a drift-vortex initial condition. In order to illustrate the asymptotic 'robustness' of the perturbation solution, we shall take $\epsilon = 0{\cdot}2$ in (2.1).

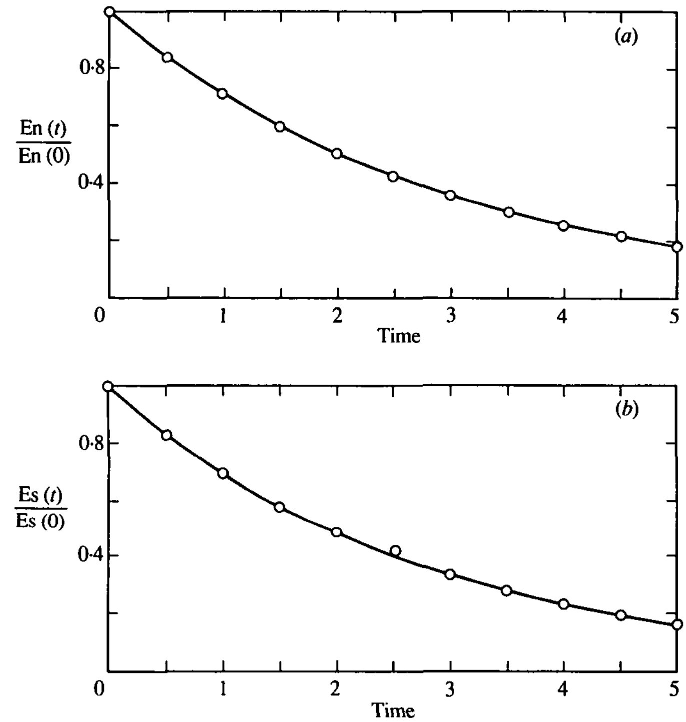

FIGURE 2. (a) Decay in the normalized globally averaged energy (see table 1) for the rightward-travelling solution. The solid line is the theoretical prediction and the open circles are the numerically determined values. (b) Decay in the normalized globally averaged enstrophy (see table 1) for the rightward-travelling solution. The solid line and circles are as described in (a).

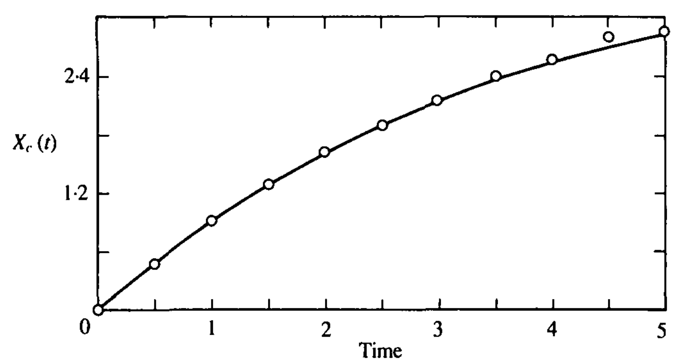

FIGURE 3. The x-co-ordinate of the rightward-travelling solution (see table 1). The solid line and circles are as described in figure 2(a).

The numerical scheme adopted for (2.1) was as follows. The Hasegawa–Mima equation was split, giving the coupled system

$$q_t + J(\varphi, q+y) = -\epsilon\Delta\varphi, \tag{3.1}$$

$$q = \Delta\varphi - \varphi. \tag{3.2}$$

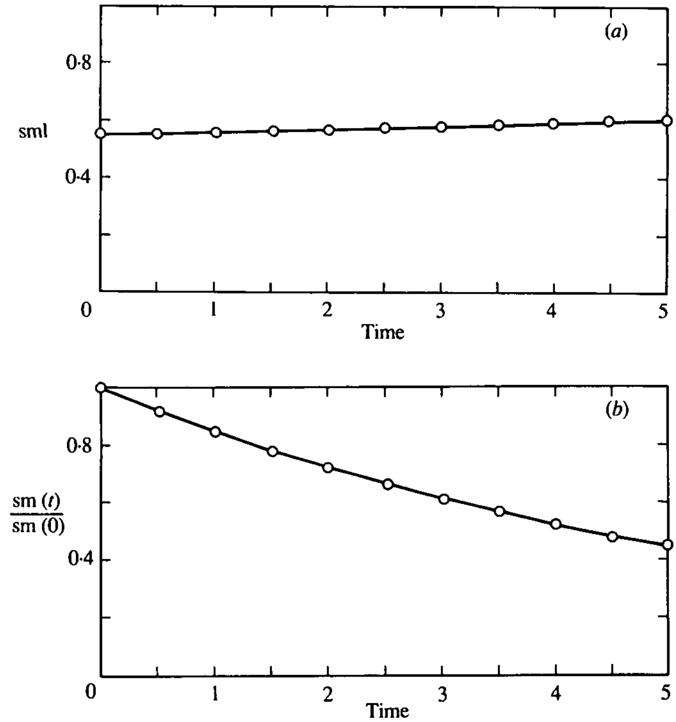

FIGURE 4. (*a*) The monotonically increasing radial co-ordinate of the stream function maximum (see table 1) for the rightward-travelling solution. (*b*) The decay in the normalized maximum of the stream-function field (see table 1) for the rightward-travelling solution. The solid lines and circles are as described in figure 2(*a*).

The variable q was integrated forward in time using a sympletic leapfrog procedure in which the Jacobian is finite-differenced using the Arakawa (1966) scheme (see also McWilliams *et al.* 1981). The updated stream function was then obtained from the inhomogeneous Helmholtz equation (3.2). The problem was solved in a 128×128 doubly periodic square domain in which each side was of length 10 non-dimensional units. The initial non-dimensional drift-vortex radius was always set equal to one. Thus the finite-difference scheme had a grid spacing of about 0·0787 units or about 25·4 grid points per drift-vortex diameter. McWilliams *et al.* (1981) showed that with this resolution the numerical translation speed of the drift-vortex is well within 5% of the true translation speed. The non-dimensional time step was chosen to be 0·005. With these finite-difference parameter values, the linearized Courant–Freidrichs–Lewy stability criteria was always satisfied for (3.1).

Two numerical simulations are described here. The first corresponds to assuming an initial rightward-travelling drift vortex with $a = c = 1$ and $\kappa = 3{\cdot}984$, and the second to assuming an initial leftward-travelling drift vortex with $a = 1$, $c = -2$ and $\kappa = 3{\cdot}883$. The comparative diagnostic calculations performed in each simulation are summarized in table 1.

The results for the rightward-travelling drift-vortex are depicted in figures 2–5. The results are in good agreement. Figures 2(*a*, *b*) illustrate the decay in the globally averaged energy and enstrophy. Figure 3 shows the x-co-ordinate

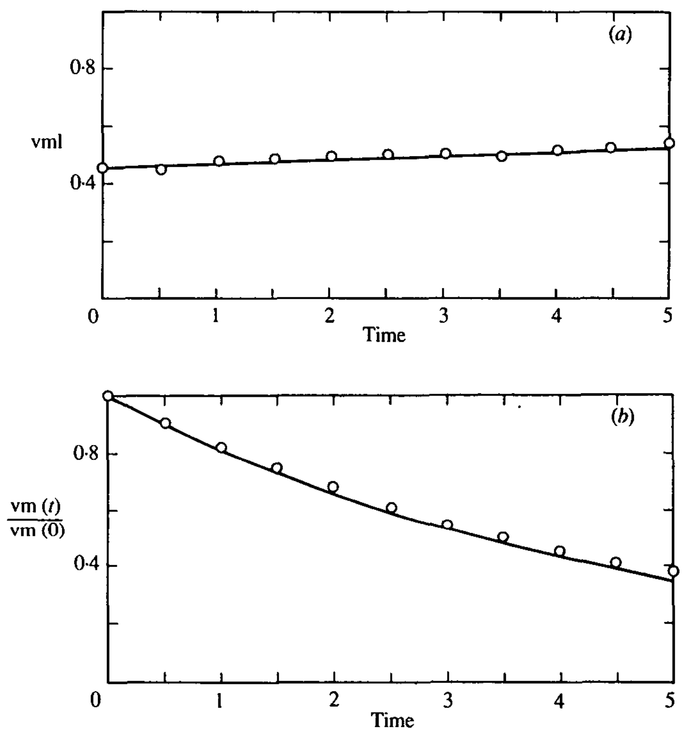

FIGURE 5. (*a*) The monotonically increasing radial co-ordinate of the vorticity maximum (see table 1) for the rightward-travelling solution. (*b*) The decay in the normalized maximum of the vorticity field (see table 1) for the rightward-travelling solution. The solid lines and circles are as described in figure 2(*a*).

of the translating and dissipating drift vortex. For the rightward-travelling drift-vortex $c(T) \rightarrow 0^+$ monotonically as $T \rightarrow +\infty$. It is when $c(T) \sim 0$ that the 'break up' into linear drift waves begins. Before this stage, the dissipating drift vortex remains coherent.

Figures 4(*a*) and 5(*a*) depict the radial co-ordinate of the maximum in the stream-function and absolute vorticity fields respectively. Throughout the decay, the vortex pair dilates in the sense that the radius monotonically increases. This trend can be inferred from figures 4(*a*) and 5(*a*). Figures 4(*b*) and 5(*b*) illustrate the decay in the magnitude of the absolute extrema in the vorticity and stream-function fields respectively. The numerically determined values in figures 4 and 5 were obtained by a quadratic interpolation procedure.

In order to test the adiabatic ansatz on which the perturbation theory presented in this paper is based, we computed a time series of vorticity–stream-function scatter diagrams for the dissipating drift vortex as determined by the numerical simulations. The diagrams for the rightward-travelling drift vortex are shown in figures 6(*a–f*). Note that it follows from (2.11) that the total or potential vorticity $\Delta\varphi - \varphi + y$ of the drift vortex is a non-analytic linear function of the co-moving streamlines $\varphi + cy$ (see figures 6(*a*) and 11(*a*) for the unperturbed diagrams). Hence if the drift vortex is, to leading order, dissipating adiabatically, this non-analytic vorticity–stream-function should be continuously maintained. Figure 6 illustrates that – at least qualitatively – the linear

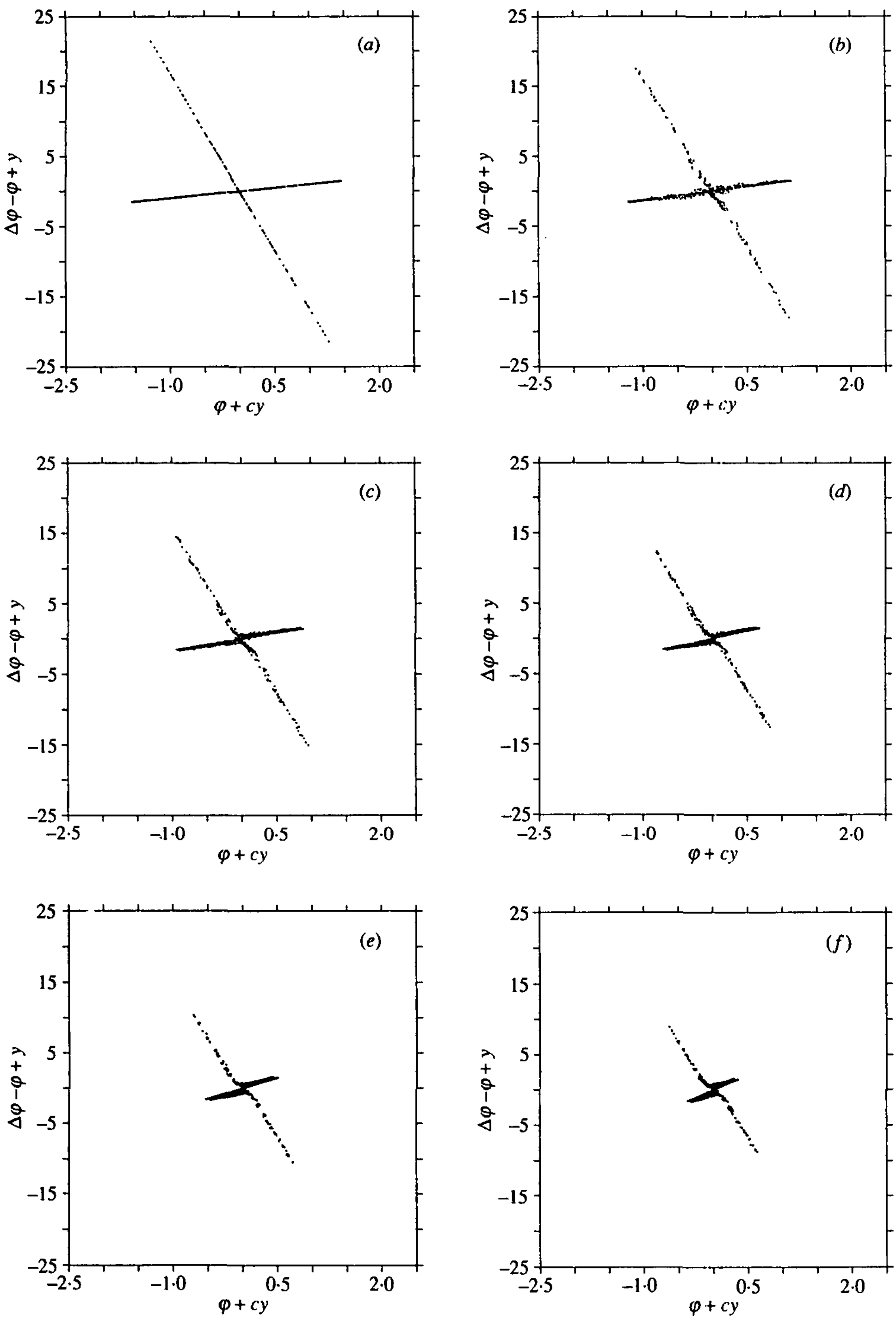

FIGURE 6. Time-series scatter diagrams of the *total* vorticity $\Delta\varphi - \varphi + y$ versus the co-moving streamlines $\varphi + cy$ for the numerical simulation of the rightward-travelling dissipating drift vortex. (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) correspond to $t = 0$, 1, 2, 3, 4 and 5 respectively. The positively sloped curve corresponds to points in the exterior ($r > a$) region, and the negatively sloped curve corresponds to points in the interior ($r < a$) region.

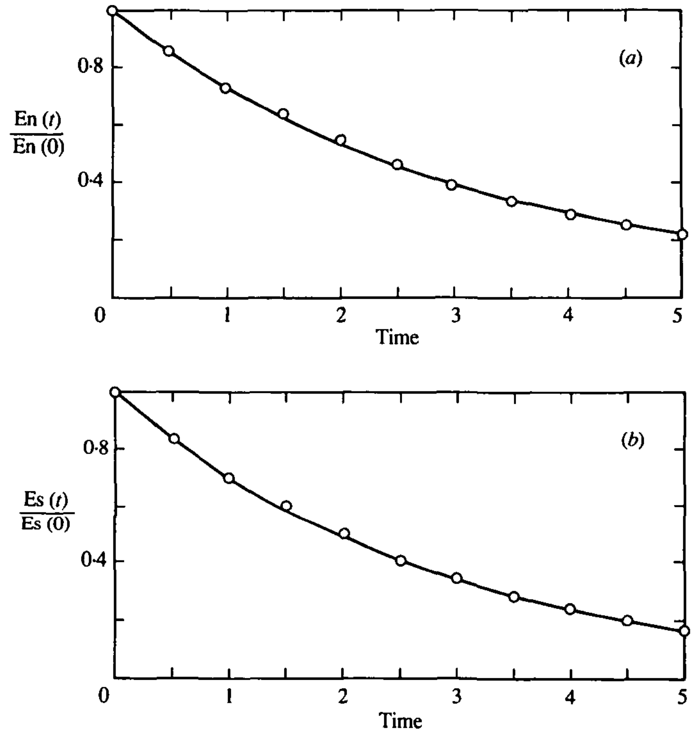

FIGURE 7. (a) Decay in the normalized globally averaged energy (see table 1) for the leftward-travelling solution. The solid line is the theoretical prediction and the open circles are the numerically determined values. (b) Decay in the normalized globally averaged enstrophy (see table 1) for the leftward-travelling solution. The solid line and circles are as described in (a).

relationship is being preserved. However, as the dissipation proceeds there is some departure from strict linearity. We believe that some of the noise is the result of the accumulation of high-wavenumber error in the numerical simulation (no numerical sub-grid-scale friction term was introduced into (3.1)), which becomes noticeable as $c(T) \rightarrow 0^+$ for the rightward-travelling solution. By way of comparison, note that in the scatter diagrams for the leftward-travelling solution (figure 11), where $c(T) \rightarrow -1^-$, much less variability occurs and the non-analytic vorticity–stream-function relationship seems to be better preserved.

Figures 7–11 depict the comparison between theory and a numerical simulation for the leftward-travelling drift-vortex. Here again, the comparison is very good. Qualitatively, the behaviour of the dissipating pair is very similar to that in the rightward-travelling situation. However, one difference is that in the present situation $c(T) \rightarrow -1^-$ as $T \rightarrow +\infty$, where $c \equiv -1$ corresponds to the phase speed of the fastest linear wave solutions to the low-wavenumber inviscid limit of (2.1).

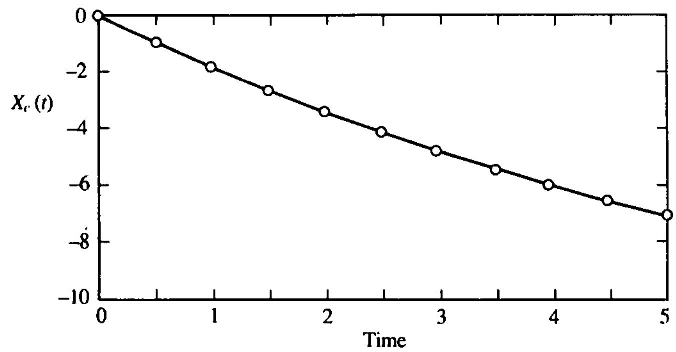

FIGURE 8. The x-co-ordinate of the leftward-travelling solution (see table 1). The solid line and circles are as described in figure 7 (*a*).

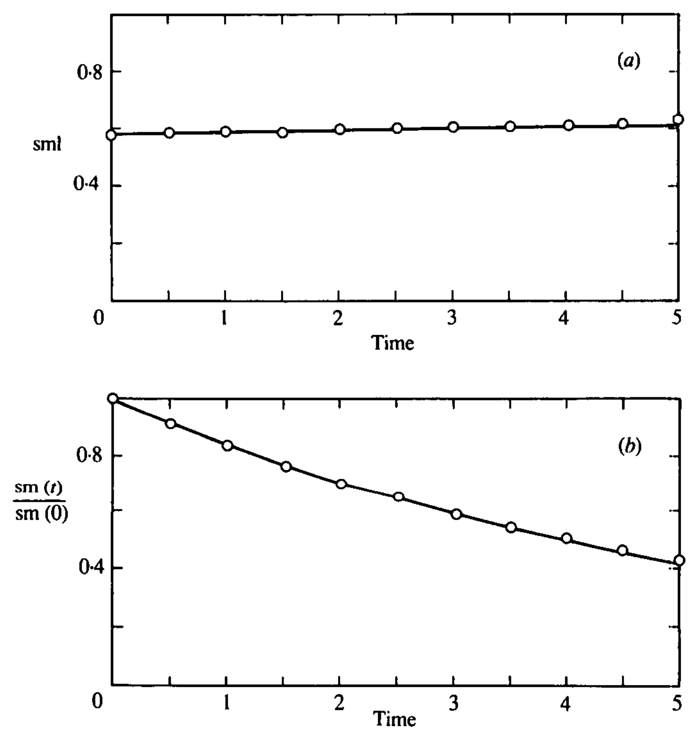

FIGURE 9. (*a*) The monotonically increasing radial co-ordinate of the stream-function maximum (see table 1) for the leftward-travelling solution. (*b*) The decay in the normalized maximum of the stream-function field (see table 1) for the leftward-travelling solution. The solid lines and circles are as described in figure 7 (*a*).

4. Summary

A multiple-scales singular perturbation theory has been presented describing the leading-order adiabatic dissipation of the solitary drift-vortex solutions of the Hasegawa–Mima equation. The vortex parameters evolve according to globally integrated energy and enstrophy balances. These evolution equations

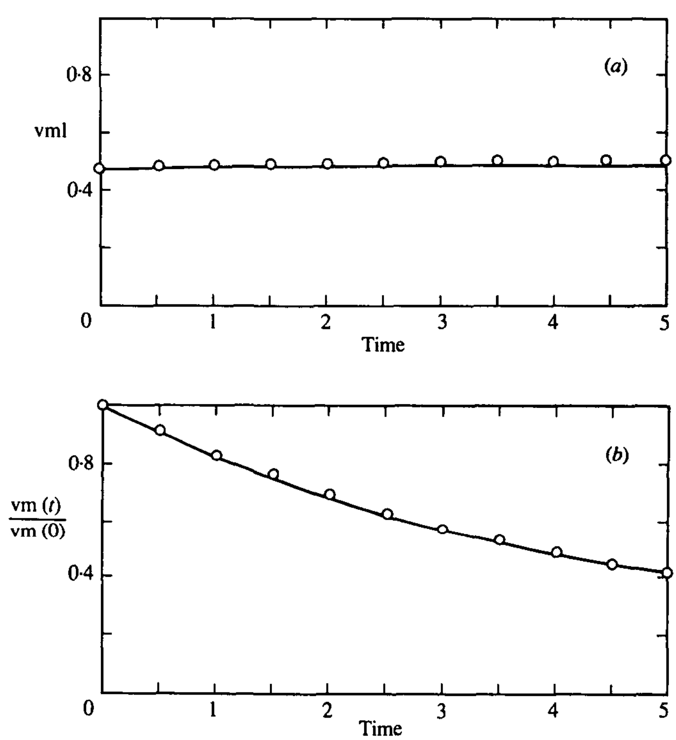

FIGURE 10. (a) The monotonically increasing radial co-ordinate of the vorticity maximum (see table 1) for the leftward-travelling solution. (b) The decay in the normalized maximum of the vorticity field for the leftward-travelling solution (see table 1). The solid lines and circles are as described in figure 7(a).

have been shown to be properly formulated solvability conditions for the asymptotic expansion presented here. A comparison has been presented between the predictions of our theory and a high-resolution numerical simulation. The two are in very good agreement.

One of the most interesting conclusions that can be made is that the non-analytic vorticity–stream-function relationship that defines the solitary-vortex pair is being continuously maintained throughout the dissipation process (at least during the main stages of the decay). As a result, there seems to be very little decaying radiation from the vortex pair in the form of linear drift waves. It is interesting to speculate that these observations may be suggesting a underlying variational principle at work here.

The point has been made by Nycander (1988) that there is no *a priori* reason to believe that the linear non-analytic vorticity–stream-function relationship of these solitary dipole-vortex waves should be maintained during a period of perturbed evolution. We believe, however, that there are indeed reasons to suppose this, and that when subjected to weak external forcing the drift vortex will respond adiabatically. Our view centres on the fact that the Hasegawa–Mima drift vortex satisfies the first-order necessary conditions for the variational problem given by

$$H = \iint_{\mathbb{R}^2} E_1 \, d\xi dy - \iint_{\mathbb{R}^2} (\lambda+1)^{-1} E_2 \, d\xi dy, \tag{4.1a}$$

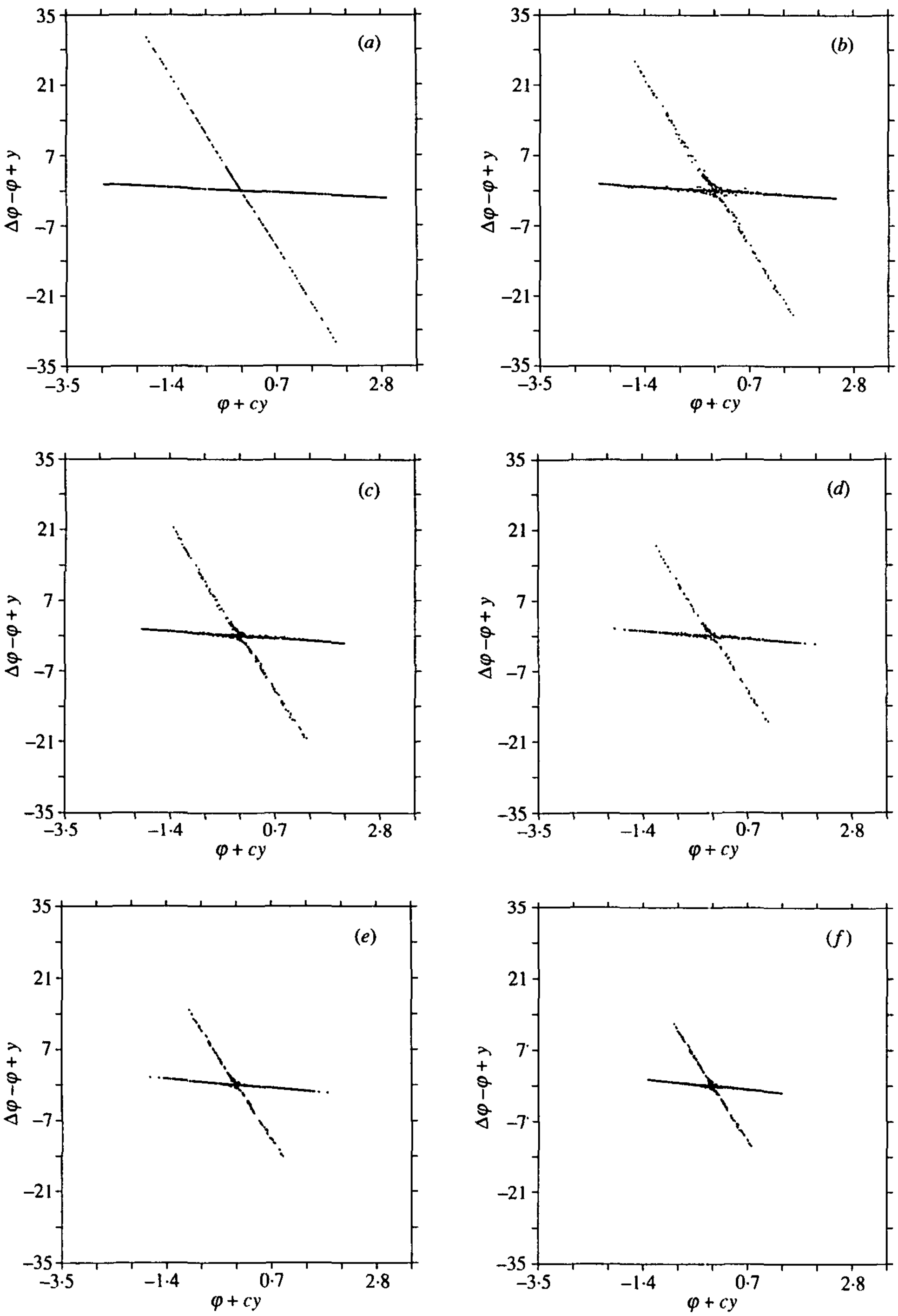

FIGURE 11. Time-series scatter diagrams of the total vorticity. $\Delta\varphi - \varphi + y$ versus the co-moving streamlines $\varphi + cy$ for the numerical simulation of the leftward-travelling dissipating drift vortex. (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) correspond to $t = 0, 1, 2, 3, 4$ and 5 respectively. The points on the more-negatively sloped curve correspond to data taken from the interior ($r < a$) region and points on the less-negatively sloped curve correspond to data taken from the exterior ($r > a$) region.

where
$$E_1 \equiv \nabla(\varphi+cy)\,.\,\nabla(\varphi+cy)+(\varphi+cy)^2-c^2(1+y^2), \tag{4.1b}$$
$$E_2 \equiv (\Delta\varphi-\varphi+y)^2-y^2. \tag{4.1c}$$

We note that (4.1*a*) can be interpreted as an energy constrained by the enstrophy pseudo-Hamiltonian (see e.g. Holm *et al.* 1985). In this context, the second term in (4.1*a*) may be interpreted as a Casimir of the non-canonical Poisson bracket in the Hamiltonian formulation of the unperturbed Hasegawa–Mima equation presented by Weinstein (1983). The linear stability proof of Laedke & Spatschek (1986) for the $c<-1$ drift vortex is a demonstration that the second variation of H is definite. It therefore seems reasonable to expect that a weakly perturbed drift vortex will evolve in such a way as to remain as 'close' as possible to this extremal state. However, it is important to add that, under large-amplitude external forcing, the adiabatic ansatz adopted in this paper will probably fail.

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