EVOLUTION OF NEAR-SINGULAR JET MODES

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1. Introduction

The purpose of this contribution is to very briefly describe some simulations we have done concerning the evolution of modal disturbances to a planar jet in which the phase velocity of the perturbation is initially equal to the maximum jet velocity. For a more complete discussion of this work see Swaters (1999, 2000). As is well known, the perturbation stream function for this configuration is algebraically singular at the jet maximum unlike the logarithmic singularity of a critical layer in a monotonic shear flow.

Perhaps surprisingly, our work clearly shows a "long" time scale oscillation in the underlying modal amplitude even when the numerical simulation is properly initialized with the leading order linear solution and first higher harmonic as determined by weakly-nonlinear asymptotics. A proper theoretical explanation for this behavior remains to be developed.

Our simulations are initialized by perturbing a well-known "jet," the *Bickley* jet (Bickley, 1937), for which there is an explicit solution for an algebraically singular perturbation (Howard and Drazin, 1964).

2. Problem formulation

The nondimensional, inviscid, incompressible two-dimensional Navier-Stokes equations can be written in the form

$$\triangle\psi_t + \psi_x \triangle\psi_y - \psi_y \triangle\psi_x = 0, \tag{1}$$

where the notation is standard.

The Bickley jet stream function, given by,

$$\psi = \psi_o(y) = -\tanh(y), \quad -\infty < y < \infty, \tag{2}$$

with corresponding velocity field

$$\mathbf{u} = \mathbf{u}_o\left(y\right) = \left(U_o\left(y\right), 0\right) = \left(\mathrm{sech}^2\left(y\right), 0\right),$$

is an exact solution to (1).

If we assume a perturbed Bickley jet solution to (1) of the form

$$\psi = \psi_o\left(y\right) + \left\{\varphi\left(y\right)\exp\left[ik\left(x - ct\right)\right] + c.c.\right\},$$

where k and c are the x-direction wavenumber and complex-valued phase velocity, respectively, where $c.c.$ means complex conjugate and neglect the quadratic perturbation terms, we obtain the Rayleigh stability equation

$$\left(U_o - c\right)\left(\partial_{yy} - k^2\right)\varphi - U_{o_{yy}}\varphi = 0, \tag{3}$$

which is solved subject to $|\varphi| \to 0$ as $|y| \to \infty$.

Howard and Drazin (1964) found the singular neutral mode solution for (3) given by

$$\varphi = D\frac{\coth\left(y\right)}{\cosh^3\left(y\right)} \quad \text{for} \quad \left(c, k\right) = \left(1, 3\right),$$

where D is a free amplitude constant. Our goal is to examine the finite-amplitude evolution of a near-singular mode for which

$$k = 3 \quad \text{and} \quad c = 1 - \varepsilon, \quad \text{where} \quad 0 < \varepsilon << 1.$$

3. Weakly-nonlinear asymptotics

It is convenient to introduce the fast phase and slow space-time variables, given by, respectively

$$\theta = x - (1 - \varepsilon)t, \quad \left(X, T\right) = \varepsilon^2\left(x, t\right), \quad \tau = \varepsilon^3 t.$$

In the outer regions, where $|y| \gtrsim O\left(1\right)$, the solution to (1) can be written in the form

$$\psi\left(\theta, y, X, T, \tau\right) = -\tanh\left(y\right) + \varepsilon^3\varphi\left(\theta, y, X, T, \tau\right). \tag{4}$$

with the straightforward asymptotic expansion

$$\varphi\left(\theta, y, X, T, \tau; \varepsilon\right) \simeq \left[\varphi^{(0)} + \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)}\right]\left(\theta, y, X, T, \tau\right) + O\left(\varepsilon^3\right).$$

After substantial algebra (see Swaters, 2000), it can be shown that, as $y \to 0$ and to $O\left(\varepsilon^2\right)$, the outer solution has the asymptotic form

$$\varphi \simeq A\left(X, T, \tau\right)\exp\left[3i\theta\right]\left\{\left[\frac{2}{3y} - \frac{7}{9}y + \frac{307}{540}y^3 - \frac{7717}{22680}y^5 + O\left(y^7\right)\right]\right.$$

$$+\varepsilon\left[\operatorname{sgn}(y)\left(\frac{128}{15}y^2+\frac{448}{225}y^4+\frac{9616}{4725}y^6\right)\right.$$

$$+\ln|y|\left(\frac{8}{5y}-\frac{28}{15}y+\frac{307}{225}y^3-\frac{7717}{9450}y^5\right)$$

$$\left.+\frac{2}{15y^3}+\frac{29}{15y}-\frac{1843}{900}y-\frac{89627}{22680}y^3-\frac{1200377}{504000}y^5+O\left(y^7\right)\right]$$

$$+\varepsilon^2\left[\frac{2}{35y^5}+\left[\frac{49}{375}-\frac{2i\left(A_T+A_X\right)}{45A}\right]\frac{1}{y^3}+\frac{8\ln|y|}{25y^3}\right.$$

$$\left.\left.+\left[\frac{4132}{875}-\frac{8i\left(A_T+A_X\right)}{15A}\right]\frac{\ln|y|}{y}+\frac{48\ln^2|y|}{25y}\right]+O(1)\right\}+c.c.. \tag{5}$$

Examination of (5) suggests secular behavior when $y\simeq O(\sqrt{\varepsilon})$.

In the region where $y\simeq O(\sqrt{\varepsilon})$, the solution to (1) is in the form

$$\psi(\theta,\chi,X,T,\tau)=-\tanh\left(\sqrt{\varepsilon}\chi\right)+\varepsilon^{\frac{5}{2}}\widetilde{\varphi}(\theta,\chi,X,T,\tau), \tag{6}$$

where $\chi=y/\sqrt{\varepsilon}$. It follows from (5) that, in the region $y\simeq O(\sqrt{\varepsilon})$, $\widetilde{\varphi}$ has the form

$$\widetilde{\varphi}\simeq\varphi^{(0)}+\varepsilon\,\varphi^{(1,0)}+\varepsilon\ln\varepsilon\;\varphi^{(1,1)}+\varepsilon^2\ln^2\varepsilon\;\varphi^{(2,0)}+\varepsilon^2\ln\varepsilon\;\varphi^{(2,1)}+O\left(\varepsilon^2\right).$$

For our purposes here, all we need are $\varphi^{(0)}$ and $\varphi^{(1,0)}$. Full details can be found in Swaters (2000). The solution for $\varphi^{(0)}$ is

$$\varphi^{(0)}(\theta,\chi,X,T,\tau)=A\Phi(\chi)\exp(3i\theta)+c.c., \tag{7}$$

with

$$\Phi(\chi)=\chi+\frac{1}{2}\left(1-\chi^2\right)\ln\left(\frac{\chi+1}{\chi-1}\right),$$

where

$$\ln\left(\frac{\chi+1}{\chi-1}\right)=\begin{cases}\ln\left|\frac{\chi+1}{\chi-1}\right| & \text{for } |\chi|>1,\\ \ln\left|\frac{\chi+1}{\chi-1}\right|+\delta\pi i & \text{for } |\chi|<1,\end{cases}$$

where δ ranges from 0 (the nonlinear critical layer) to 1 (the viscous critical layer).

The solution for $\varphi^{(1,0)}$ can be written as

$$\varphi^{(1,0)}=A^2F(\chi)\exp(6i\theta)+O(\exp(3i\theta))+c.c., \tag{8}$$

$$F(\chi)=\frac{\chi}{4\left(1-\chi^2\right)}-\ln\left(\frac{\chi+1}{\chi-1}\right)+\frac{\chi}{4}\ln^2\left(\frac{\chi+1}{\chi-1}\right).$$

4. Numerical simulation

Equation (1) was solved numerically as the system

$$q_t + J(\psi, q) = \frac{1}{R_e}\triangle q, \tag{9}$$

$$\triangle\psi = q, \tag{10}$$

where q is the vorticity and R_e is the Reynolds number. We assume a Reynolds number of $R_e = 3.125 \times 10^8$ to effectively smooth out very high wave number features without significantly altering, over the time scales of interest here, the flow evolution.

The numerical procedure we use is a second-order accurate 256×256 finite-difference leap-frog technique (see, e.g., Swaters, 1989, 1999, 2000) in which the Jacobian term is finite differenced using the Arakawa (1966) scheme. To suppress the development of the computational mode a Robert filter (Asselin, 1972) is applied at each time step with a coefficient of 0.005. The stream function was obtained at the end of each time step by inverting (10) using a direct solver.

Our simulations are done in a periodic channel domain, denoted as Ω, given by

$$\Omega = \{(x, y) \;\mid\; |x| < x_L,\; |y| < y_L\},$$

in which y_L is chosen so as to have no noticeable effect on the transverse evolution of the perturbation stream function. The stream function satisfies Dirichlet boundary conditions on $y = \pm y_L$ and is smoothly periodic along $x = \pm x_L$. The value of the vorticity on $y = \pm y_L$ was updated using second-order accurate one-sided interior domain differences.

The initial condition is a linear superposition of the leading order near-singular $k = 3$ mode and the leading order $k = 6$ harmonic, as determined by (7) and (8), and can be written as

$$\begin{aligned}\psi(x, y, t) = &-\tanh(y) + \{A\varphi_3(y)\exp(3i[x - (1-\varepsilon)\tau]) \\ &+ A^2\varphi_6(y)\exp(6i[x - (1-\varepsilon)\tau]) + c.c.\},\end{aligned} \tag{11}$$

with $\tau = 0$ or $\triangle t$ as needed, and where $\varphi_3(y)$ is the spatially uniformly valid leading order solution for the near-singular $k = 3$ mode given by

$$\varphi_3(y) = \frac{2\varepsilon^3}{3}\left[\operatorname{sech}^3(y)\coth(y) - \frac{1}{y}\right] + \varepsilon^{\frac{3}{2}}\left[\sqrt{\varepsilon}\, y + \frac{\varepsilon - y^2}{2}\ln\left(\frac{y + \sqrt{\varepsilon}}{y - \sqrt{\varepsilon}}\right)\right],$$

and where $\varphi_6(y)$, which describes the leading order transverse structure of the $k = 6$ harmonic, is given by

$$\varphi_6(y) = \varepsilon^3\left\{\frac{\varepsilon y}{4(\varepsilon - y^2)} - \sqrt{\varepsilon}\ln\left(\frac{y + \sqrt{\varepsilon}}{y - \sqrt{\varepsilon}}\right) + \frac{y}{4}\ln^2\left(\frac{y + \sqrt{\varepsilon}}{y - \sqrt{\varepsilon}}\right)\right\}.$$

We assume $A = \delta = 1.0$ and $\varepsilon = 0.05$. Thus $c = 0.95$ and the critical levels are located at $\pm\sqrt{\varepsilon} \simeq 0.22$. We choose $x_L = y_L = 4\pi/3$. With our grid spacing we had about 13 grid points in between the critical levels, i.e., in the region $|y| < \sqrt{\varepsilon}$, at least initially, for each value of x.

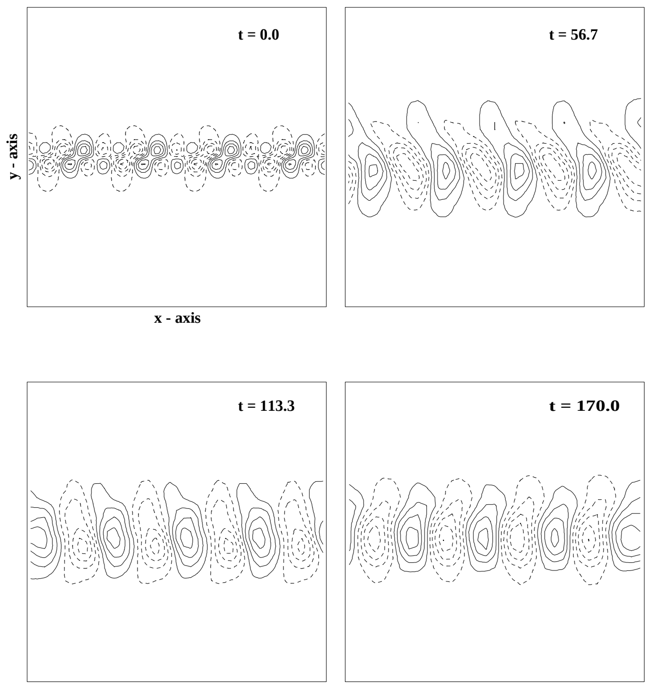
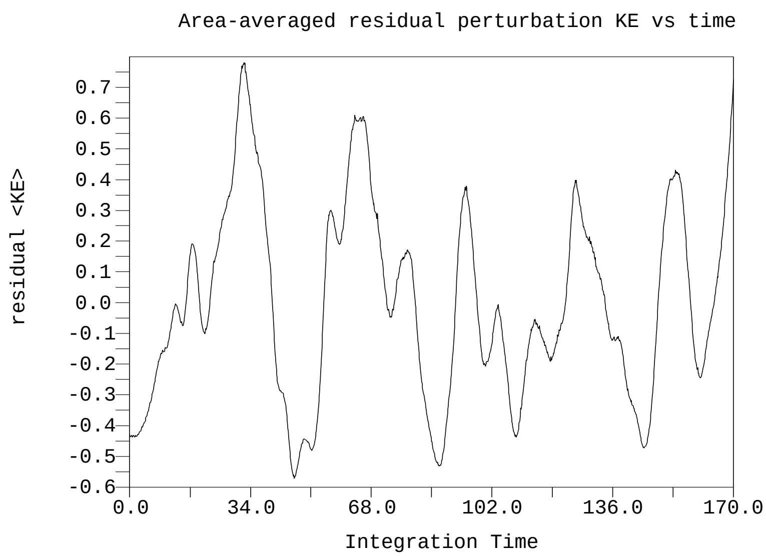


Fig. 1. Contour plots of the perturbation stream function.

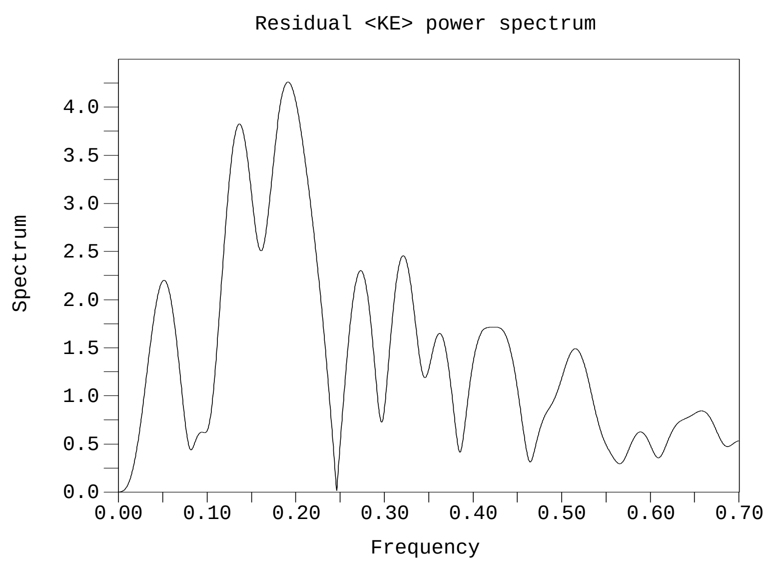
In Fig. 1 we show four contour plots of the perturbation stream function for $t = 0.0$, 56.7, 113.3 and 170.0, respectively. The solid and dashed lines correspond to positive and negative stream function values, respectively. The perturbation stream function remains rather stable over the integration although there appears to be some dilation in the individual high and lows.

As a measure of the long time variability in the underlying modal envelope, we computed the area-averaged perturbation kinetic energy normalized by its initial value and then subtracting out the slight linear trend, giving what we call the "residual" $\langle KE \rangle$ (see Swaters, 1999, 2000).



Area-averaged residual perturbation KE vs time
residual <KE>
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0.0
34.0
68.0
102.0
136.0
170.0
Integration Time

Fig. 2a.



Residual <KE> power spectrum
Spectrum
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
Frequency

Fig. 2b.

In Fig. 2*a* we show the residual $\langle KE \rangle$ versus time. One can see that there is a dominant contribution with a period of about 32 time units. This is a longer time scale than the period associated with the underlying fast phase oscillations which is about $2\pi/(kc) \approx 2.2$ time units. In Fig. 2*b* we show the power spectrum associated with the residual $\langle KE \rangle$. The highest peak is located at a frequency of about 0.19 which corresponds to a period of about 32 time units. This energy peak appears to be somewhat broad with a secondary peak at about frequency 0.13.

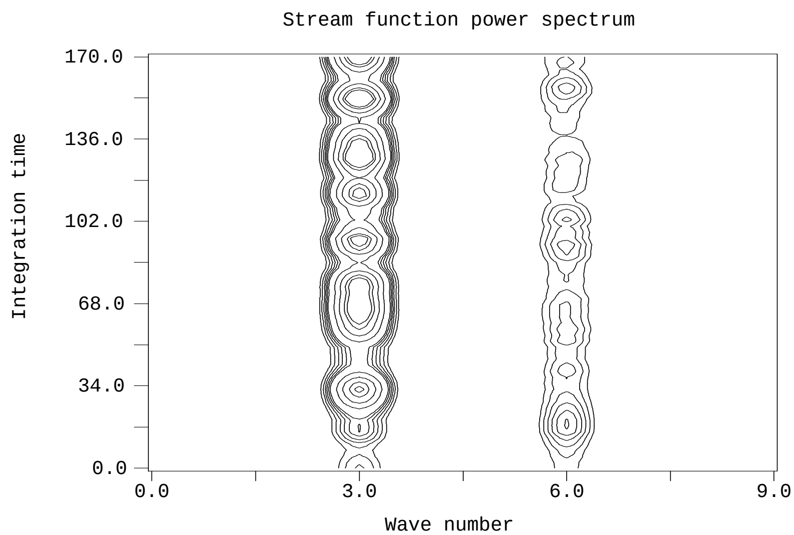


Fig. 3.

To further examine this long time variability we computed a wave number power spectrum for the perturbation stream over time. Because there was little variation as a function of y, it was convenient to average the resulting spectra over y (see Swaters, 1999, 2000) to come up with a power spectrum for the perturbation stream function which is a function of the x-direction wave number and time alone, denoted as $\mathcal{S}(k, t)$.

In Fig. 3 we present a contour plot of $\mathcal{S}(k, t)$. The peak located at $k = 3$ corresponds to the contribution from the term proportional to A in (11). The peak located at $k = 6$ corresponds to the contribution associated with the term proportional to A^2 in (11). One can see the slow time modulation of the peaks. The time scale of this modulation is consistent with the variability seen in Fig. 2a.

It is interesting, and perhaps somewhat unexpected, that our simulations suggest a relatively stable "slow" neutral oscillation in the perturbation stream function amplitude. A remaining challenge is to derive nonlinear amplitude evolution equations which have oscillatory solutions for these near-singular modes.

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