Modal Interpretation for the Ekman Destabilization of Inviscidly Stable Baroclinic Flow in the Phillips Model

GORDON E. SWATERS

Applied Mathematics Institute, and Department of Mathematical and Statistical Sciences, and Institute for Geophysical Research, University of Alberta, Edmonton, Alberta, Canada

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ABSTRACT

Ekman boundary layers can lead to the destabilization of baroclinic flow in the Phillips model that, in the absence of dissipation, is nonlinearly stable in the sense of Liapunov. It is shown that the Ekman-induced instability of inviscidly stable baroclinic flow in the Phillips model occurs if and only if the kinematic phase velocity associated with the dissipation lies outside the interval bounded by the greatest and least neutrally stable Rossby wave phase velocities. Thus, Ekman-induced destabilization does not correspond to a coalescence of the barotropic and baroclinic Rossby modes as in classical inviscid baroclinic instability. The differing modal mechanisms between the two instability processes is the reason why subcritical baroclinic shears in the classical theory can be destabilized by an Ekman layer, even in the zero dissipation limit of the theory.

1. Introduction

The role of dissipation in the transition to instability in baroclinic quasigeostrophic flow can be counterintuitive (Klein and Pedlosky 1992). It is natural to assume that dissipation acts to reduce the growth rates of baroclinic flows that are inviscidly unstable and for flows that are inviscidly stable and that dissipation will lead to the decay in the perturbation amplitudes over time (in the unforced initial-value problem). However, it has been known since Holopainen (1961) and within the context of the Phillips model (Romea 1977) that subcritical baroclinic shears in the linear inviscid stability theory can be destabilized by the presence of an Ekman boundary layer and that this destabilization occurs even in the zero dissipation limit for the frictional theory; that is, there is a range of subcritical baroclinic shears (in the linear inviscid theory) that are destabilized by the presence of an Ekman boundary layer no matter how small the Ekman number is. Recent work by Krechetnikov and Marsden (2007, 2009, hereafter KM09) has described this counterintuitive dissipative destabilization within the context of the underlying Hamiltonian structure of the (inviscid) model equations. In particular, KM09 have extended Romea's (1977) work and showed that Ekman destabilization within the Phillips model can occur for baroclinic shears that are inviscidly (i.e., in the absence of dissipation) nonlinearly stable in the sense of Liapunov.

It is important to appreciate that the discontinuous behavior of the zero dissipation limit of the marginal stability boundary when an Ekman layer is present cannot be dismissed as akin to the well-known property that solutions to the Orr–Sommerfeld equation need not necessarily reduce to solutions to the Rayleigh stability equation in the infinite Reynolds number limit (Drazin and Reid 1981). As pointed out by KM09, the infinite Reynolds number limit of the Orr–Sommerfeld equation is singular in the sense that the order of the differential equation changes from fourth order to second order so that the mathematical properties of the allowed solutions cannot be expected to depend continuously as the Reynolds number increases without limit. This is not the case for the Phillips model with Ekman layers because the zero dissipation limit is not singular.

However, the "physical reason" for the Ekman-induced destabilization of inviscidly stable baroclinic quasigeostrophic flow has yet to be given. The principal purpose of this note is to provide the model interpretation for the onset of this instability. By exploiting the concept of the "kinematic wave" introduced by Lighthill and Whitham (1955a,b) and further described by Whitham (1974), it is

Corresponding author address: Gordon E. Swaters, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada.
E-mail: gordon.swaters@ualberta.ca

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shown that the onset of Ekman destabilization of inviscidly stable baroclinic flow in a zonal channel in the Phillips model occurs when the dissipative kinematic wave phase velocity lies outside the range of zonal phase velocities spanned by the neutrally stable planetary Rossby waves. [In the present context, the kinematic wave is the solution to the long-wave approximation to the governing equations when dissipation is present; see sections 2.2, 3.1, and 10.1 in Whitham (1974) for a general account, with a more complete description given in section 3.] The onset of dissipative destabilization does not correspond to a coalescence of the barotropic and baroclinic modes as in inviscid baroclinic instability; indeed, the necessary conditions for baroclinic instability need not hold. This is the reason why subcritical shears in the inviscid theory can be unstable even in the zero dissipation limit when Ekman layers are present.

The kinematic wave phase velocity stability condition described here is conceptually similar to the stability condition for the formation of roll waves in gravity-driven flow down an inclined plane when bottom friction is present (Whitham 1974, section 3.2; Baines 1984, 1995). Of course in that problem the relevant inviscid modes are internal gravity waves and not planetary Rossby waves as they are here. Baines (1984) described the onset of dissipative destabilization to the development of ''disorder'' associated with the development, for example, of shock waves and bores (when nonlinearity is taken into account) in shallow water theory, and this occurs when the phase velocities of the kinematic waves (when dissipation is present) lie outside the range of the phase velocities allowed by the inviscid modes. Mathematically, this is equivalent to demanding that the ''characteristics'' associated with the dissipative kinematic waves (that arise as solutions to a ''low order'' hyperbolic partial differential operator) must lie in the span of the characteristics associated with the inviscid waves (that arise as solutions to a higher-order hyperbolic partial differential operator, which determines the overall dynamical system properties for wave propagation). Swaters (2009) presents a similar treatment for the Ekman destabilization of baroclinic grounded abyssal flow over a sloping bottom.

The plan of this note is as follows: Section 2 gives the governing equations and very briefly describes the discontinuous behavior of the zero dissipation limit of the marginal stability boundary when Ekman layers are present in the Phillips model for baroclinic instability. In section 3, the linear stability problem is recast in a form that facilitates the introduction of the dissipation-dependent kinematic mode and the modal interpretation for the Ekman destabilization of inviscidly stable baroclinic flow is given. Additional comments are given in relation as to whether it is generic that the zero dissipation limit of the marginal stability boundary in a dissipative baroclinic instability theory does not collapse to the inviscid result. Finally, in as much as it may be physically desirable that the zero dissipation limit of the dissipative marginal stability boundary does collapse to the inviscid result, it is remarked that the parameterization in which the dissipation is proportional to the geostrophic potential vorticity (e.g., Klein and Pedlosky 1992; Pedlosky and Thomson 2003; Flierl and Pedlosky 2007) possesses this property. Concluding remarks are given in section 4.

2. Governing equations and problem formulation

The underlying geometry is a straightforward periodic zonal channel of north–south width L and east–west length $2aL$ (a is nondimensional). The Phillips model with an upper and lower Ekman layer, in standard notation (Pedlosky 1987), can be written in the nondimensional forms

$$[\Delta\varphi_1 - \mathrm{Fr}(\varphi_1 - \varphi_2)]_t + J(\varphi_1, \Delta\varphi_1 + \mathrm{Fr}\varphi_2 + \beta y) = -r\Delta\varphi_1 \quad \text{and} \tag{1}$$

$$[\Delta\varphi_2 - \mathrm{Fr}(\varphi_2 - \varphi_1)]_t + J(\varphi_2, \Delta\varphi_2 + \mathrm{Fr}\varphi_1 + \beta y) = -r\Delta\varphi_2, \tag{2}$$

where, for convenience, the upper and lower layers have equal scale thickness; the length scale is L so that the rotational Froude number $\mathrm{Fr} \equiv f^2L^2/(g'H)$, where g', H, and f are the reduced gravity, scale layer thickness, and constant Coriolis parameter, respectively; $\beta \equiv \beta^* L^2/U$, where β^* is the dimensional beta parameter and U is the velocity scale; and r is the nondimensional (nonnegative) Ekman damping parameter (Pedlosky 1987, section 4.6; again, for convenience, the upper- and lower-layer damping parameters have been chosen to be the same). Alphabetical subscripts will denote, unless otherwise specified, partial differentiation, and the 1 and 2 subscripts refer to the upper and lower layers, respectively. The geostrophic streamfunctions are given by $\varphi_{1,2}$, $J(A, B) \equiv A_xB_y - A_yB_x$, $\Delta \equiv \partial_{xx} + \partial_{yy}$, and x and y are oriented eastward and northward, respectively.

Because of the underlying Galilean invariance of (1) and (2), it is sufficient to consider the stability of the baroclinic zonal flow $\varphi_1 = \varphi_{10} \equiv -Uy$ and $\varphi_2 = \varphi_{20} \equiv 0$ [where U is a constant, which is an exact solution to (1) and (2) irrespective of whether $r = 0$], for which the linear stability problem can be written in the forms

$$\partial_t + U\partial_x[\Delta\phi_1 - \mathrm{Fr}(\phi_1 - \phi_2)] + (\beta + \mathrm{Fr}U)\partial_x\phi_1 = -r\Delta\phi_1 \quad \text{and} \tag{3}$$

$$\partial_t[\Delta\phi_2 - \beta + \mathrm{Fr}(\phi_2 - \phi_1)] + (\beta - \mathrm{Fr}U)\partial_x\phi_2 = -r\Delta\phi_2, \tag{4}$$

where $\phi_{1,2}$ are the disturbance streamfunctions: that is, $\varphi_{1,2} = \varphi_{10,20} + \phi_{1,2}$.

It is important to appreciate that the Ekman dissipation terms are generally not negative definite in the globally integrated energy balance, and this is what is responsible for the potential for dissipative destabilization of inviscidly stable baroclinic flow. Following KM09, if (1) and (2) are multiplied through by φ_1 and φ_2, respectively, and the result is added together and integrated over the spatial domain, it follows for the mean flow considered here that

$$\frac{dE}{dt} = r\iint_\Omega \varphi_1\Delta\varphi_1 + \varphi_2\Delta\varphi_2\,dx\,dy$$
$$= r\iint_\Omega \varphi_{10}\Delta\phi_1 - \nabla\phi_1\cdot\nabla\phi_1 - \nabla\phi_2\cdot\nabla\phi_2\,dx\,dy,$$

where Ω denotes the spatial domain; the globally integrated energy E is given by

$$E \equiv \frac{1}{2}\iint_\Omega \nabla\varphi_1\cdot\nabla\varphi_1 + \nabla\varphi_2\cdot\nabla\varphi_2 + \mathrm{Fr}(\varphi_1 - \varphi_2)^2\,dx\,dy;$$

and the "natural" disturbance boundary conditions $\phi_{1,2} = 0$ along $y = 0$ and 1, respectively, have been assumed as well as periodicity along $x = \pm a$. Plainly, the right-hand side associated with the energy balance equation is not definite in sign. As shown by KM09, depending on the value of r and the mathematical properties of the solutions, it is possible for the perturbation field to extract energy out of the mean flow and amplify even if the background flow is stable when $r = 0$.

This can be more clearly seen by exploiting the underlying Hamiltonian structure of (1) and (2) when $r = 0$ (see, e.g., Holm et al. 1985). The well-known normal-mode stability condition for baroclinic flow in the inviscid Phillips model [see Pedlosky (1987) and brief description later] is most properly obtained (within the context of the Hamiltonian formalism) from a variational principle built on constraining the conserved barotropic zonal momentum with appropriately chosen conserved Casimir functionals (see Panetta et al. 1987), which in the problem considered here are simply weighted enstrophy integrals (see, e.g., Swaters 1999).

When $r = 0$, the baroclinic flow $\varphi_1 = \varphi_{10} \equiv -Uy$ and $\varphi_2 = \varphi_{20} \equiv 0$ satisfy the first-order necessary conditions for an extremal to the conserved functional

$$H = \iint_\Omega \left\{\frac{\beta - \mathrm{Fr}U}{2}[\Delta\varphi_1 - \mathrm{Fr}(\varphi_1 - \varphi_2) + \beta y]^2 + \frac{\beta + \mathrm{Fr}U}{2}[\Delta\varphi_2 - \mathrm{Fr}(\varphi_2 - \varphi_1) + \beta y]^2 - y(\beta^2 - \mathrm{Fr}^2U^2)(\Delta\varphi_1 + \Delta\varphi_2)\right\}dx\,dy$$
$$= \iint_\Omega \frac{(\beta - \mathrm{Fr}U)q_1^2}{2} + \frac{(\beta - \mathrm{Fr}U)q_2^2}{2} - y(\beta^2 - \mathrm{Fr}^2U^2)(q_1 + q_2 - 2\beta y)\,dx\,dy,$$

where

$$q_1 \equiv \Delta\varphi_1 - \mathrm{Fr}(\varphi_1 - \varphi_2) + \beta y \quad \text{and}$$
$$q_2 \equiv \Delta\varphi_2 - \mathrm{Fr}(\varphi_2 - \varphi_1) + \beta y.$$

The first two terms in H are constant-weighted enstrophy integrals for the upper and lower layers, respectively. The third term in H is a constant-weighted barotropic zonal momentum integral. It is straightforward to show using (1) and (2) that each of the three terms in H is an invariant of the full nonlinear motion (so that $dH/dt = 0$) when $r = 0$.

It follows that

$$\delta H = \iint_\Omega (\beta - \mathrm{Fr}U)[q_1 - (\beta + \mathrm{Fr}U)y]\delta q_1 + (\beta + \mathrm{Fr}U)[q_2 - (\beta - \mathrm{Fr}U)y]\delta q_2\,dx\,dy$$
$$\Rightarrow \delta H|_{q_1=q_{10},q_2=q_{20}} = 0,$$

because

$$q_{10} \equiv \Delta\varphi_{10} - \mathrm{Fr}(\varphi_{10} - \varphi_{20}) + \beta y = (\beta + \mathrm{Fr}U)y,$$
$$q_{20} \equiv \Delta\varphi_{20} - \mathrm{Fr}(\varphi_{20} - \varphi_{10}) + \beta y = (\beta - \mathrm{Fr}U)y.$$

Furthermore, it follows that

$$\delta^2 H|_{q_1=q_{10},q_2=q_{20}} = \iint_\Omega (\beta - \mathrm{Fr}U)(\delta q_1)^2 + (\beta + \mathrm{Fr}U)(\delta q_2)^2\,dx\,dy.$$

Because it is necessarily the case that $\delta^2 H|_{q_1=q_{10},q_2=q_{20}}$ is an invariant of the linear stability Eqs. (3) and (4) when $r = 0$, where the identifications $\delta q_1 = \Delta\phi_1 - F(\phi_1 - \phi_2)$ and $\delta q_2 = \Delta\phi_2 - F(\phi_2 - \phi_1)$ are made, it follows that, if U is such that $\delta^2 H|_{q_1=q_{10},q_2=q_{20}}$ is definite in sign for all perturbations, then linear stability in the sense of Liapunov can be established for the inviscid problem (with respect to the enstrophy norm). In fact, for this particular problem, should such U exist, then nonlinear stability in the sense of Liapunov (with respect to the disturbance enstrophy norm) is automatically established for the inviscid problem, because H is quadratic in the dependent variables φ_1 and φ_2 so that $\delta^2 H|_{q_1=q_{10},q_2=q_{20}}$ is an invariant of the full nonlinear Eqs. (1) and (2) when $r = 0$, where the identifications $\delta q_1 = q_1 - q_{10}$ and $\delta q_2 = q_2 - q_{20}$ are made.

There is no value of U for which both $\beta + \mathrm{Fr}U < 0$ and $\beta - \mathrm{Fr}U < 0$. However, both $\beta + \mathrm{Fr}U \geq 0$ and $\beta - \mathrm{Fr}U \geq 0$ if and only if $|U| \leq \beta/\mathrm{Fr}$ (implying that $\delta^2 H|_{q_1=q_{10},q_2=q_{20}} > 0$ for all δq_1 and δq_2), which is the well-known normal-mode stability condition for the baroclinic flow $\varphi_1 = \varphi_{10} \equiv -Uy$ and $\varphi_2 = \varphi_{20} \equiv 0$ in the inviscid Phillips problem [see Pedlosky (1987) and brief description later]. As described earlier, $|U| \leq \beta/\mathrm{Fr}$ is sufficient to prove that the flow $\varphi_1 = \varphi_{10} \equiv -Uy$ and $\varphi_2 = \varphi_{20} \equiv 0$ are nonlinearly stable in the sense of Liapunov with respect to the disturbance enstrophy norm.

Conversely, of course, inviscid baroclinic instability only occurs if $|U| > \beta/\mathrm{Fr}$, in which case the two terms in $\delta^2 H|_{q_1=q_{10},q_2=q_{20}}$ are of opposite sign. In particular, the inviscidly baroclinically unstable normal-mode solutions to the linear stability problem (described later) will satisfy $\delta^2 H|_{q_1=q_{10},q_2=q_{20}} = 0$ for all $t \geq 0$.

Finally, it is remarked that the stability argument presented in KM09 exploited a variational principle based on constraining the energy E with appropriately chosen Casimir functionals. Unfortunately, the sufficient stability conditions obtained with that approach are, in fact, more restrictive (and do not reproduce the known normal-mode stability conditions) than those obtained using the zonal momentum–based argument described here (the KM09 approach yields the stability condition $-\beta/\mathrm{Fr} < U \leq 0$). The difference is a consequence of the fact that the KM09 approach corresponds to Fjørtoft's stability condition, whereas the argument presented here corresponds to Rayleigh's inflection point stability condition (see Swaters 1999).

Returning to the case when $r > 0$, if

$$L \equiv \iint_\Omega (\beta - \mathrm{Fr}U)[\Delta\phi_1 - \mathrm{Fr}(\phi_1 - \phi_2)]^2 + (\beta + \mathrm{Fr}U)[\Delta\phi_2 - \mathrm{Fr}(\phi_2 - \phi_1)]^2\,dx\,dy$$

is introduced (L is just $\delta^2 H|_{q_1=q_{10},q_2=q_{20}}$ written in terms of ϕ_1 and ϕ_2; however, when $r \neq 0$, it is not appropriate to label it as the second variation of a conserved functional), it follows from (3) and (4) that

$$\begin{aligned}\frac{dL}{dt} &= -2r\iint_\Omega (\beta - \mathrm{Fr}U)[\Delta\phi_1 - \mathrm{Fr}(\phi_1 - \phi_2)]\,\Delta\phi_1 + (\beta + \mathrm{Fr}U)[\Delta\phi_2 - \mathrm{Fr}(\phi_2 - \phi_1)]\Delta\phi_2\,dx\,dy\\ &= -2r\iint_\Omega \{(\beta - \mathrm{Fr}U)(\Delta\phi_1)^2 + (\beta + \mathrm{Fr}U)(\Delta\phi_2)^2 + \mathrm{Fr}[\beta|\nabla(\phi_1 - \phi_2)|^2 + U(|\nabla\phi_2|^2 - |\nabla\phi_1|^2)]\}\,dx\,dy.\end{aligned}$$

Hence, in the baroclinically inviscidly stable case where $|U| \leq \beta/F$ ($\Rightarrow L > 0$), the dissipation integral on the right-hand side is not definite in sign and a dissipation-induced instability (i.e., $dL/dt > 0$) is possible depending on the properties of the solution to the linear stability problem and the values of r and U (as shown later).

The linear stability Eqs. (3) and (4) have the normal-mode solution [Romea (1977); for the f-plane version of this problem, see Pedlosky (1987, section 7.12)]

$$(\phi_1,\ \phi_2) = (\mu, 1)A\sin(n\pi y)\exp[ik(x - ct)] + \text{c.c.},$$

where the nondimensional channel walls are located at $y = 0$ and 1, respectively; c.c. means the complex conjugate of the preceding term; $n = 1, 2, \ldots$; the along channel wavenumber is $k = m\pi/a \geq 0$ with $m = 0, 1, \ldots$; A is a free amplitude coefficient; $c = c_R + ic_I$ is the complex-valued zonal phase velocity determined by the dispersion relation

$$c = \frac{U\lambda^2(\lambda^2 + 2\mathrm{Fr}) - 2(\lambda^2 + \mathrm{Fr})(\beta + ir\lambda^2/k) \pm \sqrt{4\mathrm{Fr}^2(\beta + ir\lambda^2/k)^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}}{2\lambda^2(\lambda^2 + 2\mathrm{Fr})}, \qquad (5)$$

where $\lambda^2 \equiv k^2 + n^2\pi^2 > 0$; the branch cut is taken along the negative real axis; and μ is given by

$$\mu = [c(\lambda^2 + \mathrm{Fr}) - \mathrm{Fr}U + \beta + ir\lambda^2/k]/(c\mathrm{Fr}). \qquad (6)$$

It is interesting to observe that the dispersion relation (5) is identical in form to the inviscid dispersion relation (Pedlosky 1987), with β replaced by $\beta + ir\lambda^2/k$. Thus, in some sense, Ekman dissipation may be thought of as equivalent to introducing a complex-valued background vorticity gradient in the normal-mode equations.

Stability occurs when $c_I \leq 0$; that is,

$$\mathrm{Im}\sqrt{4\mathrm{Fr}^2(\beta + ir\lambda^2/k)^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4} \\ \leq 2r\lambda^2(\lambda^2 + \mathrm{Fr})/k. \quad (7)$$

After a little algebra [see Eq. (3.5) in Romea (1977) and Eq. (4.6) in KM09; see the appendix for derivation details], this is equivalent to

$$U^2 \leq U_c^2 \equiv \frac{4}{(2\mathrm{Fr} - \lambda^2)}\left[\frac{\mathrm{Fr}^2\beta^2}{\lambda^2(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^2}{k^2}\right], \quad (8)$$

if $0 < \lambda^2 < 2\mathrm{Fr}$ and the flow is unconditionally stable for λ^2 outside this interval. The marginal stability boundary is given by $|U| = U_c$ and in the limit that Ekman dissipation vanishes

$$\lim_{r\to 0} U_c = U_{c0} \equiv \frac{2F\beta}{\lambda(\lambda^2 + \mathrm{Fr})\sqrt{2\mathrm{Fr} - \lambda^2}}. \quad (9)$$

As observed by Romea (1977), the stability boundary (9) is not the same as the stability boundary obtained by setting $r = 0$ directly in (5): that is, the classical inviscid Phillips instability problem, which is given by (see Pedlosky 1987)

$$|U| = U_B \equiv \frac{2\mathrm{Fr}\beta}{\lambda^2\sqrt{4\mathrm{Fr}^2 - \lambda^4}}, \quad (10)$$

for $0 < \lambda^2 < 2\mathrm{Fr}$ (and the flow is also unconditionally stable for λ^2 outside this interval). Over the range of potentially unstable wavenumbers, it follows that $U_B > U_{c0}$ (they are never equal unless $\mathrm{Fr} = 0$) because

$$\begin{aligned} U_B &= \frac{2\mathrm{Fr}\beta}{\lambda^2\sqrt{4\mathrm{Fr}^2 - \lambda^4}} = \frac{2\mathrm{Fr}\beta}{\lambda\sqrt{2\mathrm{Fr} - \lambda^2}\sqrt{2\mathrm{Fr}\lambda^2 + \lambda^4}} \\ &= \frac{2\mathrm{Fr}\beta}{\lambda\sqrt{2\mathrm{Fr} - \lambda^2}\sqrt{(\lambda^2 + \mathrm{Fr})^2 - \mathrm{Fr}^2}} \\ &> \frac{2\mathrm{Fr}\beta}{\lambda(\lambda^2 + \mathrm{Fr})\sqrt{2\mathrm{Fr} - \lambda^2}} = U_{c0} \end{aligned} \quad (11)$$

[see also Fig. 1 in Romea (1977) and Fig. 5 in KM09].

Baroclinic shears in the interval $U_{c0} < |U| \leq U_B$ do not satisfy the necessary conditions for inviscid baroclinic instability but are dissipatively destabilized by the presence of the Ekman layer. This was first shown by Romea (1977) in the context of the linear normal-mode stability problem for the Phillips model. Normal-mode solutions associated with baroclinic shears in the interval $U_{c0} < |U| \leq U_B$ have the property that $dL/dt > 0$, with $L > 0$.

Examining (8) and (10), it is apparent that, as r increases, eventually $U_c > U_B$ so that for large enough r the viscid stability boundary is "above" the inviscid stability boundary. It follows from (8) and (10) that

$$\frac{U_c^2}{U_B^2} = 1 - \frac{\mathrm{Fr}^2}{(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^6(\lambda^2 + 2\mathrm{Fr})}{4k^2\mathrm{Fr}^2\beta^2},$$

which implies that $U_c > U_B$ whenever

$$0 \leq r < r_{\max} \equiv \frac{2\beta\mathrm{Fr}^2|k|}{\lambda^3(\lambda^2 + \mathrm{Fr})\sqrt{\lambda^2 + 2\mathrm{Fr}}}.$$

An estimate for the order of magnitude of $r_{\max}$ in comparison to β and Fr can be obtained by examining its value at the point of inviscid marginal stability associated with the minimum of U_B located at $\lambda = \lambda_{\min} \equiv \sqrt{\sqrt{2}\mathrm{Fr}}$ for which $U_B|_{\lambda=\lambda_{\min}} = \beta/\mathrm{Fr}$ (Pedlosky 1987). It follows that

$$r_{\max}|_{\lambda=\lambda_{\min}} = \frac{2^{1/4}\beta}{(1 + \sqrt{2})^{3/2}\sqrt{\mathrm{Fr}}} \simeq O(\beta/\sqrt{\mathrm{Fr}}),$$

where $O(k) \sim \lambda_{\min}$ has been assumed. For these scalings, the dissipation plays an equal role in the dynamics as all the other terms in (3) and (4). Thus, r does not need to be asymptotically small in comparison to β and Fr for the flow to be inviscidly baroclinically stable but dissipatively destabilized.

3. Modal interpretation

The goal here is to provide the "physical mechanism" behind the dissipative destabilization of inviscidly stable flows in terms of the wave modes and its relationship to the onset of classical baroclinic instability. The linear stability problem (3) and (4) can be combined into the single equation,

$$(\partial_t + c_-\partial_x)(\partial_t + c_+\partial_x)\phi = -\frac{2r(\lambda^2 + \mathrm{Fr})}{(\lambda^2 + 2\mathrm{Fr})}(\partial_t + c_0\partial_x)\phi, \quad (12)$$

where

$$c_\pm \equiv \frac{U\lambda^2(\lambda^2 + 2\mathrm{Fr}) - 2(\lambda^2 + \mathrm{Fr})\beta \pm \sqrt{4\lambda^6(\lambda^2 + 2\mathrm{Fr})r^2/k^2 + 4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}}{2\lambda^2(\lambda^2 + 2\mathrm{Fr})} \quad \text{and} \quad (13)$$

$$c_0 \equiv \frac{U}{2} - \frac{\beta}{\lambda^2 + \mathrm{Fr}}, \qquad (14)$$

where ϕ is either ϕ_1 or ϕ_2 and a normal-mode form for the solution has been implicitly assumed so that it is possible to write $\Delta = -\lambda^2$. Writing the linear stability equation in the form of (12) helps to facilitate the comparison with the theory presented in Lighthill and Whitham (1955a,b) and Whitham (1974). To explicitly focus on the modal interpretation for the dissipation-induced instability of inviscidly stable flow, it will henceforth be assumed that $|U| \le U_B$ and $0 < \lambda^2 < 2\mathrm{Fr}$ (λ^2 outside this interval is not of interest), which implies that the quantity within the square root in (13) is strictly positive and thus $c_\pm$ are real valued and ordered as $c_- \le c_+$.

The c_+ and c_- correspond to Doppler-shifted neutrally stable baroclinic and barotropic planetary Rossby wave modes, respectively, which include the strictly neutral frequency shift associated with the dissipation [the term proportional to r^2 within the square root in (13)]. The $r \to 0$ limit of $c_\pm$ is exactly the inviscid Rossby wave solution associated with (5). The c_0 mode corresponds to a kinematic wave, as described by Lighthill and Whitham (1955a,b) and Whitham (1974).

The kinematic wave phase velocity c_0 may be interpreted as the average of the individual Rossby wave velocities associated with the linear stability equations for the upper and lower layers, respectively, in the "uncoupled" inviscid limit. The uncoupled inviscid approximations to (3) and (4) are, respectively,

$$(\partial_t + U\partial_x)(\Delta - \mathrm{Fr})\phi_1 + (\beta + \mathrm{Fr}U)\partial_x\phi_1 = 0 \quad \text{and}$$
$$\partial_t(\Delta - \mathrm{Fr})\phi_2 + (\beta - \mathrm{Fr}U)\partial_x\phi_2 = 0.$$

The upper-layer equation can be explicitly identified as an equivalent-barotropic approximation, but the presence of the FrU coefficient in the lower-layer equation makes a similar identification for it not exact. Assuming a normal-mode solution, the phase velocity associated with the upper layer, denoted as c_1, is given by $c_1 = U - (\beta + \mathrm{Fr}U)/(\lambda^2 + \mathrm{Fr})$; the phase velocity associated with the lower layer, denoted as c_2, is given by $c_2 = (\mathrm{Fr}U - \beta)/(\lambda^2 + F)$. It follows immediately that $c_0 = (c_1 + c_2)/2$. This is not just algebraically coincidental. In fact, c_0 arises precisely in this manner in the derivation of (12).

In the limit that $r \to \infty$ (when dissipation dominates the dynamics), the leading order balance associated with (12) is

$$(\partial_t + c_0\partial_x)\phi = \frac{\lambda^2}{2k^2(\lambda^2 + \mathrm{Fr})}\phi_{xx},$$

which is parabolic. Thus, in the limit of "large" dissipation, the disturbance field is stable (because $r > r_{\max}$, which is introduced in section 2), propagates zonally with phase velocity c_0, and diffuses with a dissipation coefficient that is dependent on the wavenumber. Perturbations with "short" zonal wavelength dissipate less rapidly than waves with a "long" zonal wavelength (recall that $\lambda \ge \pi$ for all $k \ge 0$).

Lighthill and Whitham (1955a,b) introduced the concept of the kinematic wave as a traveling wave solution to a first-order (1 + 1 dimensional) conservation law in which a functional relationship exists between the "density" and "flux" in the conservation law. Kinematic waves are not necessarily "classical" or "dynamic" waves that are described by Newton's second law of motion in which the acceleration associated with particle displacement occurs against a background restoring force (e.g., gravity waves). The kinematic wave concept has been useful in understanding the dynamics and stability of, for example, roll wave formation, continuous models for traffic flow, and spatially variable chemical reactions (see Whitham 1974, section 2.2).

In the context of the present problem, the kinematic wave part of the partial differential operator in (12) is the dissipation-dependent first-order wave operator on the right-hand side (if $r = 0$, the right-hand side is zero). Within this paradigm, the second-order partial differential operator on the left-hand side of (12) is the "dynamic wave" part that describes the zonally propagating vorticity waves that arise as a consequence of meridional displacement of patches of anomalous vorticity against the background planetary vorticity gradient. In the absence of dissipation, the right-hand side of (12) is zero and what remains are only the dynamic waves.

The low-frequency/wavenumber approximation in (12) would be to neglect the left-hand side and retain only the kinematic wave part of the partial differential operator. Thus, roughly speaking, in the context of (12) and from the perspective of the solution to the pure initial-value problem, the low-frequency–wavenumber part of the solution would be governed by the right-hand side and everything else would be governed by the left-hand side. However, as written, (12) corresponds to a hyperbolic partial differential equation in which the characteristics are exclusively determined by the second-order dynamic wave terms on the left-hand side. From this perspective, for the problem to be well posed, the characteristics associated with the first-order kinematic wave operator (governing the low-frequency–wavenumber part of the solution) must be consistent with the characteristics associated with the second-order dynamic wave part of the operator [see the discussion of wave hierarchies in Whitham (1974, chapter 10)], which alone determines the signal propagation properties for the overall dynamical system; that is, the phase velocity associated with dissipative kinematic wave operator must lie in the interval spanned by the phase velocities associated with the dynamic Rossby wave operator.

Thus, the stability condition associated with (12) is that

$$c_- \le c_0 \le c_+, \tag{15}$$

which, if (13) and (14) are substituted in, can be written in the form

$$\frac{2\mathrm{Fr}^2\beta}{(\lambda^2+\mathrm{Fr})} \le \sqrt{4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2-\lambda^4)+4\lambda^6(\lambda^2+2\mathrm{Fr})r^2}, \tag{16}$$

because the left-hand inequality in (15) is satisfied for all parameter values. If both sides of (16) are squared, it follows that

$$\begin{aligned} U^2 &\le \frac{4\mathrm{Fr}^2\beta^2 - 4\mathrm{Fr}^4\beta^2/(\lambda^2+\mathrm{Fr})^2 + 4\lambda^6(\lambda^2+2\mathrm{Fr})r^2/k^2}{\lambda^4(4\mathrm{Fr}^2-\lambda^4)} \\ &= \frac{4}{(2\mathrm{Fr}-\lambda^2)}\left\{\frac{\mathrm{Fr}^2\beta^2[(\lambda^2+\mathrm{Fr})^2-\mathrm{Fr}^2]}{\lambda^4(\lambda^2+2\mathrm{Fr})(\lambda^2+\mathrm{Fr})^2} + \frac{r^2\lambda^2}{k^2}\right\} = U_c^2, \end{aligned} \tag{17}$$

recovering (8).

As shown, the velocity c_0 may be interpreted as the average of the individual Rossby wave velocities associated with the linear stability equations for the upper and lower layers in the uncoupled and inviscid limit, respectively. Thus, we may interpret the stability condition (15) as the requirement that the average of the individual layers' Rossby wave velocities for the upper and lower layers in the uncoupled and inviscid limit, respectively, must lie in the interval spanned by the phase velocities associated with the fully coupled dynamic Rossby waves.

Hence, the Ekman-induced destabilization of inviscidly stable baroclinic quasigeostrophic flow in the Phillips model occurs when the phase velocity associated with the dissipation-created kinematic wave lies outside the interval spanned by the neutral barotropic and baroclinic planetary Rossby waves (including the frequency shift associated with the dissipation). From the point of view of normal-mode theory, this is a very different scenario as compared to inviscid baroclinic instability in which the onset of destabilization occurs when a coalescence develops between the barotropic and baroclinic Rossby waves (i.e., the phase velocities become equal). Dissipative destabilization does not require any such coalescence in the barotropic and baroclinic phase velocities; indeed, it follows from (5) that

$$c_R\big|_{U=U_c} = \frac{U_c}{2} - \frac{\beta[(\lambda^2+\mathrm{Fr})^2 \mp \mathrm{Fr}^2]}{\lambda^2(\lambda^2+2\mathrm{Fr})(\lambda^2+\mathrm{Fr})} \quad \text{and} \tag{18}$$

$$c_I\big|_{U=U_c} = -\frac{r(1\mp 1)(\lambda^2+\mathrm{Fr})}{k(\lambda^2+2\mathrm{Fr})}. \tag{19}$$

At marginal stability, it is the baroclinic mode that is neutrally stable and the barotropic mode exponentially decays because of the dissipation.

Alternatively, it is possible to interpret the stability condition (15) as the requirement that the "effective diffusion coefficient" associated with (12) be nonnegative, assuming the dynamics is dominated by the kinematic wave (see Whitham 1974, section 3.1). The linear stability Eq. (12) can be written in the form

$$\frac{2r(\lambda^2+\mathrm{Fr})}{(\lambda^2+2\mathrm{Fr})}(\partial_t + c_0\partial_x)\phi = \nu^2\phi_{xx} - (\partial_t + \tilde{c}\partial x)^2\phi,$$

where

$$\tilde{c} \equiv \frac{U\lambda^2(\lambda^2+2\mathrm{Fr}) - 2(\lambda^2+\mathrm{Fr})\beta}{2\lambda^2(\lambda^2+2F)},$$

$$\nu \equiv \frac{\sqrt{4\lambda^6(\lambda^6+2\mathrm{Fr})r^2/k^2 + 4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2-\lambda^4)}}{2\lambda^2(\lambda^2+2\mathrm{Fr})}$$

(i.e., $c_\pm = \tilde{c} \pm \nu$), where it is noted that ν is real for $|U| \le U_B$ in the interval $0 < \lambda^2 < 2\mathrm{Fr}$ (λ^2 outside this interval is not of interest). Assuming the dynamics is principally determined by the left-hand side, it follows that $\partial_t \simeq -c_0\partial_x$, which if substituted into the right-hand side yields

$$\frac{2r(\lambda^2+\mathrm{Fr})}{(\lambda^2+2\mathrm{Fr})}(\partial_t + c_0\partial_x)\phi = [\nu^2 - (\tilde{c}-c_0)^2]\phi_{xx}.$$

For stability, it follows that

$$(\tilde{c}-c_0)^2 \le \nu^2 \Leftrightarrow c_0 - \nu \le \tilde{c} \le c_0 + \nu,$$

which is exactly (15). Of course, instability occurs if the effective diffusion coefficient is negative: that is, $(\tilde{c}-c_0)^2 > \nu^2$.

Although the current results have been obtained for Ekman dissipation, it is natural to speculate whether it is generic that the marginal stability boundary associated with the zero dissipation limit of a dissipative baroclinic instability theory collapses to the inviscid result. The answer would seem to be that the generic situation is that the stability boundary associated with the zero dissipation limit of a dissipative baroclinic instability theory does not collapse to the inviscid result.

For example, if Ekman friction in the Phillips model were to be replaced by horizontal turbulent friction, then, within the context of the quasigeostrophic assumptions implicit in the derivation (Pedlosky 1987), the terms $-r\Delta\varphi_{1,2}$ on the right-hand side of (1) and (2), respectively, would be replaced by $\mathrm{Re}^{-1}\Delta^2\varphi_{1,2}$, where Re is the Reynolds number. The resulting stability theory for a periodic zonal channel with free slip along the meridional channel walls is exactly as described earlier, with r simply replaced by λ^2/Re.

However, objections can be raised to this particular example. For example, whether no-slip (rather than free-slip) boundary conditions should be applied on the meridional channel walls $y = 0$ and L when $\mathrm{Re} < \infty$ might be considered an issue. If no-slip boundary conditions are applied, the resulting dispersion relation for the normal modes will no longer be (5) with r replaced by λ^2/Re. In the no-slip case, the infinite Re limit of (3) and (4), with horizontal friction present, is singular, because the (spatial) order of the partial differential equations is reduced in the limit (e.g., from fourth order to second order with respect to y). Consequently, the dependence of the no-slip solutions on Re cannot be expected to be continuous as $\mathrm{Re} \to \infty$; again (for mathematical reasons similar to the Orr–Sommerfeld to Rayleigh transition), the stability boundary associated with the zero dissipation limit will not necessarily collapse to the inviscid result.

In as much as it is desirable that the stability boundary associated with the zero dissipation limit of a dissipative baroclinic instability theory does collapse to the inviscid result, the parameterization in which the dissipation is assumed proportional to the geostrophic potential vorticity (e.g., Klein and Pedlosky 1992; Pedlosky and Thomson 2003; Flierl and Pedlosky 2007) has this property. This parameterization would have (3) and (4) replaced by

$$(\partial_t + U\partial_x)[\Delta\phi_1 - \mathrm{Fr}(\phi_1 - \phi_2)] + (\beta + \mathrm{Fr}U)\partial_x\phi_1 = -r[\Delta\phi_1 - \mathrm{Fr}(\phi_1 - \phi_2)] \quad \text{and} \tag{20}$$

$$\partial_t[\Delta\phi_2 - \mathrm{Fr}(\phi_2 - \phi_1)] + (\beta - \mathrm{Fr}U)\partial_x\phi_2 = -r[\Delta\phi_2 - \mathrm{Fr}(\phi_2 - \phi_1)], \tag{21}$$

for which the dispersion relation associated with the normal modes is given by

$$c = \frac{\lambda^2(U - 2ir/k)(\lambda^2 + 2\mathrm{Fr}) - 2(\lambda^2 + \mathrm{Fr})\beta \pm \sqrt{4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}}{2\lambda^2(\lambda^2 + 2\mathrm{Fr})}. \tag{22}$$

It follows from (22) that the stability occurs when

$$\mathrm{Im}\sqrt{4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)} \le 2r\lambda^2(\lambda^2 + 2\mathrm{Fr})/k. \tag{23}$$

Clearly, if $\lambda = 0$ or $\lambda^2 \ge 2\mathrm{Fr}$, then (23) is satisfied for all shears. If $0 < \lambda^2 < 2\mathrm{Fr}$, the flow is stable, provided that

$$|U| \le U_P \equiv \frac{2\sqrt{\mathrm{Fr}^2\beta^2 + r^2\lambda^4(\lambda^2 + 2\mathrm{Fr})^2/k^2}}{\lambda^2\sqrt{4\mathrm{Fr}^2 - \lambda^4}}, \tag{24}$$

and unstable otherwise. Unlike the Ekman-layer result (8), in the limit that $r \to 0$, U_P reduces to the inviscid marginal stability boundary U_B given by (10).

From the perspective of the kinematic stability condition, the analog of (12) for (20) and (21) is

$$(\partial_t + \tilde{c}_-\partial_x)(\partial_t + \tilde{c}_+\partial_x)\phi = -2r(\partial_t + \tilde{c}_0\partial_x)\phi, \tag{25}$$

where

$$\tilde{c}_\pm \equiv \tilde{c}_0 \pm \frac{\sqrt{4r^2\lambda^4(\lambda^2 + 2\mathrm{Fr})^2/k^2 + 4\mathrm{Fr}^2\beta^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}}{2\lambda^2(\lambda^2 + 2\mathrm{Fr})} \quad \text{and} \tag{26}$$

$$\tilde{c}_0 \equiv \frac{U\lambda^2(\lambda^2 + 2\mathrm{Fr}) - 2(\lambda^2 + \mathrm{Fr})\beta}{2\lambda^2(\lambda^2 + 2\mathrm{Fr})}. \tag{27}$$

It follows that $\tilde{c}_- \le \tilde{c}_0 \le \tilde{c}_+$ when $|U| \le U_P$ if $0 < \lambda^2 < 2\mathrm{Fr}$ (and unconditionally for λ^2 outside this interval). There is no dissipative destabilization of inviscidly stable modes with this parameterization because $U_P \ge U_B$ so that, if $U \le U_B$, the phase velocity of the kinematic mode will always lie in the range of the phase velocities spanned by the Rossby waves.

From the perspective of the time evolution of the functional L (introduced in section 2), it will follow from (20) and (21) that

$$\frac{dL}{dt} = -2rL \Rightarrow L = L|_{t=0}\exp(-2rt),$$

which again underscores the fact that there can be no dissipative destabilization of inviscidly stable modes with this parameterization; that is, when the flow is inviscidly stable so that $L > 0$, there is no (positive) value of r for which $dL/dt > 0$ and the amplitude of the disturbance field, in the inviscidly stable case, always decays to zero.

4. Conclusions

It is known that the presence of Ekman layers in the Phillips model for baroclinic instability has the property that there are inviscidly (i.e., in the absence of dissipation) stable flows (i.e., subcritical shears) that are dissipatively destabilized no matter how small the Ekman number is (Romea 1977). Recent work (KM09) has shown this destabilization occurs even if the flow is inviscidly nonlinearly stable in the sense of Liapunov and that the dissipation induces a nonlinear instability. Dissipative destabilization corresponds to the onset of disorder (Baines 1984) and is a consequence of the fact that the propagation properties of the dissipation-dependent low-frequency–wavenumber waves are inconsistent with the propagation properties of the overall dynamical system, which are determined principally by inviscid dynamical processes.

In this note, a "modal interpretation" for this dissipation-induced instability was given. By exploiting the kinematic wave concept, introduced by Lighthill and Whitham (1955a,b), it was shown that the dissipative destabilization of inviscidly stable baroclinic flow corresponds to the situation where the phase velocity of the kinematic wave lies outside the range spanned by the least and greatest phase velocities of the neutral planetary Rossby waves. When the phase velocity of the kinematic waves lies inside the range inclusively spanned by the least and greatest phase velocities of the neutral planetary Rossby waves, the flow is stable. At the point of marginal stability, there is no coalescence between the barotropic and baroclinic mode. Thus, dissipative destabilization of baroclinic quasigeostrophic flow is not a "variant" of classical inviscid baroclinic instability.

It was speculated that the zero dissipation limit of the marginal stability boundary in a dissipative baroclinic instability theory generically does not collapse to the inviscid result. In as much as it may be desirable that the zero dissipation limit of the marginal stability boundary does collapse to the inviscid result, it was shown that the dissipation parameterization that is proportional to the geostrophic potential vorticity (e.g., Klein and Pedlosky 1992; Pedlosky and Thomson 2003; Flierl and Pedlosky 2007) has this property. In this parameterization, the phase velocity of the kinematic mode always lies in the range of the phase velocities spanned by Rossby waves for subcritical baroclinic shears.

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APPENDIX

Derivation of the Dissipative Stability Condition

For completeness, the dissipative stability condition (2.8) is derived here. Define the real numbers α and γ to be given by

$$\begin{aligned}\alpha\exp(i\gamma) &= 4\mathrm{Fr}^2(\beta + ir\lambda^2/k)^2 - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)\\ &= 4\mathrm{Fr}^2(\beta^2 - r^2\lambda^4/k^2) - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4)\\ &\quad + 8ir\beta\mathrm{Fr}^2\lambda^2/k,\end{aligned} \tag{A1}$$

where $\alpha \geq 0$ and $0 \leq \gamma \leq \pi$ (because the imaginary part is nonnegative). The stability condition (2.7) can be written in the form

$$\sqrt{\alpha}\,\sin(\gamma/2) \leq 2r\lambda^2(\lambda^2 + \mathrm{Fr})/k. \tag{A2}$$

Because both sides of (A2) are necessarily positive, it follows that

$$2\alpha\sin^2(\gamma/2) = \alpha[1 - \cos(\gamma)] \leq 8r^2\lambda^4(\lambda^2 + \mathrm{Fr})^2/k^2;$$

substituting in for α and γ leads to

$$\sqrt{\Gamma^2 + 64\beta^2r^2\mathrm{Fr}^4\lambda^4/k^2} \leq \Gamma + 8r^2\lambda^4(\lambda^2 + \mathrm{Fr})^2/k^2, \tag{A3}$$

where

$$\Gamma \equiv 4\mathrm{Fr}^2(\beta^2 - r^2\lambda^4/k^2) - U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4). \tag{A4}$$

For stability, the right-hand side of (A3) must be positive because the left-hand side is as well [or else (A3) can never hold and instability necessarily follows]. The right-hand side of (A3) is positive, provided that

$$4F^2\beta^2 + 4r^2\lambda^4(2\lambda^4 + 4\mathrm{Fr}\lambda^2 + \mathrm{Fr}^2)/k^2 \geq U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4). \tag{A5}$$

Inequality (A5) is clearly satisfied for all shears (i.e., U) if $\lambda = 0$ or $\lambda^2 \geq 2\mathrm{Fr}$. If $0 < \lambda^2 < 2\mathrm{Fr}$, (A5) will be satisfied, provided that

$$U^2 \leq \frac{4\mathrm{Fr}^2\beta^2 + 4r^2\lambda^4(2\lambda^4 + 4\mathrm{Fr}\lambda^2 + \mathrm{Fr}^2)/k^2}{\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}. \tag{A6}$$

This forms a necessary condition for stability. In a moment, a necessary and sufficient stability condition will be obtained that will subsume this necessary condition.

To proceed further, it is assumed that right-hand side of (A3) is positive. Thus, both sides of (A3) can be squared, and it follows that (if $r \neq 0$)

$$U^2\lambda^4(4\mathrm{Fr}^2 - \lambda^4) \le 4\lambda^4(2\mathrm{Fr} + \lambda^2)\left[\frac{\mathrm{Fr}^2\beta^2}{(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^4}{k^2}\right]. \tag{A7}$$

Again, (A7) is clearly satisfied if $\lambda = 0$ or $\lambda^2 \ge 2\mathrm{Fr}$. Hence, the flow is unconditionally stable for $\lambda = 0$ or $\lambda^2 \ge 2\mathrm{Fr}$. If $0 < \lambda^2 < 2\mathrm{Fr}$, (A7) will be satisfied, provided that

$$U^2 \le \frac{4}{(2F - \lambda^2)}\left[\frac{\mathrm{Fr}^2\beta^2}{\lambda^2(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^2}{k^2}\right], \tag{A8}$$

which is (2.8).

All that remains to be shown is that, if (A8) holds, so does (A6). Thus, (A8) [i.e., (2.8)] is the necessary and sufficient stability condition if $0 < \lambda^2 < 2\mathrm{Fr}$ and the flow is unconditionally stable for λ^2 outside this interval. It follows from (A8) that

$$\begin{aligned}\frac{4}{(2\mathrm{Fr} - \lambda^2)}\left[\frac{\mathrm{Fr}^2\beta^2}{\lambda^2(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^2}{k^2}\right] &= \frac{4(\lambda^2 + 2\mathrm{Fr})}{\lambda^2(4\mathrm{Fr}^2 - \lambda^4)}\left[\frac{\mathrm{Fr}^2\beta^2}{(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^4}{k^2}\right] \\ &= \frac{4[(\lambda^2 + \mathrm{Fr})^2 - \mathrm{Fr}^2]}{\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}\left[\frac{\mathrm{Fr}^2\beta^2}{(\lambda^2 + \mathrm{Fr})^2} + \frac{r^2\lambda^4}{k^2}\right] \\ &\le \frac{4}{\lambda^4(4\mathrm{Fr}^2 - \lambda^4)}\left\{\mathrm{Fr}^2\beta^2 + \frac{r^2\lambda^4}{k^2}[2(\lambda^2 + \mathrm{Fr})^2 - \mathrm{Fr}^2]\right\} \\ &= \frac{4\mathrm{Fr}^2\beta^2 + 4r^2\lambda^4(2\lambda^4 + 4\mathrm{Fr}\lambda^2 + \mathrm{Fr}^2)/k^2}{\lambda^4(4\mathrm{Fr}^2 - \lambda^4)},\end{aligned}$$

which is the right-hand side of (A6). Thus, if (A8) holds, so does (A6).

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