Introduction to Hamiltonian fluid dynamics and stability theory (Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, 2000), 274 pp., ISBN 1-58488-0-236

A review by S. Reich published in

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Classical mechanics can be firmly grounded on a Hamiltonian and/or Lagrangian formulation. While both approaches are essentially equivalent, Hamiltonian dynamics and the notion of symplecticness have perhaps become the prevalent foundation of mechanics. The extension of the Hamiltonian approach to infinite-dimensional systems, such as wave and fluid dynamics, has become an active area of research over the last twenty to thirty years. Even today the question of which formulation, Hamiltonian or Lagrangian, is to be preferred is largely open. However, it is without doubt that Hamiltonian dynamics has had an important impact on ideal fluid and wave dynamics. This is in particular true for geophysical fluid dynamics, as can be seen from the work of Holm, McIntyre, Morrison, Salmon, Shepherd and others.

The book under review summarizes some of the recent work on Hamiltonian fluid dynamics. In particular, it provides a rather non-technical and entertaining introduction to the Hamiltonian formulation of ideal two-dimensional fluids and stability results for steady flows and travelling waves.

Let me highlight a few of the topics covered in the book. Chapter 2 gives a very compact introduction to the basic concepts in Hamiltonian classical mechanics. The material is self-contained and is kept to the basics. Chapter 3 is concerned with the Hamiltonian structure of two dimensional ideal incompressible fluids. In a first step the non-canonical Hamiltonian structure of the vorticity formulation is stated and the various properties of the associated Euler–Poisson bracket are verified. Next this bracket is derived via explicit reduction from a Lagrangian particle formulation of fluid dynamics. The Euler–Poisson bracket leads naturally to the conservation of vorticity in terms of Casimir functionals. The chapter ends with an application of Noether's Theorem. Unfortunately, the author decided not to mention the concept of particle relabelling, which is at the very heart of the Lagrangian to Euler reduction process. The next chapter provides an extensive discussion of stability results for steady Euler flows. The stability theory of steady flows is complicated by the fact that stationary flows do not, in general, satisfy the first order necessary conditions for an energy minimum. Thus the classical stability methods break down. V. I. Arnold suggested the construction of an invariant pseudo-energy functional. For parallel shear flows Arnold's linear stability theorems reduce to Fjortoft's results. Furthermore, Arnold established sufficient conditions which would establish nonlinear stability. The author presents Arnold's stability results as well as important recent developments, such as Andrew's Theorem, from a general variational point of view and its associated Hamiltonian formulation.

An interesting generalization of the two-dimensional vorticity equation is provided by the Charney–Hasegawa–Mima (CHM) equation, which arises from the shallow-water equations for rotating fluids in the limit of small Rossby numbers. The CHM equation has dispersive linear wave solutions, called Rossby waves, and has also nonlinear steadily travelling dipole vortex solutions. These solutions play an important role in large scale evolution of the planetary atmosphere. The Hamiltonian structure of the CHM equations and its derivation are discussed in Chapter 5. A large portion of that chapter is then devoted to the stability of steady solutions. An important new feature is the existence of steadily travelling waves. The discussion of their stability leads to important modifications in the previously presented framework; these are also discussed in Chapter 5. The final chapter is concerned with the Hamiltonian structure and the associated stability theory for the celebrated Korteweg–de Vries (KdV) equations.

The book is presented in a refreshingly non-technical style with plenty of details and exercises provided. The reader should be familiar with basic fluid dynamics, classical mechanics, variational calculus, and stability theory. The text can be recommended for advanced undergraduate students and graduate students in applied mathematics and physical sciences. All in all, this is a well-written introduction to Hamiltonian fluid dynamics and basic stability results.