PHYS 485

Introductory Particle Physics

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Chapter 1

Relativistic Kinematics

In the theory of special relativity, the laws of physics are equally valid in all inertial reference systems. An inertial reference system is a system in which Newton's 1'st law – the law of inertia – is obeyed. According to the law of inertial, objects keep moving in straight lines at constant speeds unless acted upon by some force. I assume a knowledge of special relativity.

1.1 Lorentz Transformation

Two inertial systems must move with a constant velocity with respect to each other. Moreover, a system moving with a constant velocity with respect to an inertial system is also an inertial system.

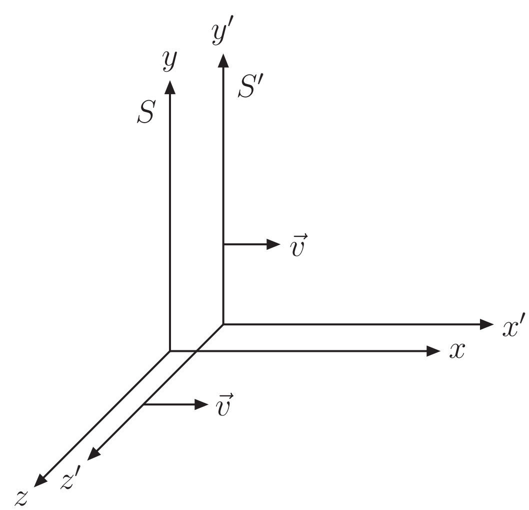


Figure 1.1: Two frames of reference in uniform translation. The x- and x'-axes are suppose to be collinear. The ct-axis can not be drawn.

Consider the initial conditions $t = t' = 0$ when $x = x' = 0$, and $\vec{v} = v\hat{x}$ is a constant. The Lorentz transformation is

$$\begin{aligned} x' &= \gamma(x - vt) \to \gamma(x - \beta t) && (1.1)\\ y' &= y && (1.2)\\ z' &= z && (1.3)\\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \to \gamma(t - \beta x)\,, && (1.4) \end{aligned}$$

where $\beta = v/c \to v$ and $\gamma = (1 - \beta^2)^{-1/2}$. In vector notation

$$\begin{aligned} \vec{x}\,' &= \gamma(\vec{x} - \vec{\beta}t) && (1.5)\\ \vec{t}\,' &= \gamma(t - \vec{\beta}\cdot\vec{x})\,. && (1.6) \end{aligned}$$

For the inverse transformation $v \to -v$.

Relativity gives rise to the following consequences, which make it quite different from Newtonian kinematics:

1. relativity of simultaneity - events simultaneous in one inertial system are not simultaneous in others,

2. Lorentz contraction - moving objects are shortened by a factor γ^{-1},

3. time dilation - moving clocks run slow by a factor γ^{-1}.

1.2 Four Vectors

We define the 4-coordinates as $x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$. Then $x^\mu \equiv (x^0, x^k) \equiv (x^0, x^1, x^2, x^3)$ is a four-vector, where $\mu = 0, 1, 2, 3$ and $k = 1, 2, 3$.

For the metric tensor in flat space-time we will use

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (1.7)$$

1.2.1 Covariant/Contravariant Indices

We will distinguish between covariant and contravariant vectors. A contravariant vector transforms as a coordinate vector under a Lorentz transformation. A covariant vector transforms as gradiant vector. The level of the indicy denotes the type of vector:

$$\begin{array}{lll} \text{covariant indicy} & \rightarrow & a_\mu \text{ (subscript)}, \\ \text{contravariant indicy} & \rightarrow & a^\mu \text{ (superscript)}. \end{array}$$

The relationship between covariant and contravariant vectors is

$$a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu \quad \Rightarrow \quad a_0 = a^0 \text{ and } a_k = -a^k \, . \tag{1.8}$$

We will drop the explict sum and use the Einstein summation convention $a_\mu = g_{\mu\nu}a^\nu$. Also, $a^\mu = g^{\mu\nu}a_\nu$, $g^{\mu\nu} = g_{\mu\nu}$, $g_\mu{}^\nu = \delta_\mu{}^\nu$, where

$$\delta_\mu{}^\nu = \begin{cases} 1 \text{ if } \mu = \nu \, , \\ 0 \text{ if } \mu \neq \nu \, . \end{cases} \tag{1.9}$$

Therefore $g^{\mu\nu}$ and $g_{\mu\nu}$ are the inverse of each other, or g is its own inverse. Also $g_{\mu\nu}g^{\mu\nu} = (g_{\mu\mu})^2 = 4$.

1.2.2 Three-vector, Four-vector and Scalar Product

Consider the arbitrary contravairant vector $a^\mu \equiv (a^0, a^1, a^2, a^3) \equiv (a^0, \vec{a})$, such that $\vec{a} = (a_x, a_y, a_z)$. The dot product is $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$. We often write a^μ as just a. The scalar product is $a \cdot b = a_\mu b^\mu = a^\mu b_\mu = a^\mu g_{\mu\nu} b^\nu = a^0 b^0 - \vec{a} \cdot \vec{b}$, which has the same value in all inertial reference systems. We call such objects "invariant".

1.2.3 Classification of Four-vectors

Four-vectos can be classified into the following types:

$$\begin{array}{ll} a_\mu a^\mu < 0, & a^\mu \text{ is space-like}, \\ a_\mu a^\mu = 0, & a^\mu \text{ is light-like or null}, \\ a_\mu a^\mu > 0, & a^\mu \text{ is time-like}. \end{array}$$

1.2.4 Lorentz Group

A four-vector transform like

$$x'^\nu = \Lambda^\nu{}_\mu x^\mu + a^\nu \, , \tag{1.10}$$

where $\Lambda^\nu{}_\mu$ is a general Lorentz transformation and a^ν is a vector giving causing a translation. The Lorentz transformation can be any of the following

1. Lorentz boost,
2. three-space rotation,
3. parity (space reflection),

4. time reflection.

A Lorentz boost along the x-axis can be written simply as

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.11}$$

1.2.5 Tensors

Let $s^{\mu\nu}$ be a 2'nd rank tensor. Then it transforms as $s^{\mu\nu\prime} = \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\sigma}s^{\kappa\sigma}$. A vector is a tensor of rank 1. A scalar is a tensor of rank 0.

1.3 Energy and Momentum

In particle physics, we think in terms of relativisitic energy and three-momentum. The energy and momentum of a particle of mass m is $E = \gamma mc^2 \to \gamma m$ and $\vec{p} = \gamma m\vec{v} \to \gamma\vec{\beta}m$.

The boost parameters from the laboratory frame to the particle's rest frame are given by

$$\gamma = \frac{E}{mc^2} \to \frac{E}{m} \quad \text{and} \quad \vec{\beta} = \frac{\vec{p}c}{E} \to \frac{\vec{p}}{E} . \tag{1.12}$$

The four-momentum is $p^\mu = (E/c, \vec{p}) \to (E, \vec{p})$. For real particles $p^2 = p_\mu p^\mu = (E/c)^2 - \vec{p}^2 = m^2c^2$ is invariant.

In the non-relativistic limit

$$\gamma = (1-\beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \ldots \quad \text{for} \quad \beta \ll 1 . \tag{1.13}$$

Thus the non-relativistic energy is $E \approx mc^2 + (1/2)mc^2\beta^2$, where mc^2 is the rest energy and the second term is the $(1/2)mv^2$ kinetic energy (classical). Also, if $\beta \approx 1$ (ultra-relativistic limit)

$$\beta^2 = 1 - \gamma^{-2} = 1 - \frac{1}{\gamma^2} , \tag{1.14}$$

$$\beta \approx 1 - \frac{1}{2}\frac{1}{\gamma^2} . \tag{1.15}$$

Relativistic kinetic energy T is given by $E = mc^2 + T$,

$$T = E - mc^2 = (\gamma - 1)mc^2 \approx \frac{1}{2}\beta^2 mc^2 . \tag{1.16}$$

There is no concept of relativistic mass in particle physics.

Classical mechanics has no massless particles:

$$m = 0 \;\Rightarrow\; p = mv = 0\,, \tag{1.17}$$
$$T = \frac{1}{2}mv^2 = 0\,, \tag{1.18}$$
$$F = ma \Rightarrow F = 0 \quad \text{always.} \tag{1.19}$$

But zero mass particles are allowed in relativistic mechanics. Considering the relativistic energy $E = \gamma mc^2$. If $m \to 0$, then we must have $\gamma \to \infty$ such that E remains finite, and $\vec{p} = \vec{\beta}\gamma mc \Rightarrow \beta\gamma \to \infty$, such that $\vec{p}$ remains finite. Now $\gamma \to \infty$ if $\beta \to 1$ and $\beta\gamma \to \infty$ if $\beta \to 1$. So if $m = 0 \Rightarrow E = |\vec{p}|c = h\nu$. A frequency determines the energy and momentum of massless particle.

1.4 Collisions and Decays

Now consider the, so called, $2 \to 2$ particle collision

$$A + B \to C + D\,.$$

The following conservation laws apply

$$E_A + E_B = E_C + E_D \quad \text{energy conservation,} \tag{1.20}$$
$$\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D \quad \text{momentum conservation,} \tag{1.21}$$
$$p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D \quad \text{each component of four} - \text{momentum conserved,} \tag{1.22}$$
$$m_A + m_B \neq m_C + m_D \quad \text{mass not conserved.} \tag{1.23}$$

In addition, the kinetic energy may not be conserved. In an elastic interaction kinetic energy, rest energy, and mass are conserved. In an inelastic interaction the kinetic energy is not conserved

We distinguish between "conserved' and "invarant" quantities. A conserved quantity has the same value after as before the interaction, eg. E and $\vec{p}$. An invariant quantity has the same value in all inertial reference frames.

Now consider the decay

$$A \to B + C\,.$$

A may decay if $m_A > m_B + m_C$. If $m_A < m_B + m_C$ A is stable and $m_A = m_B + m_C - BE$, were BE is defined ad the binding energy.

Figure 1.2: Two-body decay.

1.5 Example

Consider $M \to m_1 + m_2$ with $p = (M, 0)$, $p_1 = (E_1, \vec{p}_1)$, and $p_2 = (E_2, \vec{p}_2) = (E_2, -\vec{p}_1)$.
Assume $m_1 \neq m_2 \neq 0$.

$$\begin{aligned}
p &= p_1 + p_2 && (1.24)\\
p - p_1 &= p_2 && (1.25)\\
(p - p_1)^2 &= p_2^2 && (1.26)\\
p^2 + p_1^2 - 2p \cdot p_1 &= p_2^2 && (1.27)\\
p^2 + p_1^2 - 2EE_1 + 2\vec{p} \cdot \vec{p}_1 &= p_2^2 && (1.28)\\
M^2 + m_1^2 - 2ME_1 + 0 &= m_2^2\,. && (1.29)
\end{aligned}$$

Therefore.

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \quad \text{and} \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}\,. \tag{1.30}$$

Check $E_1 + E_2 = M$.

$$\begin{aligned}
p^2 &= (p_1 + p_2)^2 && (1.31)\\
p^2 &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 && (1.32)\\
p^2 &= p_1^2 + p_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 && (1.33)\\
M^2 &= m_1^2 + m_2^2 + 2E_1E_2 + 2\vec{p}^2 && (1.34)\\
2\vec{p}^2 &= M^2 - m_1^2 - m_2^2 - 2E_1E_2 && (1.35)\\
\vec{p}^2 &= \frac{M^2 - m_1^2 - m_2^2}{2} - \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)\left(\frac{M^2 + m_2^2 - m_1^2}{2M}\right) && (1.36)\\
&= \frac{2M^4 - 2M^2m_1^2 - 2M^2m_2^2 - M^4 + m_1^4 + m_2^4 - 2m_1^2m_2^2}{4M^2} && (1.37)\\
&= \frac{M^4 + m_1^4 + m_2^4 - 2M^2m_1^2 - 2M^2m_2^2 - 2m_1^2m_2^2}{4M^2}\,. && (1.38)
\end{aligned}$$

Special case: $m_1 = m, m_2 = 0$.

$$E_1 = \frac{M^2 + m^2}{2M} \quad E_2 = \frac{M^2 - m^2}{2M} \quad \vec{p}^2 = \frac{M^2 + m^4 - 2M^2m^2}{4M^2} \quad |\vec{p}| = \frac{M^2 - m^2}{2M} \tag{1.39}$$

Check.

$$E_1^2 - \vec{p}_1^2 = \frac{M^4 + m^4 + 2M^2m^2 - M^4 - m^4 + 2M^2m^2}{4m^2} = m^2 \tag{1.40}$$

$$E_2^2 - \vec{p}_2^2 = \frac{M^4 + m^4 - 2M^2m^2 - M^4 - m^4 + 2M^2m^2}{4m^2} = 0\,. \tag{1.41}$$

Special case: $m_1 = m_2 = m$.

$$E_1 = E_2 = \frac{M}{2}, \quad |\vec{p}| = \sqrt{\frac{M^4 - 4M^2m^2}{4M^2}} = \frac{\sqrt{M^2 - 4m^2}}{2}\,. \tag{1.42}$$

Check.

$$E^2 - \vec{p}^2 = \frac{M^2 - M^2 + 4m^2}{4} = m^2\,. \tag{1.43}$$

If $m_1 = m_2 = 0 \Rightarrow |\vec{p}| = M/2$ and $E = M/2 \rightarrow$ check $E = |\vec{p}|$.

Chapter 2

Elementary Particle Dynamics

2.1 Four Forces

Each force belongs to a physical theory. The strength of a force is ambigous notion.
Gravity: strength 10^{-42}, exchange particle graviton:

- Newton's universal gravitation,
- Einstein's general theory of relativity,
- quantum theory of gravity not worked out,
- gravity too weak in elementary particle physics.

Electromagnetic: strength 10^{-2}, exchange particle photon:

- electrodynamics,
- Maxwell theory (already relativistic),
- quantum theory in 1940.

Weak: strength 10^{-13}, exchange particles W and Z:

- weak nuclear force (β decay),
- unified with electrodyanmics.

Strong: strength 10, exchange particles gluons:

- quantum chromodynamics.

2.2 Quantum Electrodynamics (QED)

Quantum electrodynamics is the oldest, simplest, and most successful theory. It is a model theory for all theories. It has only one elementary process, or primitive vertex (Fig. 2.1).

The simplest processes involve two vertices.

$$
\begin{array}{rcll}
e^-e^- & \rightarrow & e^-e^- & \text{Möller scattering,} \\
e^-e^+ & \rightarrow & e^-e^+ & \text{Bhabha scattering (2 diagrams),} \\
e^-e^+ & \rightarrow & \gamma\gamma & \text{pair annihilation,} \\
\gamma\gamma & \rightarrow & e^-e^+ & \text{pair production,} \\
e\gamma & \rightarrow & e\gamma & \text{Compton scattering.}
\end{array}
$$

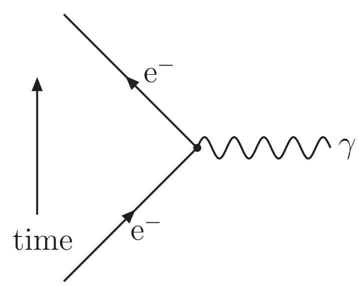


Figure 2.1: Primitive vertex in quantum electrodynamics.

These diagrams have the following properties:

- each diagram represents a number,
- the number is calculated according to Feynman rules,
- a physical process is represented by all relevant diagrams,
- the lines are not momentum or velocity vectors,
- energy and momentum is conserved at each vertex,
- external lines are real particles,
- internal lines are virtual particles,
- antiparticles are represented as a particle line running backwards in time.

There are also higher order four-vertex diagrams (representing radiative corrections).

For real particles $E^2 - p^2 = m^2$. For virtual particles article $E^2 - p^2 \neq m^2$. A vertial particle does not lie on "mass shall". Each vertex contributes a factor of $\alpha = (e^2/\hbar c) = 1/137$.

2.3 Quantum Chromodynamics (QCD)

In quantum chromodyanmics (QCD), colour plays the role of electric charge. The colour force is mediated by the exchange of gluons. Leptons do not carry colour and thus do not feel the strong force. The flavour of the quark does not change.

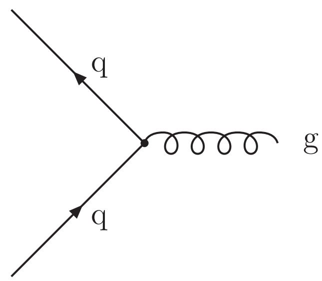


Figure 2.2: Primitive vertex in quantum chromodynamics.

The simplest process is $qq \to qq$.

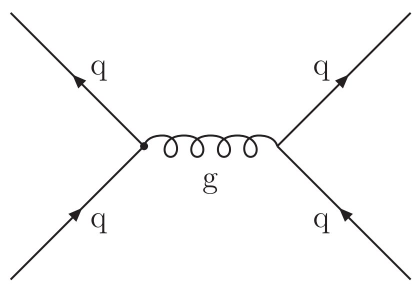


Figure 2.3: Simplest two-vertex process in quantum chromodynamics.

Colour is conserved and there are three types of colour. The colour of the quark must change. Gluons are bi-coloured (1 unit of colour + 1 negative unit of colour).

There are $3 \times 3 = 9$ possible gluons $\to$ actually only 8. Since gluons carry colour, they couple to themselves (unlike the photon which is not charged).

The analogy to the electromagnetic coupling is $\alpha_{\text{EM}} = 1/137$ is $\alpha_{\text{s}} \sim 1$. This is not a constant but "runs". α_{s} is big at large distances and small at small distance (asymptotic freedom). This effect is the opposite way around to QCD. q is the bare charge, while $q(r \to \infty)$ is the experimental measurable charge, or "effective charge".

We can not observe colour directly, but only packages of colourless objects: mesons (1 quarks and 1 antiquark), baryons (3 quarks).

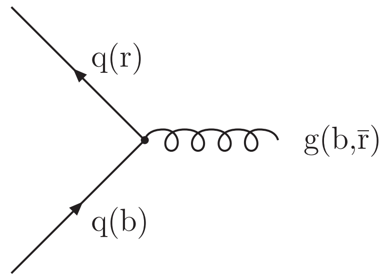


Figure 2.4: Primitive vertex with colour assignment.

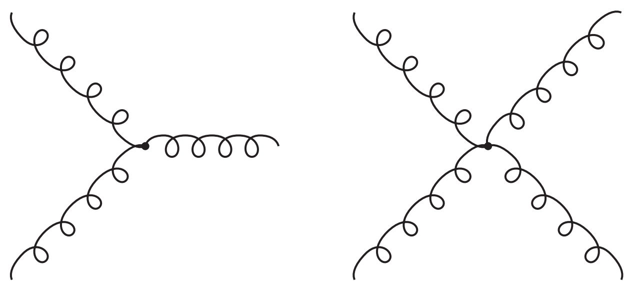


Figure 2.5: Gluon self-coupling.

2.4 Weak Interactions

All particles (q and ℓ) carry weak charge and thus interact by the weak interaction. There are two kinds of weak interactions and each has a corresponding exchange particle: 1) charged ($W^{\pm}$) and 2) neutral (Z^0).

2.4.1 Leptons

An example of a charged vertex involving leptons is β-decay.
Two examples of two-vertex charged weak processes are $\mu^- + \nu_e \to e^- + \nu_\mu$ and $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$.

Neutral vector exchange is also possible.

An example of a two-vertex neutral weak process is $\nu_\mu + e^- \to \nu_\mu + e^-$.

Electron-electron scattering can thus occur via two processes $e + e \to \gamma^* \to e + e$ and $e + e \to Z^* \to e + e$.

At each weak vertex electron, muon, and tau number are conserved separately. In otherwords, a vertex only connects members of the same generation (needs modification).

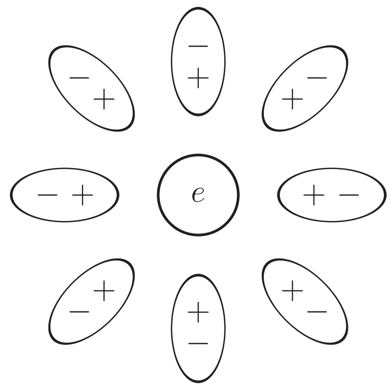


Figure 2.6: Virtual e^+e^- pairs acting as dipoles to screen the bare charge of the electron.

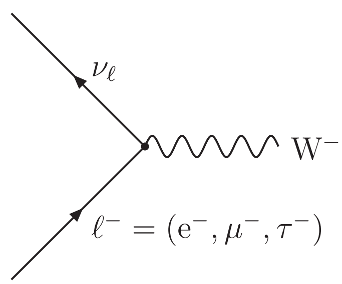


Figure 2.7: Charged weak vertex involving leptons. Emission of W^- or absorption of W^+.

2.4.2 Quarks

In a weak interaction, the quark flavour changes (not conserved) but the colour of the quark does not change. The W does not carry flavour.

Two-vertex processes can be pure leptonic, semi-leptonic, or pure hadronic.

Weak interactions also occur in hadrons.

$d + \nu_e \to u + e^- \Rightarrow \pi^- \to e^- + \bar{\nu}_e$, $n \to p + e^- + \bar{\nu}_e$ (spectator quarks) and $\Delta^0 \to p + \pi^-$ (weak and strong).

A two-vertex neutral weak process is $\nu_\mu + p \to \nu_\mu + p$.

In weak interactions, we distinguish between flavour and mass eigenstates. The mass eigenstates are represented by

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}, \tag{2.1}$$

while the flavour eigenstates (weak force couples the pairs) are represented by

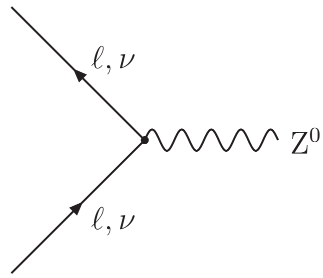


Figure 2.8: Neutral weak vertex involving leptons.

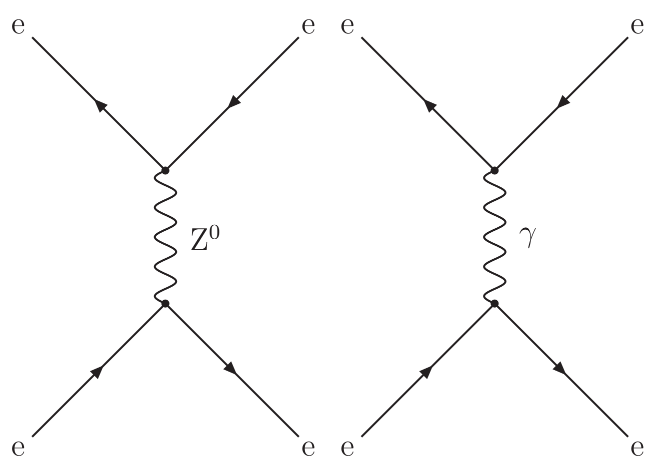


Figure 2.9: Electron-positron scattering.

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \begin{pmatrix} t \\ b' \end{pmatrix} . \tag{2.2}$$

The flavour eigenstates are linear combinations of the physical quarks

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \tag{2.3}$$

V is the 3×3 Kobayashi-Maskawa matrix. It is almost a unit matrix and the non-diagonal elements represent flavour changing processes. For example, V_{ud} measures the coupling of u to d quarks.

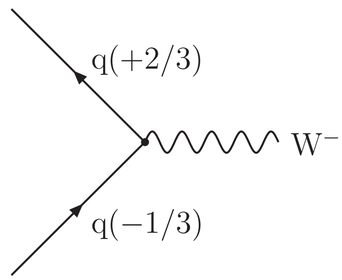


Figure 2.10: Charged weak vertex involving quarks.

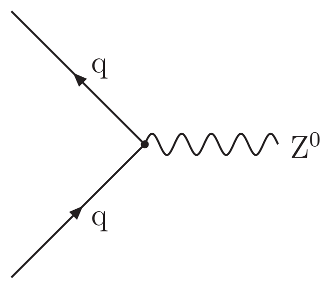


Figure 2.11: Neutral weak vertex involving quarks.

2.4.3 Weak and Electromagnetic Couplings of W and Z

The W and Z themselves couple electromagnetically and weakly via the following interactions:

$$W \to WZ, \quad WW \to WW, \quad WZ \to WZ,$$

and

$$W \to W\gamma, \quad WW \to Z\gamma, \quad W\gamma \to W\gamma.$$

2.5 Decays and Conservation Laws

Most particles want to decay to lighter particles. The following are exceptions:

- the ν and γ are stable because they are massless and can not decay to any lower mass particles since energy-momentum must be conserved,
- the e^- is stable because it is the lightest charged particle and charge must be conserved,

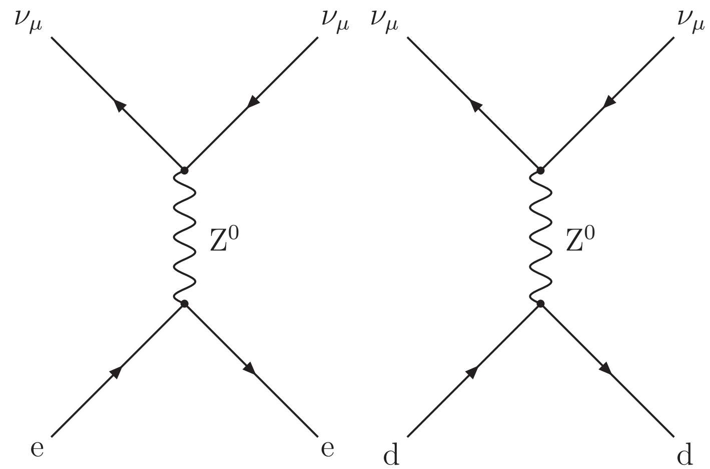
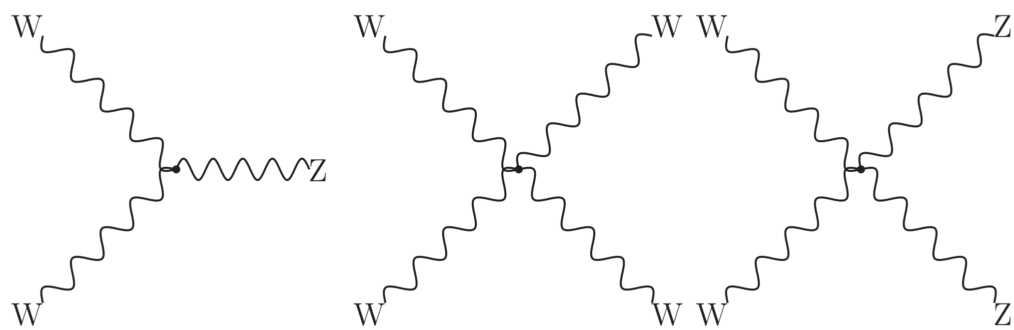


Figure 2.12: Neutral weak processes.



- the p is stable because it is the lightest baryon and baryon number must be conserved,
- e^+ and p^- are also stable,
- the n is stable when it is in matter.

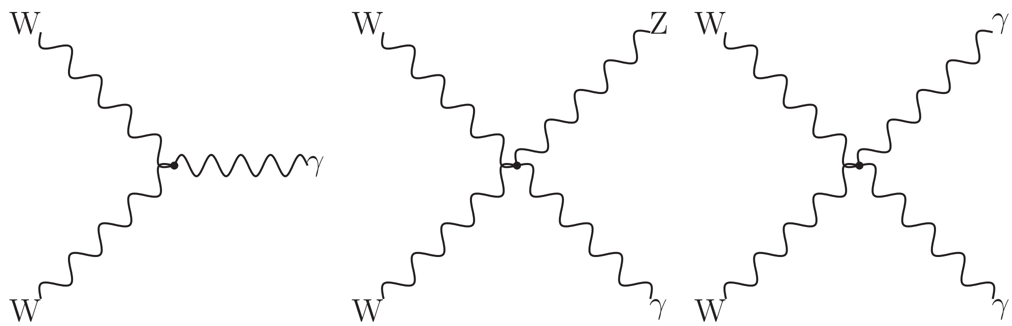
These are the particles that make up our world because they are stable. Most particles decay to several different channels.

Each unstable particle has a characteristic lifetime τ.

$$t_{1/2} = (\ln 2)\tau = 0.693\tau\,, \tag{2.4}$$

where $t_{1/2}$ is the half-life, which is the time it takes 1/2 of the sample to decay. Some typical lieftimes are

$\tau \sim 10^{-23}$ s	strong,
$\tau \sim 10^{-16}$ s	EM,
$\tau \sim 10^{-13}$ s to 15 min	weak.



For weak decays, an example of a quick decay is the tau and an example of a slow decay is the neutron. The large the mass difference, the faster the decay. Kinetic conservation laws (energy, momentum, angular momentum) apply to all interactions.

If a γ comes out of an interation, the process is electromagnetic. If a ν comes out of an interaction, the process is weak.

Dynamic conservation laws

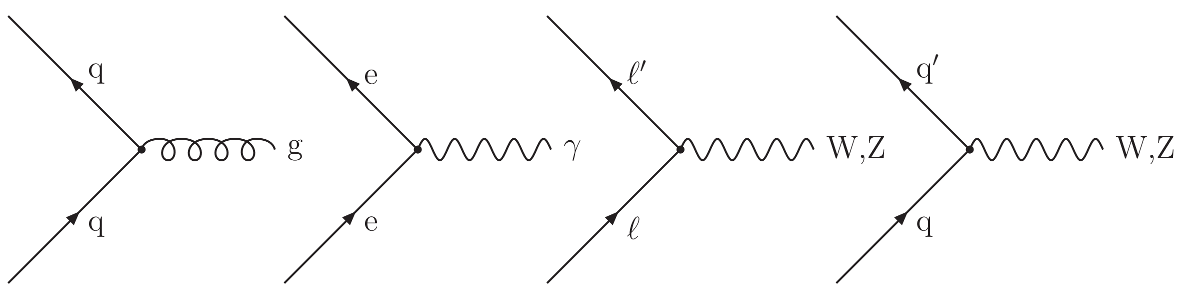


Figure 2.13: Conservation at vertices.

Anything conserved at a vertex must be conserved in the reaction as a whole.

1. Conservation of charge (electric) - all interactions. Weak interactions can change the particle charge.
2. Conservation of colour - all interactions. Strong interactions change quark colour.
3. Conservation of baryon number - all interactions. $A = +1$ baryon (1/3 quark), $A = -1$ anti-baryon ($-1/3$ antiquark), $A = 0$ everything else.
4. Electron, muon, tau number - all conserved in all interactions. Weak interactions mix leptons in the same generation. This does not happen for quarks because of Cabibbo mixing.
5. Approximate conservation of flavour - not in weak interactions.

6. OZI rule - if a diagram can be cut in two by slicing only gluon lines, the process is suppressed, eg. long life of ψ, ϕ, etc.

Chapter 3

Symmetries

3.1 Symmetries, Groups, Conservation Laws

Exploiting symmetries can be a very powerful tool. Symmetry considerations can lead to a deeper understanding and calculational simplification.

Noether's Theorem relates a symmetry to a conservation law. Every symmetry in nature yields a conservation law. Conversely, every conservation law reveals an underlying symmetry. For example,

translational invariance in time	$\Leftrightarrow$ energy conservation,
translational invariance in space	$\Leftrightarrow$ momentum conservation,
rotational invariance	$\Leftrightarrow$ angular momentum conservation,
gauge invariance	$\Leftrightarrow$ charge conservation (internal symmetry)

A symmetry is an operation you can perform on a system that leaves it invariant, that is, that carries it into a configuration indistinguishable from the original one.

3.1.1 Group

The mathematical theory of groups is a systematic study of symmetries. Groups have the following properties:

1. closure: If R_i and R_j are elements of a set, R_iR_j also in the set.

2. identity: There is an element I such that $IR_i = R_iI = R_i$, for all elements R_i.

3. inverse: For every element R_i there is an inverse R_i^{-1} such that $R_iR_i^{-1} = R_i^{-1}R_i = I$.

4. associativity: $R_i(R_jR_k) = (R_iR_j)R_k$.

If the elements commute $R_iR_j = R_jR_i$ the group is Abelian.

Finite groups (discrete)	$\leftrightarrow$ infinite groups
discrete groups	$\leftrightarrow$ continous groups

Groups usually have a matrix representation. A matrix is unitary if $U^{-1} = U^\dagger = (U^T)^* = (U^*)^T$. Thus $U^\dagger U = 1$. A matrix is orthogonal (real and unitary) if $O^{-1} = O^T$. Thus $O^T O = 1$.

For $n \times n$ matrices

U(n)	unitary,
SU(n)	unitary with determinant 1,
O(n)	orthogonal,
SO(n)	orthogonal with determinant 1.

Examples: Lorentz group 4×4 Λ matrix and SO(3) group of rotations in space.

3.2 Spin and Orbital Angular Momentum

The earth has two types of angular momenutm: orbital angular momentum rmv and spin angular momentum $I\omega$. The electron in the hydrogen atom also has two types of momentum: orbital and spin angular momentum. Spin is an intrinsic property of a particle (electron point particle).

In quantum mechanics one can only measure the magnitude of one component of momentum. For orbital angular momenutm it is conventional to take $L^2 = \vec{L} \cdot \vec{L}$ and L_z.

Measure $L^2 \to l(l+1)\hbar^2, l = 0, 1, 2, \ldots$.

Measure $L_z \to m_l\hbar, m_l = -l, -l+1, \ldots, -1, 0, 1, l-1, l$

For spin angular momenutm $s^2 = \vec{s} \cdot \vec{s} \to s(s+1)\hbar^2, s = 0, 1/2, 1, 3/2, \ldots$

$s_z \to m_s = -s, -s+1, \ldots, s-1, s$; $(2l+1)$ possibilities.

s is the spin of the particle, fixed for each particle

$s =$ half-integer	$\Rightarrow$ fermion (baryon, lepton, quark) antisymmetric (Fermic/Dirac),
$s =$ integer	$\Rightarrow$ boson (meson, mediator) symmetric (Bose/Einstein).

3.3 Addition of Angular Momentum

We will use ket notation $|lm_l\rangle$ and $|sm_s\rangle$. Total angular momentum $\vec{J} = \vec{L} + \vec{S}$ (this is conserved). Now combine two states $\vec{J} = \vec{J}_1 + \vec{J}_2$, states $|j_1 m_1\rangle$, $|j_2 m_2\rangle$, total $|jm\rangle$.

$$m = m_1 + m_2, \tag{3.1}$$

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, (j_1 + j_2) - 1, (j_1 + j_2). \tag{3.2}$$

Example: two spin-1/2 quarks in meson $(l = 0) \Rightarrow j = 0, 1$. If $l > 0 \Rightarrow j = l+1, l, l-1$ $\to$ all mesons carry integer spin (bosons).

Example: three spin-1/2 quarks in baryon $\Rightarrow j = 3/2, 1/2 \to$ all baryons carry 1/2-integer spin (fermions).

To add three angular momenta, combine two of them first.

3.4 Spin-1/2

Spin $s = 1/2$ is important: proton, neutron, electron, quarks, leptons. $m_s = 1/2$ spin up, $m_s = -1/2$ spin down. We use spinor notation.

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{3.3}$$

The most general spin state is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{3.4}$$

where α, β are complex numbers. Measurement of s_z can only return $+1/2\hbar$ or $-1/2\hbar$. $|\alpha|^2$ probability to measure $s_z = +1/2\hbar$. $|\alpha|^2 + |\beta|^2 = 1$ normalization condition.

We associate a 2×2 matrix with each component of $\vec{S}$:

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.5}$$

$\hat{S}_z$ eigenvalues are $\pm\hbar/2$. $\hat{S}_x$ eigenvalues are $\pm\hbar/2$, with normalized eigenvector

$$\chi_{\pm} = \begin{pmatrix} 1/\sqrt{2} \\ \pm 1/\sqrt{2} \end{pmatrix} . \tag{3.6}$$

An arbitrary spinor can be written as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + b \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} , \tag{3.7}$$

where $a = (1/\sqrt{2})(\alpha + \beta)$, $b = (1/\sqrt{2})(\alpha - \beta)$. The probability to measure $S_x = 1/2\hbar$ is $|a|^2$. Again $|a|^2 + |b|^2 = 1$.

We formulate a general procedure:

1. construct matrix $\hat{A}$, representing observable A,

2. allowed values of A are eigenvalues of $\hat{A}$,

3. write state of system as linear combination of eigenvectors of $\hat{A}$.

The absolute square of the coefficient of the ith eigenvector is probability that a measurement of A would yield the ith eigenvalue.

Introduce Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.8}$$

The spin is given by $\hat{S} = (\hbar/2)\vec{\sigma}$.

How does a spinor transform under rotation?

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\theta) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad U(\theta) = e^{-\vec{\theta}\cdot\vec{\sigma}/2} \tag{3.9}$$

where $U(\theta)$ is a 2×2 matrix and $\vec{\theta}$ is a vector with magnitude the angle and points along the axis of rotation.

The exponential of a matrix A is a matrix defined as

$$e^A = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots. \tag{3.10}$$

Notice $e^A e^B \neq e^{A+B}$ unless $AB = BA$.

$U(\theta)$ unitary matrix of determinant 1. All such rotation matrices consitute a group SU(2). SU(2) is the same group as SO(3) except for minus sign under rotation of 2π.

Particles of different spin belong to different representations of the rotation group

- Spin-1/2 particles transform under rotations according to the fundamental 2-D representation of SU(2).
- Spin-1 particles transform under rotations according to the fundamental 3-D representation of SU(2).
- Spin-3/2 particles transform under rotations according to the fundamental 4-D representation of SU(2).

3.5 Flavour Symmetries

The strong force experienced by proton and neutron are identical. If we could turn off electrical charge, the proton and neutron would be indistinguishable. This leads us to represent a nucleon (proton or neutron) as

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with} \quad p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.11}$$

We introduce isospin $\vec{I}$ in analogy with spin, where $\vec{I}$ is a vector in abstract space (isospin space). In ket notation the two states are $p = |1/2\,1/2\rangle$ and $n = |1/2\;-1/2\rangle$.

The strong interactions are invariant under rotations in isospin space. This is an "internal" symmetry. The operation is a rotation between different particles. Noether's theorem says that isospin is conserved in all strong interactions.

The strong interactions are invariant under an internal symmetry group SU(2), and nucleons belong to the 2-D representation (isospin 1/2).

To each multiplet we assign a particular isospin I. To each member of the multiplet we assign an I_3. For pions, $I = 1$: $\pi^+ = |1\;1\rangle$, $\pi^0 = |1\;0\rangle$, $\pi^- = |1\;-1\rangle$. For Λ, $I = 0$: $\Lambda = |0\;0\rangle$. For Δ, $I = 3/2$: $\Delta^{++} = |3/2\;3/2\rangle$, $\Delta^+ = |3/2\;1/2\rangle$, $\Delta^0 = |3/2\;-1/2\rangle$, $\Delta^- = |3/2\;-3/2\rangle$, Thus the multiplicity of a multiplet is $2I+1$, and I_3 is related to the particle charge. For u, d, s quarks only

$$Q = I_3 + 1/2(A+S)\,. \tag{3.12}$$

where I_3 is conserved in EM interactions, A is baryon number, S is strangeness. Q, A, S all conserved in EM interactions. I not conserved in EM interactions. I_3 not conserved in weak interactions.

In the quark model, $u = |1/2\ 1/2\rangle$ and $d = |1/2\ -1/2\rangle$. All other flavours carry isospin 0.

With the descovery of strange baryons SU(3) was proposed as a symmetry group. The fundamental representation could be the quarks u, d, s. Nowadays we needs to expand the representation to SU(6) for quarks. Although iospin is a good symmetry $m_p \approx m_n$, flavour symmetry SU(6) is a poor symmetry.

3.6 Parity

Parity is invariant in strong and EM interactions, but is maximually violated in weak interactions.

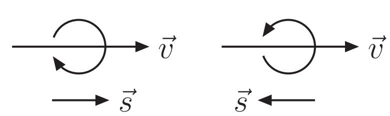


Figure 3.1: Helicity. The left diagram is right-handed (helicity +1, the right diagram is left-handed (helicity −1).

Helicity ($\equiv m_s/s$) is not Lorentz invariant, but if the particle is massless and travels at speed of light, helicity is Lorentz invariant. All ν are left-handed, all $\bar{\nu}$ are right-handed.

For reflection one needs to choose a mirror plane. We can instead consider inversion (reflection followed by rotation). The inversion operator P is called the parity operator. The parity operator causes the following affects:

$$\begin{array}{ll}
\hat{P}(\vec{a}) = -\vec{a} & \text{vector (polar vector),} \\
\hat{P}(\vec{c} = \vec{a}\times\vec{b}) = \vec{c} & \text{pseudovector (axial vector),} \\
\hat{P}(a^2) = \hat{P}(\vec{a}\cdot\vec{a}) = a^2 & \text{scalar,} \\
\hat{P}(\vec{a}\cdot\vec{c}) = -\vec{a}\cdot\vec{c} & \text{pseudoscalar.}
\end{array}$$

By definition $\hat{P}\hat{P} = \hat{P}^2 = I$. $\hat{P}$ and $\hat{I}$ for the two element parity group. Thus the eigenvalues of $\hat{P}$ are ± 1. Scalar and pseudo-vectors have eigenvalues +1, while vectors and pseudo-scalars have eigenvalues −1.

Hadrons are eigenstates of $\hat{P}$. q has intrinsic parity $P = +1$, while $\bar{q}$ has intrinsic parity $P = -1$. For an excited state, there is an extra factor $(-1)^l$ Parity is a multiplicative quantum number.

An examples: mesons

$$\begin{aligned}
&l = 0,\ s = 0 \text{ total, } J^P = 0^-, \text{ eg. } \pi, K, \eta, \eta', \\
&l = 0,\ s = 1 \text{ total, } J^P = 1^-, \text{ eg. } \rho, K^*, \omega, \phi, \\
&l = 1,\ s = 0 \text{ total, } J^P = 1^+, \\
&l = 1,\ s = 1 \text{ total, } J^P = 0^+, 1^+, 2^+.
\end{aligned}$$

3.7 Charge Conjugation

Charge conjugation converts a particle into its antiparticle $C|p\rangle = |\bar{p}\rangle$. C changes the sign of all internal quantum numbers: Q, A, L, S, C, B, T while leaving $M, E, \vec{p}, \vec{s}$ untouched. By definition $C^2 = I \Rightarrow$ eigenvalues ± 1.

For eigenstate, $C|p\rangle = \pm|p\rangle = |\bar{p}\rangle \Rightarrow |p\rangle$ and $|\bar{p}\rangle$ are the same physical state. Only particles which are thier own antiparticles can be eigenstates of C (intrinsic C partity), for example $\gamma, \pi^0, \eta, \eta', \rho^0, \phi, \omega, \psi, \ldots$. For the photon $C = -1$. For a system, eigenvalues $(-1)^{l+s}$.

Examples: mesons

$$\begin{aligned}
&l = 0,\ s = 0 \text{ total, } J^{PC} = 0^{-+}, \text{ eg. } \pi^0, \eta, \eta', \\
&l = 0,\ s = 1 \text{ total, } J^{PC} = 1^{--}, \text{ eg. } \rho, \omega, \phi, \\
&l = 1,\ s = 0 \text{ total, } J^{PC} = 1^{+-}, \\
&l = 1,\ s = 1 \text{ total, } J^{PC} = 0^{++}, 1^{++}, 2^{++}.
\end{aligned}$$

C is not a valid quantum number for the entire supermultiplet, but only for the centeral member.

C is a multiplicative quantum number. $\hat{C}$ is not a symmetry of the weak interactions. Because so few particles are eigenstates of C, its direct application is limited. Combining C with isospin rotation gives G-parity which is more useful.

3.8 CP

Although C and P are individually violated in weak interactions, often the combined operation of CP is a good symmetry.

3.8.1 Neutral Kaons

K^0 ($S = +1$) and $\bar{K}^0$ ($S = -1$) are eigenstates of strangeness produced in strong interactions.

K^0 can change to $\bar{K}^0$ through a second-order weak interaction represented by the box diagrams shown in fig. 3.2. There are other higher-mass states that could also do this: $D^0 \leftrightarrow \bar{D}^0$, $B^0 \leftrightarrow \bar{B}^0$, and $B_s^0 \leftrightarrow \bar{B}_s^0$.

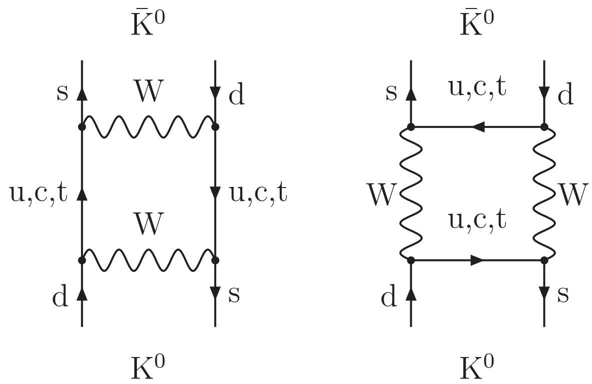


Figure 3.2: Neutral kaon box diagrams.

As as a result the K^0 and $\bar{K}^0$ are not the observed particles. The observed particles must be a linear combination of these strangeness eigenstates. We can form eigenstates of CP. Since the K's are pusedoscalars,

$$P|K^0\rangle = -|K^0\rangle, \quad P|\bar{K}^0\rangle = -|\bar{K}^0\rangle \,. \tag{3.13}$$

By definition, also

$$C|K^0\rangle = |\bar{K}^0\rangle, \quad C|\bar{K}^0\rangle = |K^0\rangle \,. \tag{3.14}$$

Therefore

$$CP|K^0\rangle = -|\bar{K}^0\rangle, \quad CP|\bar{K}^0\rangle = -|K^0\rangle \,. \tag{3.15}$$

Define two eigenstates of CP (K_1 even and K_2 odd):

$$CP|K_1\rangle = |K_1\rangle, \quad CP|K_2\rangle = -|K_2\rangle \,. \tag{3.16}$$

Therefore

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle, \quad |K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle \,. \tag{3.17}$$

Assuming CP is conserved in weak interactions, K_1 will decays to $CP = +1$ states and K_2 to $CP = -1$ states. Typically, neutral kaons decay to two or three pions:

$$\begin{array}{ll} 2\pi \text{ state}, & P = +1 \text{ and} \\ 3\pi \text{ state}, & P = -1. \end{array}$$

Both two and three pion states have $C = +1$.

Experimentally,

$$K_1 \to 2\pi, \quad \tau_1 = 0.9 \times 10^{-10}\ \text{s}, \quad l_1 \sim 3 \times 10^{-2}\ \text{m},$$
$$K_2 \to 3\pi, \quad \tau_2 = 5.2 \times 10^{-8}\ \text{s}, \quad l_2 \sim 2 \times 10^{1}\ \text{m}.$$

The 2π decay is faster (τ_1 "shorter") because more energy is released in the decay. If we start with a $|K^0\rangle = (1/\sqrt{2})(|K_1\rangle + |K_2\rangle)$ beam, the K_1 component will quickly decay and we will be left with a pure K_2 beam.

Experimentally a small mass difference is observed

$$m_2 - m_1 = 3.5 \times 10^{-6}\ \text{eV}. \tag{3.18}$$

To summarize

- $K^0, \bar{K}^0$ are eigenstates of strangeness produced in strong interactions.
- $K^0, \bar{K}^0$ are antiparticles of each other.
- $K^0, \bar{K}^0$ decay by weak interactions.
- K^0 can change to its antiparticle $\bar{K}^0$ by 2'nd order weak interaction (box diagram). What we observe in lab decays form linear combinations of K^0 and $\bar{K}^0$.
- K_1, K_2 are linear combination of $K^0, \bar{K}^0$, CP eigenstates.
- K_1, K_2 are each their own antiparticles.

3.8.2 CP Violation

Are the CP eigenstates also eigenstates of the weak interaction? After creating a long-lived kaon beam there are still some two pion decays observed. Thus, the long-lived neutral K is not a perfect eigenstate of CP. Form

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_2\rangle + \epsilon|K_1\rangle)\,, \tag{3.19}$$

where $\epsilon = 2.3 \times 10^{-3}$ measures the departure from CP invariance. This is a small CP violation, although P is maximumly violated.

Mixing has also been observed in the $D^0/\bar{D}^0$ and $B_s^0/\bar{B}_s^0$ systems. Until very recently, CP violation was only seen in kaon system, but now it is observed in the $B^0/\bar{B}^0$ system. CP violation is incorporated into the Standard Model via a phase in KM matrix.

CP violation permits an unequal treatment of particles and antiparticles. It could be responsible for the dominance of matter over antimatter in the universe. Allows definition of charge and thus a mechanism for us to tell if another universe is matter or antimatter.

3.9 Time Reversal and CTP Theorem

The laws governing collisions work just as well forward as backward in time. The initial conditions may give a clue to the arrow of time. The principle of detailed balance follows from time-reversal invariance. All particles are eigenstates of P, some particles are eigenstates of C, but no particles are eigenstates of T.

T is hard to measure, eg. $n + p \to d + \gamma$ is the same as the reverse $d + \gamma \to n + p$. Thus T invariance has been tested in strong and EM interactions. However, T invariance is hard to test in weak interactions, eg. $\Lambda \to p^+ + \pi^+$ is a weak decay but the reverse $p^+ + \pi^+ \to \Lambda$ is going to be masked by the strong interaction $p^+ + \pi^+ \to \Lambda + X$.

We must measure quantities which should be exactly zero if T is perfect symmetry. A non-zero elementary particle dipole moment (of neutron) would be evidence of T violation. $\vec{s}$ spin direction, $\vec{d}$ (vector) dipole moment points along axis of $\vec{s}$ (pseudovector). $\vec{s}$ changes sign under T reversal but $\vec{d}$ does not $\Rightarrow$ $\vec{d} \neq 0 \Rightarrow T$ violation. $|\vec{d}| < e \cdot (6 \times 10^{-25}$ cm) [Ramsey 1982].

It is believed that T is not a perfect symmetry but CTP is. This is the result of QFT using very general assumptions (Lorentz invariance, quantum mechanics, interactions represented by fields). CTP invarinace $\Rightarrow$ particle and antiparticle have same mass and lifetime.

$$\Delta m = \frac{|m_{K^0} - m_{\bar{K}^0}|}{m_{K^0}} < 6 \times 10^{-19}\,. \tag{3.20}$$

Chapter 4

Feynman Calculus

In this chapter we will learn how to calculate lifetimes and cross sections. In particular, we will learn the following steps:

1. Calculate the amplitude $\mathcal{M}$ using Feynman diagrams.

2. Insert $\mathcal{M}$ into Fermi's Golden Rule.

4.1 Lifetimes and Cross Sections

There are several probes of elementary particle interactions: bound state (Schorödinger), and decays and scattering (relativistic).

4.1.1 Decay and Lifetime

For decays, we will work in the rest frame of the particle and assume it is also the laboratory frame. Not all particles of a given type live the same length of time. We will calculate the average or mean lifetime τ. We define the decay rate Γ to be the probability per unit time that a particle will disintegrate. $N(t)$ is the number of particles at time t. Then $N\Gamma dt$ of them would decay in the next instant dt. Thus the decrease in the number of particles is $dN = -\Gamma N dt$. Solving this differential equation gives

$$N(t) = N(0)e^{-\Gamma t}\,, \tag{4.1}$$

where the mean lifetime is $\tau = 1/\Gamma$.

If a particle can decay be several channels

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i \tag{4.2}$$

and $\tau = \frac{1}{\Gamma_{tot}}$. Γ_i/Γ_{tot} is called the branching ratio for ith decay rate. For decays, we need to determine Γ_i and the rest is simple.

4.1.2 Cross Section

The cross section is the effective area of the target seen by the beam in an interaction. The cross section depends on the beam and the target.

$$\sigma_{tot} = \sum_{i=1}^{n} \sigma_i \,, \tag{4.3}$$

where σ_i is the cross section when measuring outgoing state i. σ_{tot} is an inclusive cross section, while σ_i is an exclusive cross section.

Normally the cross section is inversely proportional to the beam velocity $1/v$. Often we plot $\sigma(E)$ versus E and look for resonances in the cross section.

The event rate is the cross section times luminosity $\mathcal{L}$,

$$dN = \mathcal{L} d\sigma \,, \tag{4.4}$$

where dN is the number of particles per unit time, $\mathcal{L}$ is the number of particles passing per unit time per unit area, and $d\sigma$ is the area.

The number of particles per unit time scattered into solid angle $d\Omega$ is

$$dN = \mathcal{L} d\sigma = \mathcal{L} D(\theta) d\Omega \,, \tag{4.5}$$

$$\frac{d\sigma}{d\Omega} = D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega} \,. \tag{4.6}$$

4.2 Golden Rule

Two ingredients are necessary in calculating decay:

1. The amplitude (or matrix element) $\mathcal{M}$. This contains the dynamical information (physics) and is calculated using Feynman diagrams (rules).

2. The phase space. This depends on the masses, energies, and momenta.

Fermi's Golden Rule for decays is

$$\text{transition rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phasespace}) \,. \tag{4.7}$$

4.2.1 Golden Rule for Decays

Consider the decay $1 \to 2 + 3 + 4 + \ldots + n$. The rate is

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c d^3 \vec{p_2}}{(2\pi)^3 2E_2} \right) \left(\frac{c d^3 \vec{p_3}}{(2\pi)^3 2E_3} \right) \ldots \left(\frac{c d^3 \vec{p_n}}{(2\pi)^3 2E_n} \right) \right] (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 \ldots - p_n) \tag{4.8}$$

where $p_i = (E_i/c, \vec{p}_i), E_i^2 - (\vec{p}_i c)^2 = m_i^2 c^4 \Rightarrow E_i$ function of $\vec{p}_i c$ (E_i short hand). The delta function δ cause the energy-momentum to be conserved in the decay. For a decay at rest $p_1 = (m_1, 0)$. S is the product of statistics factors $1/j!$, group of j identical particles in final state.

This is the differential rate, normally we integrate over all the outgoing momenta. For two final state particles $1 \to 2+3$,

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \frac{c d^3\vec{p}_2}{(2\pi)^3 2E_2} \frac{c d^3\vec{p}_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^4(p_1 - p_2 - p_3) \,, \tag{4.9}$$

$$\Gamma = \frac{S}{\hbar m_1} \left(\frac{c}{4\pi}\right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{E_2 E_3} \delta^4(p_1 - p_2 - p_3) d^3\vec{p}_2 d^3\vec{p}_3 \,, \tag{4.10}$$

where $\mathcal{M} = \mathcal{M}(\vec{p}_2, \vec{p}_3)$

Example, $\pi^0 \to \gamma\gamma$.

$E_1 = mc^2$, $\vec{p} = 0$, $m_2 = m_3 = 0$

$$\begin{aligned} \delta^4(p_1 - p_2 - p_3) &= \delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right) \delta^3(0 - \vec{p}_2 - \vec{p}_3) && (4.11) \\ &= \delta(mc - |\vec{p}_2| - |\vec{p}_3|) \delta^3(\vec{p}_2 + \vec{p}_3) \,. && (4.12) \end{aligned}$$

$$\begin{aligned} \Gamma &= \frac{S}{\hbar m} \left(\frac{1}{4\pi}\right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{|\vec{p}_2||\vec{p}_3|} \delta(mc - |\vec{p}_2| - |\vec{p}_3|) \delta^3(\vec{p}_2 + \vec{p}_3) d^3\vec{p}_2 d^3\vec{p}_3 && (4.13) \\ &= \frac{S}{\hbar m} \left(\frac{1}{4\pi}\right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2} \delta(mc - 2|\vec{p}_2|) d^3\vec{p}_2 \,. && (4.14) \end{aligned}$$

Now $\mathcal{M} = \mathcal{M}(\vec{p}_2)$ only, actually $\mathcal{M} = \mathcal{M}(|\vec{p}_2|)$ ($\mathcal{M}$ must be a scalar). Since $d^3\vec{p}_2 = |\vec{p}_2|^2 d|\vec{p}_2| \sin\theta d\theta d\phi$ and $\int \sin\theta d\theta d\phi = 4\pi$,

$$\Gamma = \frac{S}{8\pi\hbar m} \int_0^\infty |\mathcal{M}|^2 \delta(mc - 2|\vec{p}_2|) d|\vec{p}_2| \,. \tag{4.15}$$

Using $\delta(mc - 2|\vec{p}_2|) = \delta(g(x)), g(x) = mc - 2x,\ g(x_1) = 0 = mc - 2x_1 \Rightarrow x_1 = mc/2$, $g'(x) = -2\ g'(x_1) = -2$,

$$\delta(g(x)) = \frac{1}{|g'(x_1)|} \delta(x - x_1) \,. \tag{4.16}$$

Therefore

$$\delta(mc - 2|\vec{p}_2|) = \frac{1}{2} \delta\left(|\vec{p}_2| - \frac{mc}{2}\right) \tag{4.17}$$

and

$$\begin{aligned}\Gamma &= \frac{S}{16\pi\hbar m}\int_0^\infty |\mathcal{M}|^2\delta\left(|\vec{p}_2| - \frac{mc}{2}\right) d|\vec{p}_2| \\ &= \frac{S}{16\pi\hbar m}|\mathcal{M}|^2\,, \end{aligned} \tag{4.18}$$

where $\vec{p}_3 = -\vec{p}_2$, $|\vec{p}_2| = mc/2$. $S = 1/2$ for two identical photons in the final state.

Could determine without knowing $\mathcal{M}$. For three-body decay must insert $\mathcal{M}$ before calculation.

4.2.2 Golden Rule for Scattering

Consider the scattering process $1 + 2 \to 3 + 4 + \ldots + n$. The cross section is

$$\begin{aligned} d\sigma &= |\mathcal{M}|^2 \frac{\hbar^2 S}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2 c^2)}}\left[\left(\frac{cd^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{cd^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\cdots\left(\frac{cd^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right] \\ &\quad\times(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4 \ldots - p_n)\,. \end{aligned} \tag{4.19}$$

In a typical situation we integrate over everything except for the angles of particle three to obtain the differential cross section $d\sigma/d\Omega$

Example: two-body scattering in CMS $1 + 2 \to 3 + 4$.

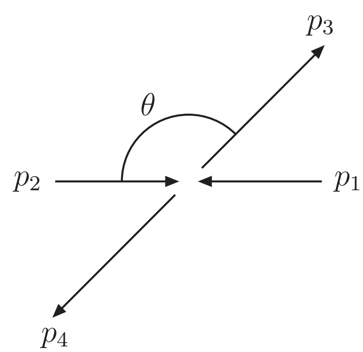


Figure 4.1: Definition of the kinematics and scattering angle for $1 + 2 \to 3 + 4$ in the center-of-mass frame.

Kinematices gives

$\vec{p}_2 = -\vec{p}_1 \Rightarrow p_1 \cdot p_2 = E_1E_2/c^2 - \vec{p}_1 \cdot \vec{p}_2 = E_1E_2/c^2 + |\vec{p}_1|^2$. After some algebra $\sqrt{(p_1\cdot p_2)^2 - (m_1m_2c^2)^2} = (E_1 + E_2)|\vec{p}_1|/c$.

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1 + E_2)|\vec{p}_1|}\frac{d^3\vec{p}_3 d^3\vec{p}_4}{E_3E_4}\delta^4(p_1 + p_2 - p_3 - p_4)\,, \tag{4.20}$$

where

$$\delta^4(p_1+p_2-p_3-p_4) = \delta\left(\frac{E_1}{c}+\frac{E_2}{c}-\frac{E_3}{c}-\frac{E_4}{c}\right)\delta^3(-\vec{p}_3-\vec{p}_4)\,. \tag{4.21}$$

Integrating over $\vec{p}_4 \Rightarrow \vec{p}_4 \to -\vec{p}_3$ gives

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1+E_2)|\vec{p}_1|}\frac{\delta(E_1/c+E_2/c-\sqrt{m_3^2c^2+\vec{p}_3^2}\sqrt{m_4^2c^2+\vec{p}_3^2})}{\sqrt{m_3^2c^2+\vec{p}_3^2}\sqrt{m_4^2c^2+\vec{p}_3^2}}d^3\vec{p}_3\,. \tag{4.22}$$

Writing $d^3\vec{p}_3 = p^2dpd\Omega$, where $p=|\vec{p}_3|$ and $d\Omega = \sin\theta d\theta d\phi$ gives

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_1|}\int_0^\infty |\mathcal{M}|^2\frac{\delta(E_1/c+E_2/c-\sqrt{m_3^2c^2+p^2}-\sqrt{m_4^2c^2+p^2})p^2dp}{\sqrt{m_3^2c^2+p^2}\sqrt{m_4^2c^2+p^2}}\,. \tag{4.23}$$

Let $a = E_1/c+E_2/c$ and $u=\sqrt{m_3^2c^2+p^2}+\sqrt{m_4^2c^2+p^2}$.

$$\begin{aligned}
\frac{du}{dp} &= \frac{1}{2}(m_3^2c^2+p^2)^{-1/2}2p+\frac{1}{2}(m_4^2c^2+p^2)^{-1/2}2p \\
&= \frac{p}{\sqrt{m_3^2c^2+p^2}}+\frac{p}{\sqrt{m_4^2c^2+p^2}} \\
&= \frac{p\sqrt{m_4^2c^2+p^2}}{\sqrt{(m_3^2c^2+p^2)(m_4^2c^2+p)}}+\frac{p\sqrt{m_3^2c^2+p^2}}{\sqrt{(m_3^2c^2+p^2)(m_4^2c^2+p^2)}} \\
&= \frac{pu}{\sqrt{(m_3^2c^2+p^2)(m_4^2c^2+p^2)}}\,.
\end{aligned} \tag{4.24}$$

Therefore

$$\delta(a-u)\frac{du}{dp}\frac{p^2}{pu}dp = \delta(a-u)\frac{p}{u}du \to \frac{p}{a}\,. \tag{4.25}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2\frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}\,, \tag{4.26}$$

where $|\vec{p}_i| \equiv |\vec{p}_1| = |\vec{p}_2|$ and $|\vec{p}_f| \equiv |\vec{p}_2| = |\vec{p}_3|$.

We have carried the calculation through without knowing $\mathcal{M}$. Lifetimes have units of time (s). So decay rates have units of inverse time (s^{-1}). A cross sections has dimensions of area cm^2 (1 b $= 10^{-24}$ cm^2). $\mathcal{M}$ has units which depend on the number of particles involved.

Chapter 5

Quantum Electrodynamics

The wave equations used in physics are

- Schrödinger equation - nonrelativistic.
- Klein-Gordon equation - relativistic spin-0.
- Dirac equation - relativistic spin-1/2.
- Proca equation - relativistic spin-1.

In this chapter we will be mainly concerned with the spin-1/2 electron and thus work with the Dirac equation.

5.1 Dirac Equation

Consider the clasical (nonrelativistic) energy-momentum relation

$$\frac{\vec{p}^2}{2m} + V = E\,. \tag{5.1}$$

Substituting the quantum mechanical operators

$$\vec{p} \to \frac{\hbar}{i}\vec{\nabla}, \quad E \to i\hbar\frac{\partial}{\partial t} \tag{5.2}$$

and act on a wave function ψ gives

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}\,, \tag{5.3}$$

which is the Schrödinger equation.

Starting with the relativistic energy-momentum relation $E^2 - \vec{p}^2c^2 = m^2c^4$ or $p^2 - m^2c^2 = 0$, we can proceed in an analogous fashion. The four component version of the operator is

$$p_\mu \rightarrow i\hbar\frac{\partial}{\partial x^\mu} = i\hbar\partial_\mu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) , \tag{5.4}$$

which is a covariant four-vector $(E/c, -\vec{p})$. Operating on a wavefunction gives

$$\begin{aligned}
-\hbar^2\partial^\mu\partial_\mu\psi - m^2c^2\psi &= 0 && (5.5)\\
\Box\psi + \left(\frac{mc}{\hbar}\right)^2\psi &= 0 && (5.6)\\
-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi &= \left(\frac{mc}{\hbar}\right)^2\psi , && (5.7)
\end{aligned}$$

which is the Klein-Gordon equation.

There are some apparent problems with the Klein-Gordon equation:

1. The interpretation of $|\psi|^2$ as a probability and
2. negative-energy solutions.

Both these problems are linked to the second-order derivative in t. These are not really problems when we consider that the number of particles is not conserved. We must allow for pair production and annihilation.

We now look for an equation with all first-order derivatives. Factorizing the energy-momentum relation gives

$$p^\mu p_\mu - m^2c^2 = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0 , \tag{5.8}$$

where β^κ and γ^λ are eight coefficients to be determined. Expanding gives

$$\beta^\kappa\gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa - \gamma^\lambda)p_\kappa - m^2c^2 = 0 . \tag{5.9}$$

Therefore $\beta^\kappa = \gamma^\kappa$ and $p^\mu p_\mu = \gamma^\kappa\gamma^\lambda p_\kappa p_\lambda$. Summing over Einstien indices gives

$$\begin{aligned}
(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = \; & (\gamma^0)^2(p^0)^2 + (\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 + (\gamma^3)^2(p^3)^2 \\
& + (\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)p_0p_2 \\
& + (\gamma^0\gamma^3 + \gamma^3\gamma^0)p_0p_3 + (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2 \\
& + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3 . \qquad (5.10)
\end{aligned}$$

This is not possible if γ^κ are c-numbers. We choose the γ's to be matrices with the conditions $(\gamma^0)^2 = 1, (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1, \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \mu \neq \nu$ or more compactly

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} . \tag{5.11}$$

The curly brackets represents the anticommutator defined as $\{A, B\} \equiv AB + BA$.

The γ are 4×4 matrices (not unique set)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix} . \tag{5.12}$$

We have picked a representation that is block 2×2 matrices. $\sigma^i = \sigma_i$ since they are not the space components of a four-vector.

Now $p^\mu p_\mu - m^2c^2 = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$. We first examine the case $\gamma^\mu p_\mu - mc = 0$. Substituting $p_\mu \to i\hbar\partial_\mu$ and act on a wave function gives the Dirac equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0\,, \tag{5.13}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{5.14}$$

is bi-spinor or Dirac spinor. ψ is not a four-vector, but does transform a certain way under a Lorentz transformation.

5.2 Solutions to Dirac Equation

The simplist solution to the Dirac equation is for a particle at rest. In tis case, $\vec{p} = 0$ and thus

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0\,. \tag{5.15}$$

Which says that the wave function is uniform over all space. The Dirac equation becomes

$$i\hbar\gamma^0\partial_0\psi - mc\psi = 0 \quad \text{or} \quad \frac{i\hbar}{c}\gamma^0\frac{\partial\psi}{\partial t} - mc\psi = 0\,. \tag{5.16}$$

In block two-component form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = -i\frac{mc^2}{\hbar}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} , \tag{5.17}$$

where

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} . \tag{5.18}$$

In two-component form

$$\frac{\partial\psi_A}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_A \quad \text{and} \quad -\frac{\partial\psi_B}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_B\,. \tag{5.19}$$

The solutions are

$$\psi_A(t) = \exp\left[-i\left(\frac{mc^2}{\hbar}\right)t\right]\psi_A(0) \quad \text{and} \quad \psi_B(t) = \exp\left[+i\left(\frac{mc^2}{\hbar}\right)t\right]\psi_B(0)\,. \tag{5.20}$$

Since $E = mc^2$ at rest, $e^{-iEt/\hbar}$ is the time dependence of a quantum state with energy E. ψ_B is a negative energy state $E = -mc^2$. We can not throw away the negative-energy state as unphysical since we need a complete set of states in quantum mechanics. We will interpret the negative-energy solution as an antiparticle with positive energy.

Each solution is a two-component spinor and thus represents a spin-1/2 particle. ψ_A is the solution for a spin-1/2 particle (electron); ψ_B is the solution for a spin-1/2 antiparticle (positron).

$$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{5.21}$$

$$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{5.22}$$

We now look for plane-wave solutions of the form

$$\psi(\vec{r}, t) = ae^{-(i/\hbar)(Et-\vec{p}\cdot\vec{r})}u(E, \vec{p}) = ae^{-(i/\hbar)(p\cdot x)}u(p)\,, \tag{5.23}$$

where $u(p)$ is a bi-spinor, and E and $\vec{p}$ are four arbitrary parameters. Operating on the wavefunction gives

$$\partial_\mu\psi = -\frac{i}{\hbar}p_\mu ae^{-(i/\hbar)p\cdot x}u \tag{5.24}$$

$$\begin{aligned} \gamma^\mu p_\mu ae^{-(i/\hbar)p\cdot x}u - mcae^{-(i/\hbar)p\cdot x}u &= 0 \\ (\gamma^\mu p_\mu - mc)u &= 0\,. \end{aligned} \tag{5.25}$$

This is the momentum space representation of the Dirac equation.

We may split this up into two coupled equations.

$$\gamma^\mu p_\mu = \gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} = \frac{E}{c}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix} = \begin{pmatrix}E/c & -\vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -E/c\end{pmatrix}. \tag{5.26}$$

$$(\gamma^\mu p_\mu - mc)u = \begin{pmatrix}\left(\frac{E}{c} - mc\right) & -\vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & \left(-\frac{E}{c} - mc\right)\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = 0 \tag{5.27}$$

$$(\gamma^\mu p_\mu - mc)u = \begin{pmatrix} \left(\frac{E}{c} - mc\right) u_A & -\vec{p}\cdot\vec{\sigma}u_B \\ \vec{p}\cdot\vec{\sigma}u_A & -\left(\frac{E}{c} + mc\right) u_B \end{pmatrix} = 0\,. \tag{5.28}$$

$$\left(\frac{E}{c} - mc\right) u_A - \vec{p}\cdot\vec{\sigma}u_B = 0, \quad \text{and} \quad \vec{p}\cdot\vec{\sigma}u_A - \left(\frac{E}{c} + mc\right) u_B = 0\,. \tag{5.29}$$

$$u_A = \frac{c}{E - mc^2}(\vec{p}\cdot\vec{\sigma})u_B \quad \text{and} \quad u_B = \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\sigma})u_A\,. \tag{5.30}$$

$$u_A = \left(\frac{c}{E - mc^2}\right)\left(\frac{c}{E + mc^2}\right)(\vec{p}\cdot\vec{\sigma})^2 u_A = \frac{c^2}{E^2 - m^2c^4}(\vec{p}\cdot\vec{\sigma})^2 u_A\,. \tag{5.31}$$

$$\vec{p}\cdot\vec{\sigma} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.32}$$

$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix} = \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix}, \tag{5.33}$$

$$(\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix} = \begin{pmatrix} p_z^2 + p_-p_+ & p_zp_- - p_-p_z \\ p_+p_z - p_zp_+ & p_+p_- + p_z^2 \end{pmatrix} \tag{5.34}$$

$$= \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} = \begin{pmatrix} \vec{p}^2 & 0 \\ 0 & \vec{p}^2 \end{pmatrix} = \vec{p}^2\,. \tag{5.35}$$

where we have used $p_- = p_+^*$ and $p_-p_+ = p_x^2 + p_y^2$.

$$u_A = \frac{\vec{p}^2c^2}{E^2 - m^2c^4}u_A \Rightarrow E^2 - m^2c^4 = \vec{p}^2c^2 \tag{5.36}$$

enforces the energy-momentum relation $E = \pm\sqrt{m^2c^4 + \vec{p}^2c^2}$. The positive root is the particle, while the negative root is the antiparticle.

There are four independent solutions

$$\text{pick} \quad u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_B = \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E + mc^2}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \tag{5.37}$$

$$\text{pick} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_B = \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E + mc^2}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \tag{5.38}$$

$$\text{pick} \quad u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_A = \frac{c}{E - mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E - mc^2}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \tag{5.39}$$

$$\text{pick} \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_A = \frac{c}{E - mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E - mc^2}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \tag{5.40}$$

The normalization is defined as $u^\dagger u = 2|E|/c$, where $u^\dagger$ is the Hermitian conjugate of u. In four-component form

$$u = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \Rightarrow u^\dagger = (\alpha^* \beta^* \gamma^* \delta^*)\,. \tag{5.41}$$

and $u^\dagger u = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2$. The four-component spinor solutions are

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ cp_z/(E+mc^2) \\ cp_+/(E+mc^2) \end{pmatrix} \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ cp_-/(E+mc^2) \\ -cp_z/(E+mc^2) \end{pmatrix}, \tag{5.42}$$

with $E = \sqrt{m^2c^4 + \vec{p}^2c^2}$.

$$u^{(3)} = N \begin{pmatrix} cp_z/(E-mc^2) \\ cp_+/(E-mc^2) \\ 1 \\ 0 \end{pmatrix} \quad u^{(4)} = N \begin{pmatrix} cp_-/(E-mc^2) \\ -cp_z/(E-mc^2) \\ 0 \\ 1 \end{pmatrix}, \tag{5.43}$$

with $E = -\sqrt{m^2c^4 + \vec{p}^2c^2}$, where $N = \sqrt{(|E| + mc^2)/c}$.

These solutions are not eigenstates of spin. The generalized spin matrix is

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma}\,, \tag{5.44}$$

with

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{5.45}$$

However, these solutions are eigenstates of helicity if $p_x = p_y = 0$ (spin along direction of motion). $u^{(1)}, u^{(3)}$ are spin up, $u^{(2)}, u^{(4)}$ are spin down. E in $u^{(3)}$ and $u^{(4)}$ cannot represent positive energy. All free particles carry positive energy. The negative-energy solution must be reinterpreted as a positive-energy antiparticle.

To express these solutions in terms of physical energy and momentum of the positron, we flip the sign of E and $\vec{p}$:

$$\psi(\vec{r}, t) = e^{(i/\hbar)(Et - \vec{p}\cdot\vec{r})} u(-E, -\vec{p})) \tag{5.46}$$

for solution (3) and (4).

We use v for the antiparticle expressed in terms of physical energy and momentum. We have

$$v^{(1)}(E,\vec{p}) \;=\; u^{(4)}(-E,-\vec{p}) = N\begin{pmatrix} cp_-/(E+mc^2) \\ -cp_z/(E+mc^2) \\ 0 \\ 1 \end{pmatrix} \tag{5.47}$$

$$v^{(2)}(E,\vec{p}) \;=\; -u^{(3)}(-E,-\vec{p}) = -N\begin{pmatrix} cp_z/(E+mc^2) \\ cp_+/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}, \tag{5.48}$$

with $E = \sqrt{m^2c^4 + \vec{p}^2c^2}$.

$u^{(1)}$ and $v^{(1)}$ are particle-antiparticle pairs under charge conjugation (as are $u^{(2)}$ and $v^{(2)}$). $u^{(1)}$ and $u^{(2)}$ represent two spin states of the electron of energy E and momentum $\vec{p}$; $(\gamma^\mu p_\mu - mc)u = 0$. $v^{(1)}$ and $v^{(2)}$ represent two spin states of the positron of energy E and momentum $\vec{p}$; $(\gamma^\mu p_\mu + mc)v = 0$.

Plane waves are not the only solution, but are eigenstates of E and $\vec{p}$ which is what we measure.

5.3 Bilinear Covariants

Under a Lorentz transformation, $\psi \to \psi' = S\psi$ (along x-axis). S is 4×4 matrix $S = a_+ + a_-\gamma^0\gamma^1$ and $a_\pm = \pm\sqrt{1/2(\gamma \pm 1)}$. In two-component form

$$\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \Rightarrow S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}. \tag{5.49}$$

We can construct bilinears of ψ that transform as a scalar, pseudo scalar, vector, pseudo vector, etc.

Scalar: $\psi^\dagger\psi$ does not transform as scalar

$$(\psi^\dagger\psi)' = (\psi')^\dagger\psi' = (S\psi)^\dagger S\psi = \psi S^\dagger S\psi \neq \psi^\dagger\psi\,. \tag{5.50}$$

$$S^\dagger S = S^2 = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} = \begin{pmatrix} a_+^2 + a_-^2 & 2a_-a_+\sigma_1 \\ 2a_-a_+\sigma_1 & a_+^2 + a_-^2 \end{pmatrix}. \tag{5.51}$$

$$a_+^2 + a_-^2 = 1/2(\gamma+1) + 1/2(\gamma-1) = \gamma\,. \tag{5.52}$$

$$2a_-a_+ = -2\sqrt{1/4(\gamma-1)(\gamma+1)} = -\sqrt{\gamma^2-1}\,. \tag{5.53}$$

$$\gamma = (1-\beta^2)^{-1/2}, \gamma^2 = \frac{1}{1-\beta^2}, \gamma^2 - 1 = \frac{1-1+\beta^2}{1-\beta^2} = \frac{\beta^2}{1-\beta^2} \tag{5.54}$$

$$-\sqrt{\gamma^2-1} = -\gamma\beta^2\,. \tag{5.55}$$

Therefore

$$S^2 = \begin{pmatrix} \gamma & -\gamma\beta^2\sigma_1 \\ -\gamma\beta^2\sigma_1 & \gamma \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta^2\sigma_1 \\ -\beta^2\sigma_1 & 1 \end{pmatrix} \neq 1\,. \tag{5.56}$$

We introduce the adjoint spinor $\bar{\psi} \equiv \psi^\dagger\gamma^0 = (\psi_1^*\psi_2^* - \psi_3^* - \psi_4^*)$. Now
Scalar: $\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$
$(\bar{\psi}\psi)' = (\psi^\dagger\gamma^0\psi)' = (\psi')^\dagger\gamma^{0\dagger}\psi' = (S\psi)^\dagger\gamma^{0\dagger}S\psi = \psi^\dagger S^\dagger\gamma^{0\dagger}S\psi = \bar{\psi}\psi$

$$\begin{aligned} S^\dagger\gamma^{0\dagger}S = S^\dagger\gamma^0 S &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \\ &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}\begin{pmatrix} a_+ & a_-\sigma_1 \\ -a_-\sigma_1 & -a_+ \end{pmatrix} \\ &= \begin{pmatrix} a_+^2 - a_-^2 & 0 \\ 0 & a_-^2 - a_+^2 \end{pmatrix} \\ &= (a_+^2 - a_-^2)\gamma^0\,. \end{aligned} \tag{5.57}$$

$$a_+^2 - a_-^2 = 1/2(\gamma+1) - 1/2(\gamma-1) = 1\,. \tag{5.58}$$

Therefore $S^\dagger\gamma^0 S = \gamma^0$.

For pseudo scalar, how does ψ transform under $\hat{P}$?
$\psi \to \psi' = \gamma^0\psi$
$(\bar{\psi}\psi)' = (\psi')^\dagger\gamma^0\psi' = (\hat{P}\psi)^\dagger\gamma^0\psi' = \psi^\dagger\hat{P}^\dagger\gamma^0\hat{P}\psi = \psi^\dagger\gamma^{0\dagger}\gamma^0\gamma^0\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi \Rightarrow$ true scalar.

Pseudo-scalar: $\bar{\psi}\gamma^5\psi$, where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\begin{aligned} \gamma^5 &= i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \\ &= i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} = i\begin{pmatrix} 0 & -\sigma_1\sigma_2\sigma_3 \\ -\sigma_1\sigma_2\sigma_3 & 0 \end{pmatrix} \\ &= i\begin{pmatrix} 0 & -i\sigma_1\sigma_1 \\ -i\sigma_1\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{5.59}$$

$(\bar{\psi}\gamma^5\psi)' = (\psi')^\dagger\gamma^0\gamma^5\psi' = \psi^\dagger\gamma^0\gamma^0\gamma^5\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\psi$
Now since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\{\gamma^0, \gamma^\mu\} = 0, \mu \neq 0$.

$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5$
Therefore
$(\bar\psi\gamma^5\psi)' = -\psi^\dagger\gamma^0\gamma^5 = -\bar\psi\gamma^5\psi$ pseudo scalar
Also $\{\gamma^\mu, \gamma^5\} = 0$.
In summary

$\bar\psi\psi$	scalar	1 component
$\bar\psi\gamma^5\psi$	pseudo scalar	1 component
$\bar\psi\gamma^\mu\psi$	vector	4 component
$\bar\psi\gamma^\mu\gamma^5\psi$	pseudo vector	4 component
$\bar\psi\sigma^{\mu\nu}\psi$	antisymmetric tensor	6 component

$16 \to (4 \times 4)$ products, where

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,. \tag{5.60}$$

$1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}$ are basis for the space of 4×4 matrices.

5.4 The Photon

We first review some classical electrodynamics. Maxwell's equations are

$$\vec\nabla \cdot \vec E = 4\pi\rho\,, \quad \vec\nabla \cdot \vec B = 0\,, \quad \vec\nabla \times \vec E + \frac{1}{c}\frac{\partial \vec B}{\partial t} = 0\,, \quad \vec\nabla \times \vec B - \frac{1}{c}\frac{\partial \vec E}{\partial t} = \frac{4\pi}{c}\vec J\,. \tag{5.61}$$

In relativistic notation, we combine $\vec E$ and $\vec B$ into an antisymmetric second-rank tensor (field strength tensor).

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -E_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{5.62}$$

where $F^{01} = -E_x, F^{12} = -B_z$, etc.

We also define the current four-vector $J^\mu = (c\rho, \vec J)$.

The inhomogeneous Maxwell's equation becomes

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu\,. \tag{5.63}$$

Since $F^{\mu\nu}$ is antisymmetric, $\partial_\mu J^\nu = \partial \cdot J = 0$.

$$\partial_0 J^0 - \partial_i J^i = \frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \vec\nabla \cdot \vec J = 0\,. \tag{5.64}$$

Therefore

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{5.65}$$

is the continuity equation. It expresses local conservation of charge.

The homogeneous Maxwell's equation becomes

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0\,. \tag{5.66}$$

For homogenous Maxwell's equation, introduce potential vector $\vec{A}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{B} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right) = 0\,. \tag{5.67}$$

$$\vec{E} = -\vec{\nabla} V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,. \tag{5.68}$$

In relativistic notation, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where $A^\mu = (V, \vec{A})$.

The inhomogeneous Maxwell's equation is

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu\,. \tag{5.69}$$

$\vec{E}$ and $\vec{B}$ are physical fields. V and $\vec{A}$ are not unique.

Consider the gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \lambda(x)\,. \tag{5.70}$$

$\partial^\mu A^{\nu'} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$.
We use the gauge freedom to impose a constraint on the potential $\partial_\mu A^\mu = 0$. This give the Lorentz condition.

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu \tag{5.71}$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu = \Box A^\nu = \frac{4\pi}{c} J^\nu \tag{5.72}$$

d'Alembertian operator is defined as

$$\Box \equiv \partial^\mu \partial_\mu = \frac{1}{c}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\,. \tag{5.73}$$

Further gauge transformations are possible provided $\Box\lambda = 0$. There remains a residual ambiguity in A^μ. We have two choices:

1. live with indeterminacy (spurious degrees of freedom) or
2. break Lorentz covariance.

In empty space, $J^\mu = 0$ so we pick $A^0 = 0$. Lorentz condition $\to \vec{\nabla} \cdot \vec{A} = 0$ Coulomb gauge. These two conditions are not Lorentz invariant.

In QED, A^μ is the wave function of the photon. For a free photon, $\Box A^\mu = 0$ is the Klein-Gordon equation for a massless particle. The plane-wave solution is $A^\mu(x) = a\varepsilon^\mu(p)e^{-(i/\hbar)p\cdot x}$, where $\varepsilon^\mu(p)$ is the polarization vector (spin of the photon) $\Rightarrow p^\mu p_\mu = 0 \Rightarrow E = |\vec{p}|c$.

ε^μ has four components but they are not independent. The Lorentz condition gives $p^\mu \varepsilon_\mu = 0$. In the Coulomb gauge (transverse gauge), $\varepsilon^0 = 0, \vec{\varepsilon} \cdot \vec{p} = 0$ transverse polarization.

If $\vec{p} = |\vec{p}|\hat{z}$, two linearly independent polarizations $\varepsilon_{(1)} = (1, 0, 0), \varepsilon_{(2)} = (0, 1, 0)$.

Massive particles have $2s + 1$ spin orientations. Massless particles have only two orientations (regardless of spin), except for $s = 0$ which has only one. For massless particles $m_s = +s$ or $m_s = -s \Rightarrow$ helicity can be ± 1.

5.5 Feynman Rules for QED

The Feynman rules appropriate for a point charge of spin-1/2 and photons are $(s = 1, 2)$

Electron	Positron
$\psi(x) = ae^{-(i/\hbar)p\cdot x}u^{(s)}(p)$	$\psi(x) = ae^{(i/\hbar)p\cdot x}v^{(s)}(p)$
$(\gamma^\mu p_\mu - mc)u = 0$	$(\gamma^\mu p_\mu + mc)v = 0$
$\bar{u} = u^\dagger \gamma^0$	$\bar{v} = v^\dagger \gamma^0$
$\bar{u}(\gamma^\mu p_\mu - mc) = 0$	$\bar{v}(\gamma^\mu p_\mu + mc) = 0$
$\bar{u}^{(1)}u^{(2)} = 0$	$\bar{v}^{(1)}v^{(2)} = 0$
$\bar{u}u = 2mc$	$\bar{v}v = -2mc$
$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc)$	$\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$

Photons

$A^\mu(x) = ae^{-(i/\hbar)p\cdot x}\varepsilon^\mu_{(s)}$

$\varepsilon^\mu p_\mu = 0$

$\varepsilon^{\mu *}_{(1)}\varepsilon_{\mu(2)} = 0$

$\varepsilon^{\mu *}\varepsilon_\mu = 1$

In addition, we have the Colomb gauge $\varepsilon^0 = 0, \vec{\varepsilon} \cdot \vec{p} = 0$ and the completeness relationship

$$\sum_{s=1,2} (\varepsilon_{(s)})_i (\varepsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j \,. \tag{5.74}$$

5.5.1 Rules to calculate amplitude $\mathcal{M}$

1. Notation
   - label incoming/outgoing four-momentum p_i and s_i,
   - label internal momentum q_i,
   - assign arrows

(a) external fermions: electron, positron
(b) internal fermions: direction of flow
(c) external photons: point forward
(d) internal photons: no arrows,

2. External lines: electrons, positrons, photons

3. Vertex factors: for electron $ig_e\gamma^\mu$, $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$,

4. Propagators: each internal line
   $\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$ electrons/positrons
   $-\frac{ig_{\mu\nu}}{q^2}$ photons

5. Conservation of energy and momentum for each vertex: $(2\pi)^4\delta^4(k_1 + k_2 + k_3)$

6. Integrate over internal momentum for each internal momentum q: $\frac{d^4q}{(2\pi)^4}$

7. Cancel the delta function for overall energy-momentum conservation, what remains is $-i\mathcal{M}$,

8. Antisymmetrization - include a relative minus sign between diagrams that differ only in the interchange of two fermions.

5.6 Examples

5.6.1 Electron-Positron Scattering

$e^-e^+ \to \gamma^* \to e^-e^+$
Conservation of energy-momentum is $p_1 = p_3 + q$ and $q + p_2 = p_4$.

$$\begin{aligned}
&\int \bar{u}(3)(ig_e\gamma^\mu)u(1)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}(2)(ig_e\gamma^\nu)v(4) \\
&\cdot(2\pi)^4\delta^4(p_1 - p_3 - q)(2\pi)^4\delta^4(p_2 + q - p_4)\frac{d^4q}{(2\pi)^4} \\
&= ig_e^2\bar{u}(3)\gamma^\mu u(1)\frac{g_{\mu\nu}}{(p_1 - p_3)^2}\bar{v}(2)\gamma^\nu v(4)(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) \qquad (5.75)
\end{aligned}$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)] \qquad (5.76)$$

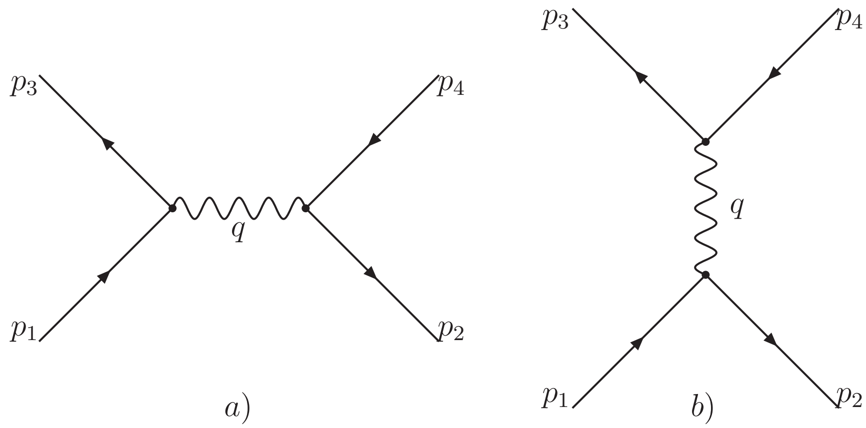


Figure 5.1: Feynman diagrams for electron-positron scattering: a) exchange ($p_1 = p_3 + q$ and $q + p_2 = p_4$) and b) annihilation.

$$(2\pi)^4 \int \bar{u}(3)(ig_e\gamma^\mu)v(4)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}(2)(ig_e\gamma^\nu)u(1)\delta^4(q-p_3-p_4)\delta^4(p_1+p_2-q)d^4q$$

$$= i(2\pi)^4\frac{g_e^2}{(p_1+p_2)^2}[\bar{u}(3)\gamma^\mu v(2)][\bar{v}(2)\gamma_\mu u(1)]\delta^4(p_1+p_2-p_3-p_4) \tag{5.77}$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}[\bar{u}(3)\gamma^\mu v(2)][\bar{v}(2)\gamma_\mu u(1)] \tag{5.78}$$

we need a relative minus sign between diagrams.

5.6.2 Compton Scattering

$\gamma + e \to \gamma + e$

This time the two diagrams add.

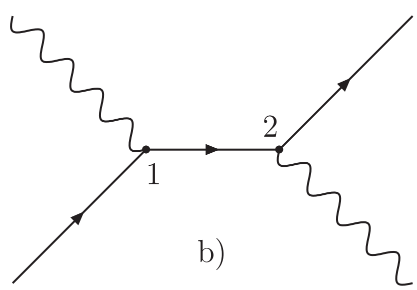


Figure 5.2: Compton scattering in s-channel.

$\not{a} \equiv a^\mu\gamma_\mu$ $\not{\varepsilon}^* = \gamma^\mu\varepsilon^*_\mu$ no conjugate on γ^μ

$$(2\pi)^4 \int \varepsilon_\mu(2)\bar{u}(4)(ig_e\gamma^\mu)\frac{i(\not{q}+mc)}{(q^2-m^2c^2)}(ig_e\gamma^\nu)u(1)\varepsilon^*_\nu(3)\delta^4(p_1-p_3-q)\delta^4(p_2+q-p_4)d^4q$$

$$= -i(2\pi)^4\frac{g_e^2}{(p_1-p_3)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\varepsilon}^*(3)u(1)] \tag{5.79}$$

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1-p_3)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\varepsilon}^*(3)u(1)] \tag{5.80}$$

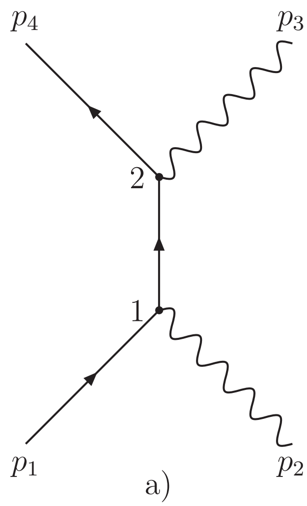


Figure 5.3: Compton scattering in t-channel.

$$(2\pi)^4 \int \varepsilon^*_\mu(3)\bar{u}(4)(ig_e\gamma^\mu)\frac{i(\not{q}+mc)}{(q^2-m^2c^2)}(ig_e\gamma^\nu)u(1)\varepsilon_\nu(2)\delta^4(p_1+p_2-q)\delta^4(q-p_3-p_4)d^4q$$

$$= -i(2\pi)^4\frac{g_e^2}{(p_1+p_2)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}^*(3)(p_1+p_2+mc)\not{\varepsilon}(2)u(1)]\delta^4(p_1+p_2-p_3-p_4) \tag{5.81}$$

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1+p_2)^2-m^2c^2}[\bar{u}(4)\not{\varepsilon}^*(3)(p_1+p_2+mc)\not{\varepsilon}(2)u(1)] \tag{5.82}$$

$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$.

5.7 Trace Theorems

We now need to calculate $|\mathcal{M}|^2$. We could use the appropriate spinors and polarization vectors corresponding to the experiment of interest. Most experiments average over initial state spins and sum over final state spins. Let $\langle|\mathcal{M}|^2\rangle$ denote the average over initial spins and sum over final spins of $|\mathcal{M}(i \to f)|^2$. In principle, calculate all possibilities and take average and sums. In practice, we compute $\langle|\mathcal{M}|^2\rangle$ directly without ever evaluating the individual amplitudes.

For electron-muon scattering,

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)] \,. \tag{5.83}$$

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^* \,. \tag{5.84}$$

In general, $G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$.

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [\bar{u}^\dagger(a)\gamma^0\Gamma_2 u(b)]^\dagger = u^\dagger(b)\Gamma_2^\dagger\gamma^{0\dagger}u(a) \,. \tag{5.85}$$

† is the same as * for a number (1×1 matrix). Since $\gamma^{0\dagger} = \gamma^0$, $(\gamma^0)^2 = 1$.

$$[\bar{u}(a)\Gamma_2 u(b)]^* = u^\dagger(b)\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a) \,, \tag{5.86}$$

where $\bar{\Gamma}_2 = \gamma^0\Gamma_2^\dagger\gamma^0$. Therefore $G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$.
Summing over spin

$$\sum_{b\ spins} G = \bar{u}(a)\Gamma_1\left[\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b)\right]\bar{\Gamma}_2 u(a) = \bar{u}(a)\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2 u(a) = \bar{u}(a)Qu(a) \,, \tag{5.87}$$

where $Q \equiv \Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2$.

$$\begin{aligned} \sum_{a\ spins}\sum_{b\ spins} G &= \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a)Qu^{(s_a)}(p_a) && (5.88)\\ &= \sum_{s_a=1,2} \bar{u}^{(a_a)}(p_a)_i Q_{ij} u^{(s_a)}(p_a)_j && (5.89)\\ &= Q_{ij}\left[\sum_{s_a=1,2} u^{(s_a)}(p_a)\bar{u}^{(s_a)}(p_a)\right]_{ji} && (5.90)\\ &= Q_{ij}(\not{p}_a + m_a c)_{ji} && (5.91)\\ &= Tr[Q(\not{p}_a + m_a c)] \,. && (5.92) \end{aligned}$$

Trace $Tr[A] = \sum_i A_{ii}$ sum of diagonal elements. $\sum_{all\ spins}[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = Tr[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(p_a + m_a c)]$. No spinors left, just matrix multiplication, taking trace.

For electron-muon scattering, $\Gamma_1 = \gamma^\mu$ and $\bar{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$.

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{4(p_1 - p_3)^4}Tr[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)]Tr[\gamma_\mu(\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc)] \,. \tag{5.93}$$

The 1/4 is because we average over initial spins and there are two of them in two spin states each: (1/2)(1/2) = 1/4. $m \equiv$ mass of electron and $M \equiv$ mass of muon.

Traces in general: A, B matrices, α number.

1. $Tr[A+B] = Tr[A] + Tr[B]$.
2. $Tr[\alpha A] = \alpha Tr[A]$.
3. $Tr[AB] = Tr[BA]$, must preserve the order because of the cyclic property of trace $Tr[ABC] = Tr[CAB] = Tr[BCA]$.
4. $g_{\mu\nu} g^{\mu\nu} = 4$.
5. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \not{a} = a_\mu\gamma^\mu, \not{b} = b_\nu\gamma^\nu$, $a_\mu b_\nu \gamma^\mu\gamma^\nu + a_\mu b_\nu \gamma^\nu\gamma^\mu = 2a_\mu b_\nu g^{\mu\nu}$, $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$.
6. Contraction theorems
   $\gamma_\mu\gamma^\mu = 4$
   $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \quad \gamma_\mu\not{a}\gamma^\mu = -2\not{a}$
   $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda} \quad \gamma_\mu\not{a}\not{b}\gamma^\mu = 4a \cdot b$
   $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu \quad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$
7. Trace of an odd number of γ's is zero.

$$Tr[I] = 4 \tag{5.94}$$
$$Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{5.95}$$
$$Tr[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \tag{5.96}$$

$$Tr[\not{a}\not{b}] = 4a \cdot b \tag{5.97}$$
$$Tr[\not{a}\not{b}\not{c}\not{d}] = 4(a \cdot bc \cdot d - a \cdot cb \cdot d - a \cdot db \cdot c) \tag{5.98}$$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (even number of γ matrices)

$$Tr[\gamma^5] = 0 \tag{5.99}$$
$$Tr[\gamma^5\gamma^\mu\gamma^\nu] = 0 \quad Tr[\gamma^5\not{a}\not{b}] = 0 \tag{5.100}$$
$$Tr[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4i\epsilon^{\mu\nu\lambda\sigma} \quad Tr[\gamma^5\not{a}\not{b}\not{c}\not{d}] = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma \tag{5.101}$$

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} -1 \text{ if } \mu\nu\lambda\sigma \text{ even permutation of } 0123 \\ +1 \text{ if } \mu\nu\lambda\sigma \text{ odd permutation of } 0123 \\ 0 \text{ if any 2 indices are the same} \end{cases} \tag{5.102}$$

This is slightly different from the Levi-Civita symbol ϵ_{ijk}. Also notice $\epsilon^{0123} = -1$, where $\epsilon_{0123} = +1$.

Return to electron-muon scattering

$$Tr[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_3+mc)] = Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3]+mcTr[\gamma^\mu\not{p}_1\gamma^\nu]+mcTr[\gamma^\mu\gamma^\nu\not{p}_3]+(mc)^2Tr[\gamma^\mu\gamma^\nu] \tag{5.103}$$

$$Tr[\gamma^\mu\not{p}_1\gamma^\nu] = Tr[\gamma^\mu\gamma^\nu\not{p}_3] = 0 \tag{5.104}$$

$$Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{5.105}$$

$$\begin{aligned} Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3] &= (p_1)_\lambda(p_3)_\sigma Tr[\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma] \\ &= (p_1)_\lambda(p_3)_\sigma 4(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}) \\ &= 4(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1\cdot p_3) + p_3^\mu p_1^\nu) \end{aligned} \tag{5.106}$$

Therefore

$$Tr[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_3+mc)] = 4[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (mc)^2g^{\mu\nu} - (p_1\cdot p_3)g^{\mu\nu}]\,. \tag{5.107}$$

For the second trace, $m \to M, 1 \to 2, 3 \to 4$.

$$Tr[\gamma_\mu(\not{p}_2+Mc)\gamma_\nu(\not{p}_4+Mc)] = 4[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + (Mc)^2g_{\mu\nu} - (p_2\cdot p_4)g_{\mu\nu}] \tag{5.108}$$

Therefore

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{4g_e^4}{(p_1-p_3)^4}\,[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) \\ &\quad +(Mc)^2(p_1\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_1\cdot p_4) \\ &\quad +(p_3\cdot p_4)(p_1\cdot p_2) + (Mc)^2(p_1\cdot p_3) - (p_2\cdot p_4)(p_1\cdot p_3) \\ &\quad +(mc)^2(p_2\cdot p_4) + (mc)^2(p_2\cdot p_4) + 4(mc)^2(Mc)^2 \\ &\quad -4(mc)^2(p_2\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) \\ &\quad -4(Mc)^2(p_1\cdot p_3) + 4(p_1\cdot p_3)(p_2\cdot p_4)\Big] \qquad (5.109) \\ &= \frac{8g_e^4}{(p_1-p_3)^4}\,[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) \\ &\quad -(Mc)^2(p_1\cdot p_3) - (mc)^2(p_2\cdot p_4) + 2(mc)^2(Mc)^2\Big]\,. \qquad (5.110) \end{aligned}$$

5.8 Cross Sections and Lifetimes

We now plug $|\mathcal{M}|^2$ into cross section formula.

5.8.1 Mott and Rutherford Scattering

Lets do μ at rest in lab. Consider $M \gg m$, assume M recoil neglected, M at rest.

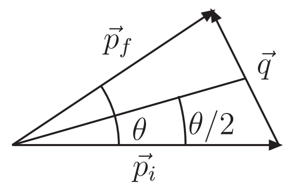


Figure 5.4: Definition of the three-momentum transfer.

$$p_1 = (E/c, \vec{p}_1), p_2 = (Mc, 0), p_3 = (E/c, \vec{p}_3), p_4 = (Mc, 0) \tag{5.111}$$

since $E_1 = E_3 = E$

$$|\vec{p}_1| = |\vec{p}_3| \equiv |\vec{p}|; \vec{p}_1 \cdot \vec{p}_3 = \vec{p}^2 \cos\theta \tag{5.112}$$

$$\begin{aligned}(p_1 - p_3)^2 &= (E_1 - E_3)^2 - (\vec{p}_1 - \vec{p}_3)^2 = -\vec{p}_1^2 - \vec{p}_3^2 + 2\vec{p}_1 \cdot \vec{p}_3 \\ &= -2\vec{p}^2 + 2\vec{p}^2 \cos\theta = -2\vec{p}^2(1 - \cos\theta) \\ &= -4\vec{p}^2 \sin^2\theta/2 \end{aligned} \tag{5.113}$$

$$\begin{aligned}(p_1 \cdot p_3) &= (E/c)(E/c) - \vec{p}_1 \cdot \vec{p}_3 = \vec{p}^2 + m^2c^2 - \vec{p}^2 \cos\theta \\ &= m^2c^2 + 2\vec{p}^2 \sin^2\theta/2 \end{aligned} \tag{5.114}$$

$$p_1 \cdot p_2 = ME, p_1 \cdot p_4 = ME, p_2 \cdot p_3 = ME, p_3 \cdot p_4 = ME \tag{5.115}$$

$$p_2 \cdot p_4 = (Mc)^2 \tag{5.116}$$

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{8g_e^4}{16\vec{p}^4 \sin^4\theta/2}\Big[(ME)^2 + (ME)^2 - (Mc)^2(m^2c^2 + 2\vec{p}^2 \sin^2\theta/2) \\ &\quad -(Mc)^2(mc)^2 + 2(mc)^2(Mc)^2\Big] \\ &= \left(\frac{g_e^2}{\vec{p}^2 \sin^2\theta/2}\right)^2 \Big[(ME)^2 - (Mc)^2\vec{p}^2 \sin^2\theta/2\Big] \end{aligned} \tag{5.117}$$

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \left(\frac{g_e^2 Mc}{\vec{p}^2\sin^2\theta/2}\right)^2\left[(E/c)^2 - |\vec{p}|^2\sin^2\theta/2\right] \\ &= \left(\frac{g_e^2 Mc}{\vec{p}^2\sin^2\theta/2}\right)^2\left[(mc)^2 + |\vec{p}|^2(1-\sin^2\theta/2)\right] \\ &= \left(\frac{g_e^2 Mc}{\vec{p}^2\sin^2\theta/2}\right)^2\left[(mc)^2 + |\vec{p}|^2\cos^2\theta/2\right] \qquad (5.118)\end{aligned}$$

Now $g_e = \sqrt{4\pi\alpha}$,

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi Mc}\right)^2\langle|\mathcal{M}|^2\rangle \qquad (5.119)$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{\hbar}{8\pi Mc}\right)^2\left(\frac{4\pi\alpha Mc}{\vec{p}^2\sin^2\theta/2}\right)^2[(mc)^2 + \vec{p}^2\cos^2\theta/2] \qquad (5.120) \\ \frac{d\sigma}{d\Omega} &= \left(\frac{\alpha\hbar}{2\vec{p}^2\sin^2\theta/2}\right)^2\left[(mc)^2 + \vec{p}^2\cos^2\frac{\theta}{2}\right], \qquad (5.121)\end{aligned}$$

which is Mott formula.

If the electron is nonrelativistic, $\vec{p}^2 \ll (mc)^2$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{(e^2/\hbar c)\hbar}{2(mv)^2\sin^2\theta/2}\right)^2(mc)^2 \\ &= \left(\frac{e^2}{2mv^2\sin^2\theta/2}\right)^2, \qquad (5.122)\end{aligned}$$

which is Rutherford formula.

Interestingly enough there are no particle decays in QED of fundamental particles. Fermion goes in, a fermion must come out. Can not convert one fermion to another. Decays of composite particles $\pi^0 \to \gamma + \gamma$ is really pair annhilation (scattering process).

Chapter 6

Electrodynamics of Quarks and Hadrons

Electrodynamics also applies to quarks, but quarks are not free particles. Charge 2/3 and -1/3.

6.1 Hadron Production in e^+e^- Scattering

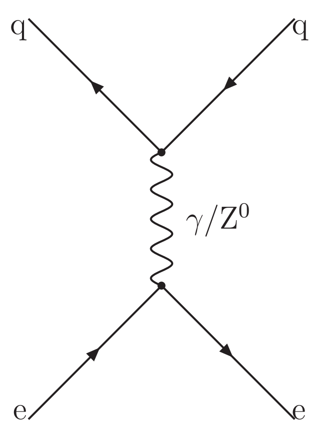


Figure 6.1: Quark pair production.

$e^+ + e^- \rightarrow \gamma \rightarrow q + \bar{q} \rightarrow$ hadrons EM
$e^+ + e^- \rightarrow Z^0 \rightarrow q + \bar{q} \rightarrow$ hadrons weak

When quarks are separate by more than 10^{-15} m new quarks are formed. These quarks make mesons and baryons. They look like two back to back jets along the direction of the primordial quarks.

Three-jet events indicate the existence of the gluon.

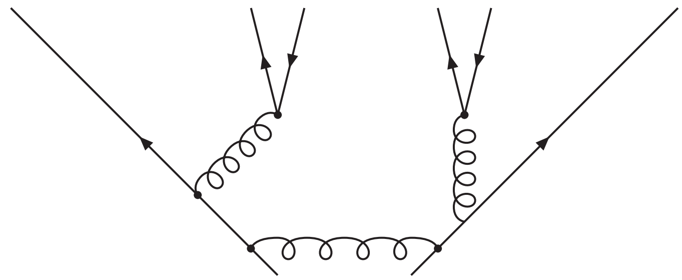


Figure 6.2: Gluon radiation.

$e^+e^- \to \gamma \to q\bar{q}$,
Calculation similar to $e^+e^- \to \gamma \to \mu^+\mu^-$

$$\mathcal{M} = \frac{Qg_e^2}{(p_1+p_2)^2}[\bar{v}(2)\gamma^\mu u(1)][\bar{u}(3)\gamma_\mu v(4)] \tag{6.1}$$

$Q = 2/3$ for u, c, t; $Q = -1/3$ for d, s, b.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2 Tr[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_2 - mc)]Tr[\gamma_\mu(\not{p}_4 - Mc)\gamma_\nu(\not{p}_3 + Mc)]\,, \tag{6.2}$$

where $m \equiv$ mass electron; $M \equiv$ mass quark.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 8\left[\frac{Qg_e^2}{(p_1+p_2)^2}\right]^2[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\\ &\quad +(mc)^2(p_3\cdot p_4)+(Mc)^2(p_1\cdot p_2)+2(mc)^2(Mc)^2)]\,. \end{aligned} \tag{6.3}$$

In terms of the incident (CM) electron energy E, and the angle θ between incoming electron and outgoing quark

$$\langle|\mathcal{M}|^2\rangle = Q^2 g_e^4\left\{1+\left(\frac{mc^2}{E}\right)^2+\left(\frac{Mc^2}{E}\right)^2+\left[1-\left(\frac{mc^2}{E}\right)^2\right]\left[1-\left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\} \tag{6.4}$$

$$\sigma = \frac{\pi Q^2}{3}\left(\frac{\hbar c\alpha}{E}\right)^2\sqrt{\frac{1-(Mc^2/E)^2}{1-(mc^2/E)^2}}\left[1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)\right]\left[1+\frac{1}{2}\left(\frac{mc^2}{E}\right)\right]\,. \tag{6.5}$$

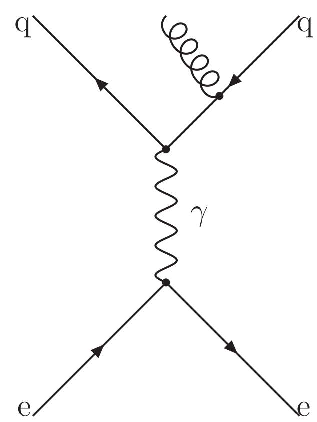


Figure 6.3: Single gluon radiation.

There is a threshold at $E > Mc^2$.

If $E > Mc^2 \gg mc^2$,

$$\sigma = \frac{\pi}{3}\left(\frac{\hbar Q c\alpha}{E}\right)^2 . \tag{6.6}$$

As E increases, we encounter a series of thresholds corresponding to approximately $2M_q$ for each quark type.

Consider the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{6.7}$$

$$R(E) = \frac{3\sum Q_i^2 e^2}{e^2} = 3\sum Q_i^2 \tag{6.8}$$

The 3 is because of the three colours, the sum is over the quarks below threshold.

At low energies: u, d, s

$$R = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right] = 2\,. \tag{6.9}$$

Between the c- and b-quark threshold

$$R = 2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3} \approx 3.3\,. \tag{6.10}$$

Above b threshold

$$R = \frac{10}{3} + 3\left(\frac{-1}{3}\right)^2 = \frac{11}{3} \approx 3.7\,. \tag{6.11}$$

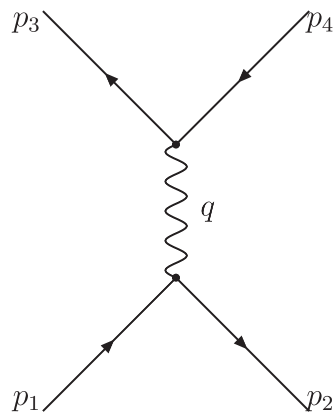


Figure 6.4: Annihilation.

These are in good agreement with experiment . This is futher evidence for quarks (lacking of top) and evidence for colour. Factor of three is necessary which implies there are three colours.

Chapter 7

Weak Interactions

Three classica problems: 1) beta decay of muon, 2) neutron decay, 3) charged pion decay.

In weak interactions there are no massless mediators like the γ and g. The mediators $W^{\pm}$ and Z^0 are heavy with $M_W \sim 80$ GeV and $M_Z \sim 91$ GeV. For massive spin-1 particles, there are three spins $m_s = -1, 0, +1$. The Lorentz condition exhausts the number of degrees of freedom (do not also invoke Coulomb gauge).

The propagator is

$$\frac{-ig_{\mu\nu}}{q^2} \to \frac{-i(g_{\mu\nu} - q_\mu q_\nu/(M^2c^2))}{q^2 - M^2c^2}\,, \tag{7.1}$$

where $M \equiv M_W$ or M_Z. Usually

$$\frac{ig_{\mu\nu}}{(Mc)^2}\,, \tag{7.2}$$

for $q^2 \ll (Mc)^2$.

Impose Lorentz condition $\varepsilon^\mu p_\mu = 0 \to 4$ to 3 components of ε^μ (massive boson). Impose Coulomb gauge $\to$ 3 to 2 components of ε^μ (massless boson).

7.1 Charged Leptonic Weak Interactions

Charged weak interaction for leptons
Emission of W^- or absoption of W^+. $\ell^- \to \nu_e W^-$ also $\nu_e \to \ell^- W^+$. Weak vertex factor (V-A coupling)

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\,. \tag{7.3}$$

The reversed and crossed interactions are also possible. A W^+ can be absorbed instead. $g_W = \sqrt{4\pi\alpha_W}$ is the weak coupling constant. γ^μ vector and $\gamma^\mu\gamma^5$ axial vector $\to$ violation of parity (maximum violation). ν has only one spin state in spin averages or sums.

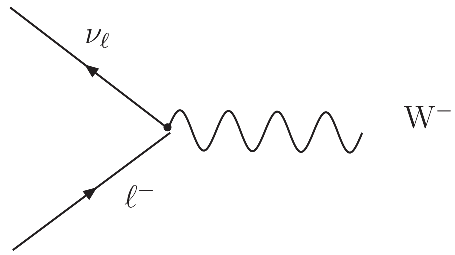


Figure 7.1: Charged weak vertex.

7.2 Decay of the Muon

Muon decay is the cleanest of all weak interactions, both theoretically and experimentally.

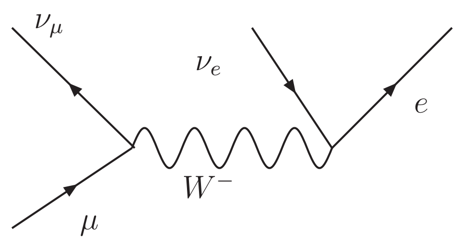


Figure 7.2: 3-body decay $\mu \to e + \nu_\mu + \bar{\nu}_e$.

g_W and M_W do not appear separately so it is common to group them together. Fermi coupling constant is

$$G_F = \frac{\sqrt{2}}{8}\left(\frac{g_W}{M_W c^2}\right)^2 (\hbar c)^2 . \tag{7.4}$$

By measuring the muon lifetime and mass we can determine G_F. If we know M_W we can predict $g_w = 0.66$. The weak fine structure constant $\alpha_W = g_W^2/4\pi = 1/29 > 1/137$. Weak interactions are weak not because g_W is small but because the mediator is massive. $E \ll M_W c^2 \to$ weak interactions are weak. $E \approx M_W c^2 \to$ weak interactions stronger than EM interactions.

7.3 Charged Weak Interactions of Quarks

Lepton generations

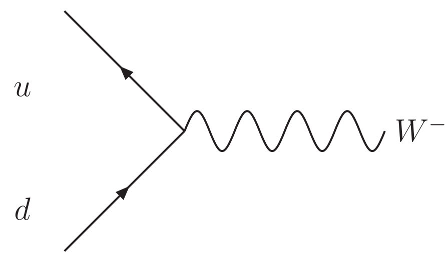
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \tag{7.5}$$

$e^- \to \nu_e + W^-$, $\mu^- \to \nu_\mu + W^-$, $\tau^- \to \nu_\tau + W^-$.

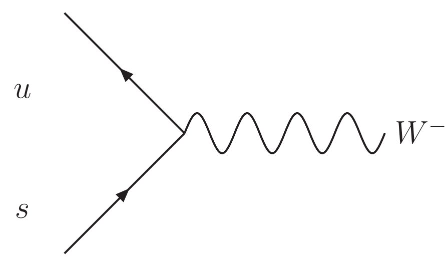
No cross generation coupling. Conservation of lepton-type number
Quark generations

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \tag{7.6}$$

Cross generation coupling possible. $d \to u + W^- \cos\theta_C$
$s \to u + W^- \sin\theta_C$ also.



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_C \tag{7.7}$$



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_C \tag{7.8}$$

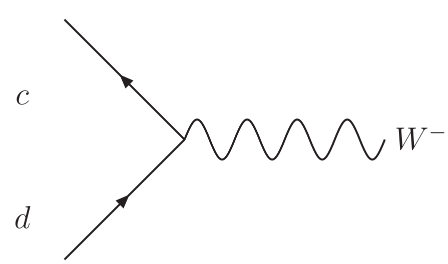
$\theta_C = 13.1°$ (small)
Examples of leptonic, semileptonic, and nonleptonic decays are.

Leptonic decays $\pi^- \to \ell^- + \bar{\nu}_\ell$, $K^- \to \ell^- + \bar{\nu}_\ell$

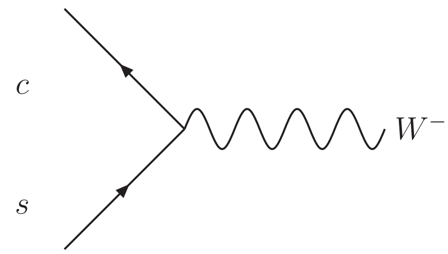
Semileptonic decays $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$, $\bar{K}^0 \to \pi^+ + \mu^- + \bar{\nu}_\mu$, $n \to p + e^- + \bar{\nu}_e$.

Nonleptonic decays $K^- \to \pi^0 + \pi^-$, $\lambda \to p + \pi^-$

Now, included the c quark.



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)(-\sin\theta_C) \tag{7.9}$$



$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_C \tag{7.10}$$

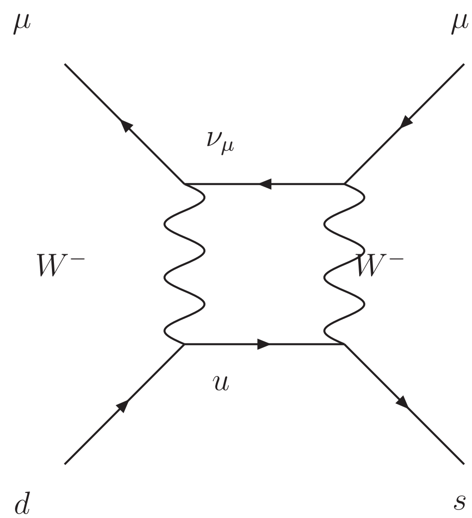


Figure 7.3: $K^0(d\bar{s})$

$K^0 \rightarrow \mu^+ + \mu^-$ Calculated rate far greater than experiment. This diagram cancels previous diagram.

The GIM mechanisum has two cancelling diagrams so the decay is highly suppressed. Cabibbo-GIM scheme: correct states to use in weak interactions are d', s'.

$$d' = d\cos\theta_C + s\sin\theta_C \tag{7.11}$$
$$s' = -d\sin\theta_C + s\cos\theta_C \tag{7.12}$$

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \tag{7.13}$$

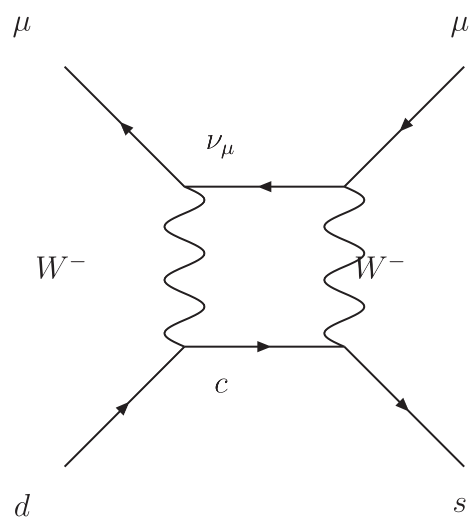


Figure 7.4: $K^0(d\bar{s})$

The primed denotes the weak states (Cabibbo rotated), while the un-primed denote the physical quark states (states of specific flavour).

W's couple to Cabibbo-rotated states

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix} \tag{7.14}$$

similarly to W coupling to lepton states

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \tag{7.15}$$

$$\begin{pmatrix} u \\ d' \end{pmatrix} = \begin{pmatrix} u \\ d\cos\theta_C + s\sin\theta_C \end{pmatrix} \tag{7.16}$$

$$\begin{pmatrix} c \\ s' \end{pmatrix} = \begin{pmatrix} c \\ -d\sin\theta_C + s\cos\theta_C \end{pmatrix}. \tag{7.17}$$

We generalization the Cabibbo-GIM scheme to handle three generations of quarks. The weak interaction quark generations are

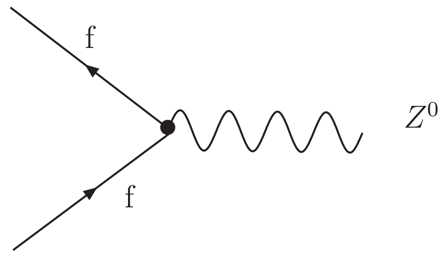
$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}, \begin{pmatrix} t \\ b' \end{pmatrix}. \tag{7.18}$$

Kobayahi-Maskawa matrix

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{7.19}$$

U_{ud} specifies coupling of u to d ($d \rightarrow u + W^-$). Not all the entries are independent. There are three generalized Cabibbo angles and one phase factor. The SM offers no insight into KM matrix. We thus have four more parameters of the SM. The matrix is almost diagonal.

7.4 Neutral Weak Interactions



where f is any lepton or quark. Same f comes out that goes in.

$\mu^- \rightarrow e^- + Z^0$ not allowed

$s \rightarrow d + Z^0$ flavour changing neutral current

First evidence of $\bar{\nu}_\mu + e \rightarrow \bar{\nu}_\mu + e$ in 1973 suggested mediation of Z^0.

$W^\pm$ vertex factor

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \tag{7.20}$$

is universal (always) V-A form.

Z^0 vertex factor

$$\frac{-ig_Z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5) \tag{7.21}$$

c_V^f, c_A^f depend on particular quark or lepton (f) involved. They are a function of a single parameter.

$$c_V^f = c_V^f(\theta_W) \tag{7.22}$$
$$c_A^f = c_A^f(\theta_W)\,, \tag{7.23}$$

where $\theta_W \equiv$ weak mixing angle (Weinberg angle).

g_W, g_Z depend on g_e and θ_W

$$g_W = \frac{g_e}{\sin\theta_W} \quad g_Z = \frac{g_e}{\sin\theta_W\cos\theta_W} \tag{7.24}$$

θ_W is a free parameter of the SM. $\theta_W = 28.7°$ ($\sin^2\theta_W = 0.23$).

Z^0 propagator

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu/M_Z^2c^2)}{q^2 - M_Z^2c^2} \tag{7.25}$$

Normally $q^2 \ll M_Z^2 c^2 \to \frac{ig_{\mu\nu}}{(M_Z c)^2}$. Also $M_W = M_Z \cos\theta_W$.

M_W, M_Z discovered in 1983 as predicted. Hard to see weak interaction effects (particularly Z^0) because masked by EM and strong interactions. Must use ν or go to energies near M_Z, M_W. UA1, UA2 had enough energy to observe M_W, M_Z. Constructed SLC (1987) and LEP (1989) to study M_W, M_Z. Motivated by accuracy of predictions to measurements.

$e^+ + e^- \to Z^0 \to f + \bar{f}$ gives infinite cross section at $\sqrt{s} = M_Z c^2$.

But Z^0 not stable particle. Modify propagator.

$$\frac{1}{q^2 - (M_Z c)^2} \to \frac{1}{q^2 - (M_Z c)^2 + i\hbar M_Z \Gamma_Z} \tag{7.26}$$

where $\Gamma_Z = 1/\tau_Z$ is decay rate. $\sigma_Z/\sigma_\gamma \approx 200$ at the Z^0 pole. Below the Z^0 pole ($\sim 1/2 M_Z c^2$) the weak contributions is less than 1%. γ/Z^0 interference measurable.

7.5 Electroweak Unification

7.5.1 Chiral Fermion States

Our aim is to unify the EM and weak interactions. There are two rather big differences between the interactons:

1. EM is strong, while weak is weak $\to$ explain by massive propagator.

2. EM has a vector γ^μ structure, while the eweak has a V-A $\gamma^\mu(1-\gamma^5)$ structure (for $W^\pm$).

We will try to absorb the extra $(1-\gamma^5)$ into the particle spinors.

Left handed (helicity -1)

$$u_L(p) \equiv \frac{(1-\gamma^5)}{2} u(p) \tag{7.27}$$

$$\frac{1}{2}(1-\gamma^5)u(p) = \begin{cases} 0 & \text{if } u(p) \text{ carries helicity } +1 \\ u(p) & \text{if } u(p) \text{ carries helicity } -1 \end{cases} \tag{7.28}$$

(for $m = 0$ only). $1/2(1-\gamma^5)$ is a projection operator which picks out helicity -1. This is only approximate for ultrafelativistic massive particles (terminology is approximate).

For antiparticle

$$v_L(p) \equiv \frac{(1+\gamma^5)}{2} v(p)\,. \tag{7.29}$$

The right-handed spinors are

$$u_R(p) \equiv \frac{(1+\gamma^5)}{2} u(p) \quad \text{and} \quad v_R(p) \equiv \frac{(1-\gamma^5)}{2} v(p)\,. \tag{7.30}$$

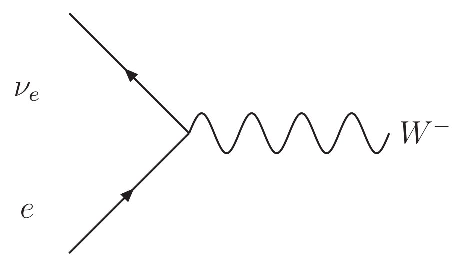
For adjoint spinors $[(\gamma^5)^\dagger = \gamma^5, \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu]$.

$$\bar{u}_L = u_L^\dagger\gamma^0 = u^\dagger\frac{(1-\gamma^5)}{2}\gamma^0 = u^\dagger\gamma^0\frac{(1+\gamma^5)}{2} = \bar{u}\frac{(1+\gamma^5)}{2} \tag{7.31}$$

$$\bar{v}_L = \bar{v}\frac{(1-\gamma^5)}{2}, \quad \bar{u}_R = \bar{u}\frac{(1-\gamma^5)}{2}, \quad \bar{v}_R = \bar{v}\frac{(1+\gamma^5)}{2}\,. \tag{7.32}$$

These spinors are called chiral fermions states. This is just notation and terminology so far.

Now consider the contribution to $\mathcal{M}$ from the vertex. For the negative charged weak current



$$j_\mu^- = \bar{\nu}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)e\,. \tag{7.33}$$

Now

$$\left(\frac{1-\gamma^5}{2}\right)^2 = \frac{1}{4}\left[1-2\gamma^5+(\gamma^5)^2\right] = \frac{1-\gamma^5}{2} \tag{7.34}$$

and

$$\gamma_\mu\left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu \tag{7.35}$$

so

$$\gamma_\mu\left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\,. \tag{7.36}$$

Insert this in the above current.

Therefore $j_\mu^- = \bar{\nu}_L\gamma_\mu e_L$ is a pure vectorial vertex. It couples only to left-handed electrons and neutrions. Notice that

$$u = \left(\frac{1-\gamma^5}{2}\right)u + \left(\frac{1+\gamma^5}{2}\right)u = u_L + u_R \tag{7.37}$$

and $\bar{u} = \bar{u}_L + \bar{u}_R$.

Now consider the electromagnetic current

$$j_\mu^{em} = -\bar{e}\gamma_\mu e = -(\bar{e}_L + \bar{e}_R)\gamma_\mu(e_L + e_R) = -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R\,. \tag{7.38}$$

The crossed term vanishes

$$\bar{e}_L\gamma_\mu e_R = \bar{e}\left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)e = \bar{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\left(\frac{1+\gamma^5}{2}\right)e \tag{7.39}$$

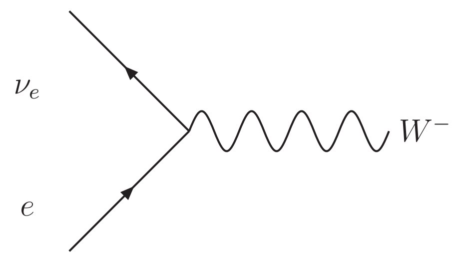
but

$$(1-\gamma^5)(1+\gamma^5) = 1 - (\gamma^5)^2 = 0\,. \tag{7.40}$$

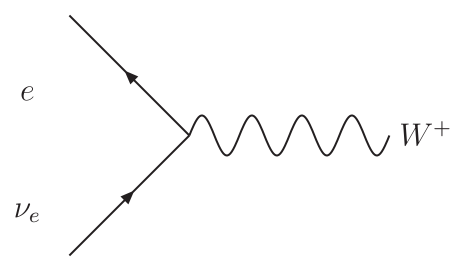
Recap: $j_\mu^- = \bar{\nu}_L\gamma_\mu e_L$ weak couples to left-hand fermions only. $j_\mu^{em} = -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R$ EM couples to both. Therefore $(1-\gamma^5)/2$ characterizes particle, not interaction. All interactions (strong, EM, weak) are vectorial.

7.5.2 Weak Isospin and Hypercharge

Negative charged weak current $j_\mu^- = \bar{\nu}_L\gamma_\mu e_L$, $e^- \to \nu_e + W^-$.



Positive charged weak current $j_\mu^+ = \bar{e}_L\gamma_\mu\nu_L$, $\nu_e \to e^- + W^+$.



Introduce left-handed doublet (like isospin)

$$\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{7.41}$$

$$j_\mu^\pm = \bar{\chi}_L\gamma_\mu\tau^\pm\chi_L \tag{7.42}$$

where

$$\tau^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tau^- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tau^\pm = \frac{1}{2}(\tau^1 \pm i\tau^3) \tag{7.43}$$

where τ^1 and τ^3 are Pauli spin matrices.

Now we need a third weak current (for full weak isospin symmetry). Let

$$\frac{1}{2}\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.44}$$

$$j^3_\mu = \bar{\chi}_L \gamma_\mu 1/2\tau^3 \chi_L = 1/2\bar{\nu}_L\gamma_\mu\nu_L - 1/2\bar{e}_L\gamma_\mu e_L\,. \tag{7.45}$$

This current only couples left-handed particles (neutral weak interactions involve right-handed as well). Consider the weak analog of hypercharge Y $(= S + A)$ (in parallel to isospin). Recall $Q = I_3 + 1/2Y$. Introduce the weak hypercharge current

$$j^Y_\mu = 2j^{em}_\mu - 2j^3_\mu = -2\bar{e}_L\gamma_\mu e_L - 2\bar{e}_R\gamma_\mu e_R - \bar{\nu}_L\gamma_\mu\nu_L + \bar{e}_L\gamma_\mu e_L \tag{7.46}$$

$$j^Y_\mu = -2\bar{e}_R\gamma_\mu e_R - \bar{e}_L\gamma_\mu e_L - \bar{\nu}_L\gamma_\mu\nu_L\,. \tag{7.47}$$

This is invariant as far as weak isospin is concerned. $\bar{e}_R\gamma_\mu e_R$ is untouched, while $\bar{e}_L\gamma_\mu e_L + \bar{\nu}_L\gamma_\mu\nu_L = \bar{\chi}_L\gamma\chi_L$ invariant.

The underlying symmetry group is $SU(2)_L \otimes U(1)_Y$. $SU(2)_L$ for weak isopsin (left-handed states only) and $U(1)_Y$ for weak hypercharge (both chiralities).

Extend to all leptons and quarks

$$\chi_L \to \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L, \begin{pmatrix} t \\ b' \end{pmatrix}_L. \tag{7.48}$$

There are three weak isospin currents $\vec{j}_\mu = 1/2\bar{\chi}_L\gamma_\mu\vec{\tau}\chi_L$ and one weak hypercharge current $J^Y_\mu = 2j^{em}_\mu - 2j^3_\mu$, where $j^{em}_\mu = \sum_{i=1}^2 Q_i(\bar{u}_{iL}\gamma_\mu u_{iL} + \bar{u}_{iR}\gamma_\mu u_{iR})$. The sum is over both particles in the doublet.

7.5.3 Electroweak Mixing

GWS model asserts

$$-i\left[g_W\vec{j}_\mu \cdot \vec{W}^\mu + \frac{g'}{2}j^Y_\mu B^\mu\right]. \tag{7.49}$$

This contains all of electrodynamics and weak interactions. In weak isospin space

$$\vec{j}_\mu \cdot \vec{W}^\mu = j^1_\mu W^{\mu 1} + j^2_\mu W^{\mu 2} + j^3_\mu W^{\mu 3}\,. \tag{7.50}$$

For charged currents $j^\pm_\mu = j^1_\mu \pm i j^2_\mu$

$$\vec{j}_\mu \cdot \vec{W}^\mu = \frac{1}{\sqrt{2}} j_\mu^+ W^{\mu +} + \frac{1}{\sqrt{2}} j_\mu^- W^{\mu -} + j_\mu^3 W^{\mu 3} \tag{7.51}$$

where

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^+ \mp i W_\mu^-)\,. \tag{7.52}$$

This is the wave function of the W$^\pm$ particle.

$$-i g_W (1/\sqrt{2}) j_\mu^- W^{\mu -} = \frac{-i g_W}{2\sqrt{2}} \left[\bar{\nu}\gamma_\mu (1-\gamma^5) e\right] W^{\mu -} \tag{7.53}$$

Vertex is given by

$$\Rightarrow \frac{-i g_W}{2\sqrt{2}} \gamma_\mu (1-\gamma^5) \tag{7.54}$$

The couplings to W$^\pm$ can be read off from the coefficients of W$_\mu^\pm$. The underlying $SU(2)_L \otimes U(1)$ symmetry broken in GWS theory. W^3 and B must mix $\rightarrow \gamma, Z^0$:

$$A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W \tag{7.55}$$
$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W \tag{7.56}$$

Inverting gives

$$B_\mu = A_\mu \cos\theta_W - Z_\mu \sin\theta_W \tag{7.57}$$
$$W_\mu^3 = A_\mu \sin\theta_W + Z_\mu \cos\theta_W \tag{7.58}$$

$$\begin{aligned} -i\left[g_W j_\mu^3 W^{\mu 3} + \frac{g'}{2} j_\mu^Y B^\mu -\right] &= -i\left\{\left[g_W \sin\theta_W j_\mu^3 + \frac{g'}{2}\cos\theta_W j_\mu^Y\right] A^\mu \right. \\ &\quad \left. + \left[g_W \cos\theta_W j_\mu^3 - \frac{g'}{2}\sin\theta_W j_\mu^Y\right] Z^\mu \right\}. \end{aligned} \tag{7.59}$$

Since $j_\mu^{em} = j_\mu^3 + (1/2) j_\mu^Y$ and $-i g_e j_\mu^{em} A^\mu$, the first square bracket gives $g_W \sin\theta_W = g' \cos\theta_W = g_e$. Thus the weak and EM couplings are not independent.

Now consider the second square bracket.

$$g_W \cos\theta_W = \frac{g_e \cos\theta_W}{\sin\theta_W} = \frac{g_e}{\sin\theta_W \cos\theta_W}\cos^2\theta_W = g_Z \cos^2\theta_W \tag{7.60}$$

$$\frac{g'}{2}\sin\theta_W = \frac{g_e}{2}\frac{\sin\theta_W}{\cos\theta_W} = \frac{g_e}{\sin\theta_W\cos\theta_W}\frac{\sin^2\theta_W}{2} = g_Z \frac{\sin^2\theta_W}{2} \tag{7.61}$$

where

$$g_Z = \frac{g_e}{\sin\theta_W \cos\theta_W}\,. \qquad (7.62)$$

Since $j^Y_\mu = 2j^{em}_\mu - 2j^3_\mu$

$$-\frac{g'}{2}\sin\theta_W j^Y_\mu = -g_Z \sin^2\theta_W j^{em}_\mu + g_Z \sin^2\theta_W j^3_\mu \qquad (7.63)$$

Thus the second bracket now gives the weak coupling to Z^0

$$-ig_Z(j^3_\mu - \sin^2\theta_W j^{em}_\mu)Z^\mu \qquad (7.64)$$

We can not determine the vector and axial vector couplings c_V and c_A. For $\nu_e \to \nu_e + Z^0$ only the j^3_μ term contributes $[j^3_\mu = (1/2)\bar{\nu}_L\gamma_\mu\nu_L - (1/2)\bar{e}_L\gamma_\mu e_L]$.

$$-\frac{ig_Z}{2}\left(\bar{\nu}_L\gamma_\mu\nu_L\right)Z^\mu = -\frac{ig_Z}{2}\left[\bar{\nu}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\nu\right]Z^\mu = -\frac{ig_Z}{2}\left[\bar{\nu}\gamma_\mu\left(\frac{1}{2}-\frac{1}{2}\gamma^5\right)\nu\right]Z^\mu \qquad (7.65)$$

$\Rightarrow c^\nu_V = 1/2$ and $c^\nu_A = 1/2$ for neutrinos.
For electrons both j^3_μ and j^{em}_μ contribute

$$\begin{aligned} & -\frac{ig_Z}{2}\left[-\bar{e}_L\gamma_\mu e_L - 2\sin^2\theta_W(-\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R)\right]Z^\mu \\ = & -\frac{ig_Z}{2}\left[-\bar{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)e + 2\sin^2\theta_W\left(\bar{e}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)e + \bar{e}\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)e\right)\right]Z^\mu \\ = & -\frac{ig_Z}{2}\left[\bar{e}\gamma_\mu\left(-\frac{1}{2} + 2\sin^2\theta_W + \frac{1}{2}\gamma^5\right)e\right]Z^\mu \end{aligned} \qquad (7.66)$$

$\Rightarrow c^e_V = -1/2 + 2\sin^2\theta_W$ and $c^e_A = -1/2$ for charged leptons.

Since θ_W is undetermined, there are two independent couplings (g_e and g_W, or g_e and g_Z). Thus we have not completely unified the two theories, but we now have a single integrated theory.

Chapter 8

Gauge Theories

8.1 Lagrangian Formulation of Classical Particle Mechanics

Newton's second law is

$$\vec{F} = m\vec{a}\,. \tag{8.1}$$

If the force is conservative, $\vec{F} = -\vec{\nabla}U$ and

$$m\frac{d\vec{v}}{dt} = -\vec{\nabla}U\,. \tag{8.2}$$

An alternative formulation is to use the Lagrangian $L = T - U$, where

$$T = \frac{1}{2}mv^2\,. \tag{8.3}$$

In general, $L = L(q_i, \dot{q}_i); q_i = x, y, z; \dot{q}_i = v_x, v_y, v_z$. The fundamental law of motion (Euler-Lagrange equation) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad (i = 1, 2, 3)\,. \tag{8.4}$$

For $L = 1/2m\vec{v}^2 - U$,

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1} = mv_x \quad \text{and} \quad \frac{\partial L}{\partial q_1} = -\frac{\partial U}{\partial x}\,. \tag{8.5}$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\frac{dv_x}{dt} = \frac{\partial L}{\partial q_1} = -\frac{\partial U}{\partial x} \tag{8.6}$$

which is Newton's law.

8.2 Lagrangians in Relativistic Field Theory

A classical particle is localized. We are interested in its position as function of time $x(t), y(t)$, and $z(t)$. A field occupies a region of space. We are interested in a function of position and time $\phi_i(x, y, z, t)$. In particle mechanics $L = L(q_i, \dot{q}_i)$. In field theory $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is the Lagrangian density, where

$$\partial_\mu \phi_i \equiv \frac{\partial \phi_i}{\partial x^\mu}\,. \tag{8.7}$$

Relativity must treat space and time on equal footing.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \to \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right), \quad \frac{\partial L}{\partial q_i} \to \frac{\partial \mathcal{L}}{\partial \phi_i} \quad (i = 1, 2, 3\ldots) \tag{8.8}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i} \tag{8.9}$$

is the generalized Euler-Lagrange equation.

8.2.1 Klein-Gordon Lagrangian for Scalar (Spin-0) Field

For a single scalar ϕ,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2 \tag{8.10}$$

The Euler-Lagrante equations are

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -\left(\frac{mc}{\hbar}\right)^2 \phi \tag{8.11}$$

$$\partial_\mu \partial^\mu \phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0\,, \tag{8.12}$$

which is the Klein-Gordon equation (this is the field equation).

8.2.2 Dirac Lagrangian for Spinor (Spin-1/2) Field

For a single spinor field ψ,

$$\mathcal{L} = i(\hbar c)\bar{\psi}\gamma^\mu \partial_\mu \psi - (mc^2)\bar{\psi}\psi\,. \tag{8.13}$$

ψ and $\bar{\psi}$ are eight independent fields.

The Euler-Lagrange equations for $\bar{\psi}$ are

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0, \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\hbar c \gamma^\mu \partial_\mu \psi - mc^2 \psi \tag{8.14}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\hbar c \partial\!\!\!/ \psi - mc^2 \psi \,. \tag{8.15}$$

This gives the Dirac equation

$$i \partial\!\!\!/ \psi - \left(\frac{mc}{\hbar} \right) \psi = 0 \,. \tag{8.16}$$

The Euler-Lagrange equations for ψ are

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\hbar c \bar{\psi} \gamma^\mu, \quad \frac{\partial \mathcal{L}}{\partial \psi} = -mc^2 \bar{\psi} \tag{8.17}$$

$$i\hbar c \partial_\mu \bar{\psi} \gamma^\mu + mc^2 \bar{\psi} = 0 \tag{8.18}$$

$$i \partial_\mu \bar{\psi} \gamma^\mu + \left(\frac{mc}{\hbar} \right) \bar{\psi} = 0 \,, \tag{8.19}$$

which is the adjoint Dirac equation.

8.2.3 Proca Lagrangian for Vector (Spin-1) Field

For a single vector field A^μ,

$$\mathcal{L} = \frac{-1}{16\pi} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{8\pi} \left(\frac{mc}{\hbar} \right)^2 A^\nu A_\nu \tag{8.20}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi} (\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{8.21}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{4\pi} \left(\frac{mc}{\hbar} \right)^2 A^\nu \tag{8.22}$$

The Euler-Lagrange equations are

$$-\frac{1}{4\pi} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{4\pi} \left(\frac{mc}{\hbar} \right)^2 A^\nu \tag{8.23}$$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar} \right)^2 A^\nu = 0 \,, \tag{8.24}$$

which is the Proca equation.
We will find it useful to define $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Then

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi} \left(\frac{mc}{\hbar} \right)^2 A^\nu A_\nu \tag{8.25}$$

and the field equation is

$$\partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0\,. \tag{8.26}$$

Notice that the Lagrangian comes out of thin air. In relativistic field theory, $\mathcal{L}$ is taken as axiomatic. Also, notice that we can multiply $\mathcal{L}$ by a constant, add a constant, or add a divergence and the field equations are unchanged.

8.2.4 Maxwell Lagrangian for Massless Vector Field with J^μ

Start with the Lagrangian

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^\mu A_\mu\,. \tag{8.27}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. J^μ is just some function (respects conservation of charge). The second term is the source term.
The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{-2}{16\pi}(\partial^\mu A^\nu + \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu) = -\frac{1}{4\pi} F^{\mu\nu} \tag{8.28}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{1}{c} J^\nu \tag{8.29}$$

$$\frac{-1}{4\pi}\partial_\mu F^{\mu\nu} = -\frac{1}{c} J^\nu \Rightarrow \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{8.30}$$

The Lagrangian can be for a free field or with interactions. L has units of energy, so $\mathcal{L}$ has units of energy per unit volume. ϕ and A^μ have units of $\sqrt{ML}/T$. ψ has units of $L^{-3/2}$.

Units of $(\partial_\mu \phi)(\partial^\mu \phi)$ are

$$\left(\frac{ML}{T^2}\right)\left(\frac{1}{L^2}\right) = \frac{M(L/T)^2}{L^3} = \frac{E}{L^3} \tag{8.31}$$

Units of $(mc/\hbar)^2 \phi^2$ are

$$\left(\frac{ML/T}{ET}\right)^2 \frac{ML}{T^2} = \left(\frac{1}{L}\right)^2 \frac{ML}{T^2} = \frac{E}{L^3} \tag{8.32}$$

Units of $(\hbar c)\bar{\psi}\gamma^\mu \partial_\mu \psi$ are

$$\left(ET\frac{L}{T}\right) L^{-3/2} \frac{L^{-3/2}}{L} = \frac{E}{L^3} \tag{8.33}$$

Units of $(mc^2)\bar{\psi}\psi$ are

$$\left(\frac{ML^2}{T^2}\right) L^{-3} = \frac{M(L/T)^2}{L^3} = \frac{M}{L^3} \tag{8.34}$$

8.3 Local Gauge Invariance

The Dirac Lagrangian is

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi\,. \tag{8.35}$$

This Lagrangian is invariant under a global gauge transformation (phase transformation)

$$\psi \to e^{i\theta}\psi\,, \tag{8.36}$$

where θ is a real number.

$$\psi^{\dagger} \to e^{-i\theta}\psi^{\dagger}, \quad \bar{\psi} \to e^{-i\theta}\bar{\psi}, \quad \mathcal{L} \to \mathcal{L} \tag{8.37}$$

If θ is a function of x^{μ}, $\psi \to e^{i\theta(x)}\psi$ is a local gauge transformation.

$$\partial_{\mu}\left(e^{i\theta}\psi\right) = i\left(\partial_{\mu}\theta\right)e^{i\theta}\psi + e^{i\theta}\partial_{\mu}\psi \tag{8.38}$$

$$\mathcal{L} \to \mathcal{L} - \hbar c\left(\partial_{\mu}\theta\right)\bar{\psi}\gamma^{\mu}\psi \tag{8.39}$$

Let

$$\lambda(x) \equiv -\frac{\hbar c}{q}\theta(x)\,, \tag{8.40}$$

where q is the charge of the particle involved. Under $\psi \to e^{-iq\lambda(x)/\hbar c}\psi$

$$\mathcal{L} \to \mathcal{L} + \left(q\bar{\psi}\gamma^{\mu}\psi\right)\partial_{\mu}\lambda\,. \tag{8.41}$$

Let's demand the complete Lagrangian to be invariant under local gauge transformations (new principle of physics). We need to add something to $\mathcal{L}$ to cancel the extra term. Try

$$\mathcal{L} = \left[i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi\right] - (q\bar{\psi}\gamma^{\mu}\psi)A_{\mu}\,, \tag{8.42}$$

where A_{μ} is a new field (gauge field), which transforms under local gauge transformations as

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda\,. \tag{8.43}$$

It is like a gauge transformation in EM. Now the Lagrangian is invariant but we had to introduce a new vector field. The new term in the Lagrangian represents an interaction (A^{μ} couples to ψ).

Must add a free term to $\mathcal{L}$ for gauge field

$$\mathcal{L} = \frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2 A^{\mu}A_{\mu} \tag{8.44}$$

which is the Proca Lagrangian.

$F^{\mu\nu}$ invariant under a gauge transformation, but $A^\mu A_\mu$ is not invariant. So to restore invariance we must require $m_A = 0$. This gauge fields must be massless.

$$\mathcal{L} = \left[i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi\right] + \left[\frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] - i\left(q\bar{\psi}\gamma^\mu\psi\right)A_\mu\,, \tag{8.45}$$

where the current density is

$$J^\mu = cq(\bar{\psi}\gamma^\mu\psi)\,. \tag{8.46}$$

We see that the requirement of local gauge invariance plus the free Dirac Lagrangian gives electrodynamics and specifies the current.

We now have a prescription. Replace every derivative (∂_μ) in the original (free) Lagrangian by the covariant derivative:

$$\mathcal{D}_\mu \equiv \partial_\mu + \frac{iq}{\hbar c}A_\mu\,, \tag{8.47}$$

$$\mathcal{D}_\mu\psi \rightarrow e^{-iq\lambda/\hbar c}\mathcal{D}_\mu\psi\,. \tag{8.48}$$

This gives the minimum coupling rule: substituting $\mathcal{D}_\mu$ for ∂_μ converts a globally invariant Lagrangian to a locally invariant one (device).

Proof

$$\psi \;\rightarrow\; e^{-iq\lambda(x)/\hbar c}\psi \tag{8.49}$$

$$\partial_\mu\psi \;\rightarrow\; e^{-iq\lambda/\hbar c}\left[\partial_\mu - \frac{iq}{\hbar c}(\partial_\mu\lambda)\right]\psi \tag{8.50}$$

Define

$$\mathcal{D}_\mu \equiv \partial_\mu + \frac{iq}{\hbar c}A_\mu \tag{8.51}$$

$$\begin{aligned}\mathcal{D}_\mu\psi &\rightarrow \left[\partial_\mu + \frac{iq}{\hbar c}(A_\mu + \partial_\mu\lambda)\right]e^{-iq\lambda/\hbar c}\psi \\ &= e^{-iq\lambda/\hbar c}\left[\partial_\mu - \frac{iq}{\hbar c}\partial_\mu\lambda + \frac{iq}{\hbar c}A_\mu + \frac{iq}{\hbar c}\partial_\mu\lambda\right]\psi \\ &= e^{-iq\lambda/\hbar c}\left[\partial_\mu + \frac{iq}{\hbar c}A_\mu\right]\psi \\ &= e^{-iq\lambda/\hbar c}\mathcal{D}_\mu\psi\,.\end{aligned} \tag{8.52}$$

Consider the global phase transformation as a unitary 1×1 matrix. $\psi \rightarrow U\psi$, where $U^\dagger U = 1, U = e^{i\theta}$. The group of all such matrices is $U(1) \Rightarrow U(1)$ gauge invariance $\Rightarrow$ generalize to $SU(2), SU(3)$.

Lagrangian with ψ and $\bar{\psi}$ is asymmetric. A more symmetric notation is

$$\mathcal{L} = \frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - mc^2\bar{\psi}\psi\,, \tag{8.53}$$

which is invariant under U(1).

Apply Nother's theorem. Consider an infinitesimal U(1) transformation

$$\psi \to (1+i\theta)\psi \quad \bar{\psi} \to \bar{\psi}(1-i\theta)\,. \tag{8.54}$$

Invariance requires the Lagrangian to be uncharged

$$\begin{aligned}
\delta\mathcal{L} = 0 &= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\bar{\psi}} + \delta(\partial_\mu\bar{\psi})\frac{\partial\psi}{\partial(\partial_\mu\psi)} \\
&= \frac{\partial\mathcal{L}}{\partial\psi}(i\theta\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(i\theta\partial_\mu\psi) - (i\theta\bar{\psi})\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - (i\theta\partial_\mu\bar{\psi})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} \\
&= i\theta\left[\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\right]\psi + i\theta\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi\right) \\
&\quad -i\theta\bar{\psi}\left[\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right)\right] - i\theta\partial_\mu\left(\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) \\
&= i\theta\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi\right) - i\theta\partial_\mu\left(\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) \\
&= i\theta\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi - \bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right] \\
&= -\frac{1}{2}\theta\partial_\mu[\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\psi] \\
&= -\theta\partial_\mu(\bar{\psi}\gamma^\mu\psi)
\end{aligned} \tag{8.55}$$

$\to \partial_\nu J^\mu = 0$, where $J^\mu = -e\bar{\psi}\gamma^\mu\psi$.

Global U(1) symmetry implies current conservation.

8.4 Yang-Mills Theory

Consider two spin-1/2 fields with Lagrangian

$$\mathcal{L} = \left[i\hbar c\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 - m_1c^2\bar{\psi}_1\psi_1\right] + \left[i\hbar c\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m_2c^2\bar{\psi}_2\psi_2\right]\,. \tag{8.56}$$

This is the addition of two independent Lagrangians. Combine ψ_1 and ψ_2 into a two-component column vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{8.57}$$

This has eight components. The adjoint spinor is $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$. Thus

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - c^2\bar{\psi}M\psi\,, \tag{8.58}$$

where the mass matrix is

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{8.59}$$

If $m_1 = m_2$,

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi\,, \tag{8.60}$$

where ψ is a two-component vector.

Now consider global invariance $\psi \to U\psi$, where U is 2×2 unitary matrix ($U(2)$ symmetry). $\bar{\psi} \to \bar{\psi}U^\dagger \Rightarrow \bar{\psi}\psi$ is invariant.

Any complex number of modulus 1 can be written as $e^{i\theta}$, where θ is real. Any unitary matrix can be written as $U = e^{iH}$, where H is Hermitian $H^\dagger = H$. The most general 2×2 Hermitian matrix is

$$H = \theta I + \vec{\tau}\cdot\vec{a}\,, \tag{8.61}$$

where $\vec{\tau}$ are the Pauli matrices and a_1, a_2, a_3, θ are four real numbers. Therefore

$$U = e^{i\theta}e^{i\vec{\tau}\cdot\vec{a}}\,. \tag{8.62}$$

This is a $U(2) = U(1)\otimes SU(2)$ symmetry, we know how the first part behaves.

$$\psi \to e^{i\vec{\tau}\cdot\vec{a}}\psi \tag{8.63}$$

is a global $SU(2)$ transformation. The matrix $e^{i\vec{\tau}\cdot\vec{a}}$ has determinant 1. This implies the Lagrangian is invariant under a global $SU(2)$ gauge transformation.

Now consider local invariance: $\vec{a} = \vec{a}(x)$. Let

$$\vec{\lambda}(x) = -(\hbar c/q)\vec{a}(x)\,. \tag{8.64}$$

$\psi \to S\psi$, where $S \equiv e^{-iq\vec{\tau}\cdot\vec{\lambda}(x)/\hbar c}$ local $SU(2)$ transformation. $\mathcal{L}$ is not invariant so replace the derivative in $\mathcal{L}$ by covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + \frac{iq}{\hbar c}\vec{\tau}\cdot\vec{A}_\mu\,. \tag{8.65}$$

There are three gauge fields.

For $\mathcal{D}_\mu\psi \to S(\mathcal{D}_\mu\psi)$, $\vec{A}_\mu \to \vec{A}'_\mu$ where

$$\vec{\tau}\cdot\vec{A}'_\mu = S(\vec{\tau}\cdot\vec{A}_\mu)S^{-1} + i\left(\frac{\hbar c}{q}\right)(\partial_\mu S)S^{-1} \tag{8.66}$$

Proof

$$\mathcal{L}_F = i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi \tag{8.67}$$

$$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}\vec{\tau}\cdot\vec{A}_\mu \tag{8.68}$$

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\vec{\tau}\cdot\vec{A}\psi \tag{8.69}$$

$\mathcal{L} \to \mathcal{L}'$

$$\begin{aligned}
\mathcal{L}' &= i\hbar c(\bar{\psi}S^\dagger)\gamma^\mu\partial_\mu(S\psi) - mc^2(\bar{\psi}S^\dagger)(S\psi) - q(\bar{\psi}S^\dagger)\gamma^\mu\vec{\tau}\cdot\vec{A}'_\mu(S\psi) && (8.70)\\
&= i\hbar c\bar{\psi}S^\dagger\gamma^\mu(\partial_\mu S)\psi + i\hbar c\bar{\psi}S^\dagger\gamma^\mu S(\partial_\mu\psi) - mc^2\bar{\psi}\psi - q\bar{\psi}S^\dagger\gamma^\mu\vec{\tau}\cdot\vec{A}'_\mu S\psi && (8.71)\\
&= \mathcal{L} - q\psi\gamma^\mu\left[S^\dagger\vec{\tau}\cdot\vec{A}'_\mu S - \vec{\tau}\cdot\vec{A}_\mu - i\left(\frac{\hbar c}{q}\right)S^\dagger\partial_\mu S\right]\psi\,. && (8.72)
\end{aligned}$$

Therefore

$$S^\dagger\vec{\tau}\cdot\vec{A}'_\mu S = \vec{\tau}\cdot\vec{A}_\mu + i\left(\frac{\hbar c}{q}\right)S^\dagger\partial_\mu S \tag{8.73}$$

$$\vec{\tau}\cdot\vec{A}'_\mu = S(\vec{\tau}\cdot\vec{A}_\mu)S^{-1} + i\left(\frac{\hbar c}{q}\right)(\partial_\mu S)S^{-1} \tag{8.74}$$

End of proof.

Remember

$$S = \exp\left[-iq\vec{\tau}\cdot\vec{\lambda}(x)/(\hbar c)\right] \tag{8.75}$$

S and S^{-1} do not commute with $\vec{\tau}\cdot\vec{A}_\mu$

$$\Rightarrow \partial_\mu S \neq -i\left(\frac{q}{\hbar c}\right)(\vec{\tau}\cdot\partial_\mu\lambda)S\,. \tag{8.76}$$

S does not commute with $\vec{\tau}\cdot\partial_\mu\lambda$.

Consider case for small $|\vec{\lambda}|$

$$S \approx 1 - \frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}, \qquad S^{-1} \approx 1 + \frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}, \qquad \partial_\mu S \approx -\frac{iq}{\hbar c}\vec{\tau}\cdot(\partial_\mu\vec{\lambda}) \tag{8.77}$$

To lowest order in $\vec{\lambda}$

$$\vec{\tau}\cdot\vec{A}'_\mu \approx \vec{\tau}\cdot\vec{A}_\mu + \frac{iq}{\hbar c}[\vec{\tau}\cdot\vec{A}_\mu, \vec{\tau}\cdot\vec{\lambda}] + \vec{\tau}\cdot\partial_\mu\vec{\lambda} \tag{8.78}$$

$$\vec{A}'_\mu \approx \vec{A}_\mu + \partial_\mu\vec{\lambda} + \frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}_\mu) \tag{8.79}$$

Proof 2

$$
\begin{aligned}
\vec{\tau}\cdot\vec{A}'_\mu &= \left(1-\frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}\right)(\vec{\tau}\cdot\vec{A}_\mu)\left(1+\frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}\right)+i\left(\frac{\hbar c}{q}\right)\left(-\frac{iq}{\hbar c}\vec{\tau}\cdot(\partial_\mu\vec{\lambda})\right)\left(1+\frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}\right)\\
&\approx \vec{\tau}\cdot\vec{A}_\mu-\frac{iq}{\hbar c}(\vec{\tau}\cdot\vec{\lambda})(\vec{\tau}\cdot\vec{A}_\mu)+(\vec{\tau}\cdot\vec{A}_\mu)\left(\frac{iq}{\hbar c}\right)(\vec{\tau}\cdot\vec{\lambda})+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})\\
&= \vec{\tau}\cdot\vec{A}_\mu+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})+\frac{iq}{\hbar c}[\vec{\tau}\cdot\vec{A}_\mu,\vec{\tau}\cdot\vec{\lambda}]\\
&= \vec{\tau}\cdot\vec{A}_\mu+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})+\frac{iq}{\hbar c}A^i_\mu[\tau^i,\tau^j]\lambda^j\\
&= \vec{\tau}\cdot\vec{A}_\mu+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})+\frac{iq}{\hbar c}2i\epsilon_{ijk}\tau^k A^i_\mu\lambda^j\\
&= \vec{\tau}\cdot\vec{A}_\mu+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})-\frac{q}{\hbar c}(\vec{A}_\mu\times\vec{\lambda})\cdot\vec{\tau}\\
&= \vec{\tau}\cdot\vec{A}_\mu+\vec{\tau}\cdot(\partial_\mu\vec{\lambda})+\vec{\tau}\cdot\frac{q}{\hbar c}(\vec{\lambda}\times\vec{A}_\mu) \qquad (8.80)
\end{aligned}
$$

Therefore

$$\vec{A}'_\mu=\vec{A}_\mu+\partial_\mu\vec{\lambda}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}_\mu) \qquad (8.81)$$

End of proof.

$$
\begin{aligned}
\mathcal{L} &= i\hbar c\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi-mc^2\bar{\psi}\psi\\
&= [i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi-mc^2\bar{\psi}\psi]-(q\bar{\psi}\gamma^\mu\vec{\tau}\psi)\cdot\vec{A}_\mu \qquad (8.82)
\end{aligned}
$$

which is invariant under local gauge transformations. The cost is the introduction of three new gauge fields $\vec{A}^\mu=(A^\mu_1,A^\mu_2,A^\mu_3)$. S does not commute with $\vec{\tau}\cdot\partial_\mu\lambda$.

The free Lagrangian is

$$\mathcal{L}_A=\frac{-1}{16\pi}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}\,. \qquad (8.83)$$

Again the Proca mass term $\frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2\vec{A}^\mu\cdot\vec{A}_\mu$ must be excluded for local gauge invariance. In otherwords, the gauge fields must be massless.

Free Lagrangian is not invariant, so it also needs to be modified. Define

$$\vec{F}^{\mu\nu}\equiv\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu-\frac{2q}{\hbar c}(\vec{A}^\mu\times\vec{A}^\nu)\,. \qquad (8.84)$$

This is an anti-symmetric tensor $q\to 0\Rightarrow$ free Lagrangian. Under infinitesimal local gauge transformation

$$\vec{F}^{\mu\nu}\to\vec{F}^{\mu\nu}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{F}^{\mu\nu})\,. \qquad (8.85)$$

Proof of $\mathcal{L}$ Invariance

$$\begin{aligned}
\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu} &\to \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{2q}{\hbar c}\vec{F}^{\mu\nu}\cdot(\vec{\lambda}\times\vec{F}_{\mu\nu})+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{F}^{\mu\nu})\cdot\vec{F}_{\mu\nu} \\
&= \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{2q}{\hbar c}\vec{F}_{\mu\nu}\cdot(\vec{F}^{\mu\nu}\times\vec{\lambda})+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{F}^{\mu\nu})\cdot\vec{F}_{\mu\nu} \\
&= \vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}\,,
\end{aligned} \tag{8.86}$$

where $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$. $\Rightarrow \mathcal{L}_A$ is invariant under $SU(2)$ gauge transformation.

Proof of (8.85)

$$\begin{aligned}
\vec{F}^{\mu\nu}\to\vec{F}'^{\mu\nu} &= \partial^\mu\vec{A}'^\nu-\partial^\nu\vec{A}'^\mu-\frac{2q}{\hbar c}(\vec{A}'^\mu\times\vec{A}'^\nu) \\
&\approx \partial^\mu\left(\vec{A}^\nu+\partial^\nu\vec{\lambda}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}^\nu)\right)-\partial^\nu\left(\vec{A}^\mu+\partial^\mu\vec{\lambda}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}^\mu)\right) \\
&\quad -\frac{2q}{\hbar c}\left(\vec{A}^\mu+\partial^\mu\vec{\lambda}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}^\mu)\right)\times\left(\vec{A}^\nu+\partial^\nu\vec{\lambda}+\frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}^\nu)\right) \\
&\approx \partial^\mu\vec{A}^\nu+\partial^\mu\partial^\nu\vec{\lambda}+\frac{2q}{\hbar c}(\partial^\mu\vec{\lambda})\times\vec{A}^\nu+\frac{2q}{\hbar c}\vec{\lambda}\times(\partial^\mu\vec{A}^\nu) \\
&\quad -\partial^\nu\vec{A}^\mu-\partial^\nu\partial^\mu\vec{\lambda}-\frac{2q}{\hbar c}(\partial^\nu\vec{\lambda})\times\vec{A}^\mu-\frac{2q}{\hbar c}\vec{\lambda}\times(\partial^\nu\vec{A}^\mu) \\
&\quad -\frac{2q}{\hbar c}\Big[\vec{A}^\mu\times\vec{A}^\nu+(\partial^\mu\vec{\lambda})\times\vec{A}^\nu+\vec{A}^\mu\times(\partial^\nu\vec{\lambda}) \\
&\quad +\frac{2q}{\hbar c}\left(\vec{A}^\mu\times(\vec{\lambda}\times\vec{A}^\nu)+(\vec{\lambda}\times\vec{A}^\mu)\times\vec{A}^\nu\right)\Big]+\text{higher order terms} \\
&= \vec{F}^{\mu\nu}+\frac{2q}{\hbar c}\Big[\vec{\lambda}\times(\partial^\mu\vec{A}^\nu)-\vec{\lambda}\times(\partial^\nu\vec{A}^\mu) \\
&\quad +\frac{2q}{\hbar c}\left(\vec{A}^\mu\times(\vec{\lambda}\times\vec{A}^\nu)+(\lambda\times\vec{A}^\mu)\times\vec{A}^\nu\right)\Big] \\
&= \vec{F}^{\mu\nu}+\frac{2q}{\hbar c}\Big[\vec{\lambda}\times(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu) \\
&\quad +\frac{2q}{\hbar c}\left(\vec{\lambda}(\vec{A}^\mu\cdot\vec{A}^\nu)-\vec{A}^\nu(\vec{A}^\mu\cdot\vec{\lambda})-\vec{\lambda}(\vec{A}^\nu\cdot\vec{A}^\mu)+\vec{A}^\mu(\vec{A}^\nu\cdot\vec{\lambda})\right)\Big] \\
&= \vec{F}^{\mu\nu}+\frac{2q}{\hbar c}\left[\vec{\lambda}\times(\partial^\mu\vec{A}^\nu-\partial^\nu\vec{A}^\mu)+\frac{2q}{\hbar c}\left(\vec{\lambda}\times(\vec{A}^\mu\times\vec{A}^\nu)\right)\right] \\
&= \vec{F}^{\mu\nu}+\frac{2q}{\hbar c}\vec{\lambda}\times\vec{F}^{\mu\nu}\,,
\end{aligned} \tag{8.87}$$

where $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$ and $(\vec{a}\times\vec{b})\times\vec{c}=-\vec{c}\times(\vec{a}\times\vec{b})=-\vec{a}(\vec{c}\cdot\vec{b})+\vec{b}(\vec{c}\cdot\vec{a})$.

The complete Yang-Mills Lagrangian is

$$\mathcal{L}=[i\hbar c\bar{\psi}\gamma^\mu\partial_\mu\psi-mc^2\bar{\psi}\psi]-\frac{1}{16\pi}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}-(q\bar{\psi}\gamma^\mu\vec{\tau}\psi)\cdot\vec{A}_\mu\,. \tag{8.88}$$

It describes two equal-mass Dirac fields interacting with three massless vector gauge fields.

The Dirac fields generate three currents $\vec{J}^\mu \equiv cq(\bar{\psi}\gamma^\mu\vec{\tau}\psi)$, which acts as a source for the gauge fields.

The Lagrangian for gauge fields is (reminiscent of Maxwell Lagrangian)

$$\mathcal{L} = \frac{-1}{16\pi}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu} - \frac{1}{c}\vec{J}^\mu\cdot\vec{A}_\mu\,. \tag{8.89}$$

Yang-Mills is a non-Abelian gauge theory. Non-Abelian gauge theories can be applied to colour SU(3) symmetry in strong interactions and weak isospin-hypercharge $SU(2)\otimes U(1)$ symmetry in weak interactions.

8.5 Chromodynamics

Each quark flavour comes in three colours and has the same mass (all three colours have the same mass). The eight massless gauge fields are the gluons. Apply Yang-Mills theory $\Rightarrow$ description of strong interactions.

8.6 Feynman Rules

The Lagrangian we considered describe classical fields as well as quantum fields. The passage from classical field theory to quantum field theory does not change the Lagrangians or field equations. But we must reinterpret the field variables $\Rightarrow$ quantize the fields. Particles emerge as quant of the associated fields. Each Lagrangian determines a particular set of Feynman rules. The free Lagrangian gives the propagator. The interaction terms the vertex factors. Applying Euler-Lagrange equation to free Lagrangian yields the free field equation.

Klein-Gordon (spin-0)

$$\left[\partial^\mu\partial_\mu + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0 \tag{8.90}$$

Dirac (spin-1/2)

$$\left[i\gamma^\mu\partial_\mu - \left(\frac{mc}{\hbar}\right)\right]\psi = 0 \tag{8.91}$$

Proca (spin-1)

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0 \tag{8.92}$$

The momentum equations are ($p_\mu \leftrightarrow i\hbar\partial_\mu$)

$$\left[p^2 - (mc)^2\right]\phi = 0 \tag{8.93}$$

$$\left[\not{p} - (mc)\right]\psi = 0 \tag{8.94}$$

$$\left[\left(-p^2 + (mc)^2\right) g_{\mu\nu} + p_\mu p_\nu\right] A^\nu = 0 \tag{8.95}$$

Propagator is i times the inverse
spin-0

$$\frac{i}{p^2 - (mc)^2} \tag{8.96}$$

spin-1/2

$$\frac{i}{\not{p} - mc} = \frac{i(\not{p} + mc)}{p^2 - (mc)^2} \tag{8.97}$$

spin-1

$$\frac{-i}{p^2 - (mc)^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{(mc)^2}\right] \tag{8.98}$$

Must work with the photon separately.
Maxwell, massless spin-1

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \tag{8.99}$$

Lorentz condition

$$\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = 0 \tag{8.100}$$

In momentum space

$$(-p^2 g_{\mu\nu})A^\nu = 0 \Rightarrow \frac{-i g_{\mu\nu}}{p^2}\,, \tag{8.101}$$

which is the massless spin-1 propagator.

Vertex factors: write down $i\mathcal{L}_{int}$ in momentum space and examine the fields involved. These determine the qualitative structure of the interaction.

In QED,

$$i\mathcal{L}_{int} = -i(q\bar{\psi}\gamma^\mu\psi)A_\mu \tag{8.102}$$

Define vertex of three fields $\psi, \bar{\psi}, A_\mu$ Rub out the field variables.

QED vertex

$$-i\sqrt{\frac{4\pi}{\hbar c}}q\gamma^\mu = i g_e \gamma^\mu\,, \tag{8.103}$$

where the square-root piece is do the photon wavefunction normalization In QCD,

$$\mathcal{L}_{int} = -(q\bar{\psi}\gamma^\mu\vec{\lambda}\psi)\cdot\vec{A}_\mu \tag{8.104}$$

vertex factor

$$-i\frac{g_s}{2}\gamma^\mu\vec{\lambda} \tag{8.105}$$

where the 1/2 is a convention.

Also direct gluon-gluon couplings (3-gluon vertex and 4-gluon vertex).

8.7 The Mass Term

The gauge fields need to be massless for gauge invariance. This is good for EM and QCD, but what about weak interactions with massive gauge fields? We need to identify the mass term in Lagrangian.

Consider odd example

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + e^{-(\alpha\phi)^2} \\
&= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + 1 - \alpha^2\phi^2 + \frac{1}{2}\alpha^4\phi^4 - \frac{1}{6}\alpha^6\phi^6 + \dots\,, \qquad (8.106)
\end{aligned}$$

where α is real. The mass term in a scalar Lagrangian is $-(1/2)(mc/\hbar)^2$. The mass term in the above Lagrangian is second-oder in ϕ and hence $m = \sqrt{2}\alpha\hbar/c$. The last two terms are four-leg and six-leg couplings.

Consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda^2\phi^4\,, \tag{8.107}$$

where μ and λ are real constants. The first term has the wrong sign for a mass term.

Feynman calculus, treats fields as fluctuations about ground state. Ground state is minimum of potential

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \Rightarrow U(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda^2\phi^4\,. \tag{8.108}$$

Minimization gives

$$0 = -\mu^2\phi + \lambda^2\phi^3 \Rightarrow \mu^2 = \lambda^2\phi^2 \Rightarrow \phi = \pm\frac{\mu}{\lambda} \tag{8.109}$$

Feynman calculus must be formulated in terms of deviations from one or the other ground states. Introduce new field variable $\eta \equiv \phi \pm \mu/\lambda$. Therefore $\phi = \eta \mp \mu/\lambda$.

$$\begin{aligned}
\phi^2 &= \eta^2 \mp 2\left(\frac{\mu}{\lambda}\right)\eta + \left(\frac{\mu}{\lambda}\right)^2 \qquad (8.110) \\
\frac{1}{2}\mu^2\phi^2 &= \frac{1}{2}\mu^2\eta^2 \mp \left(\frac{\mu^3}{\lambda}\right)\eta + \frac{1}{2}\mu^2\left(\frac{\mu}{\lambda}\right)^2 \qquad (8.111)
\end{aligned}$$

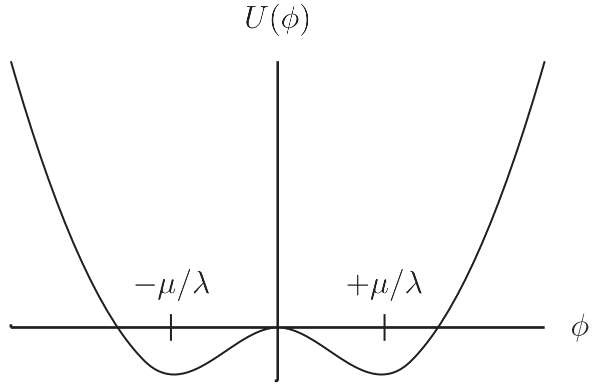


Figure 8.1: Higgs potential.

$$\phi^4 = \eta^4 + 4\left(\frac{\mu}{\lambda}\right)^2\eta^2 + \left(\frac{\mu}{\lambda}\right)^4 \mp 4\left(\frac{\mu}{\lambda}\right)\eta^3 + 2\left(\frac{\mu}{\lambda}\right)^2\eta^2 \mp 4\left(\frac{\mu}{\lambda}\right)^3\eta \quad (8.112)$$

$$-\frac{1}{4}\lambda^2\phi^4 = -\frac{1}{4}\lambda^2\eta^4 \pm \lambda\mu\eta^3 - \frac{3}{2}\mu^2\eta^2 \pm \left(\frac{\mu^3}{\lambda}\right)\eta - \frac{1}{4}\lambda^2\left(\frac{\mu}{\lambda}\right)^4 \quad (8.113)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \frac{1}{4}\lambda^2\eta^4 \pm \mu\lambda\eta^3 - \mu^2\eta^2 + \frac{1}{4}\left(\frac{\mu^2}{\lambda}\right)^2 . \quad (8.114)$$

The mass term with the correct sign is now $m = \sqrt{2}\mu\hbar/c$. There are also interaction terms. The constant term signifies nothing.

Both $\mathcal{L}$s represent the same physical system (just change in notation). The first version of $\mathcal{L}$ is not suitable for Feynman calculus (a perturbation series in ϕ would not converge because it is an expansion about an unstable point). Only in the second form can we read off the mass and vertex factors.

8.8 Spontaneous Symmetry-Breaking

Consider the last example. Invariant in $\phi \to -\phi$. Not invariant in η. The symmetry has been broken. The vacuum does not share the symmetry of the Lagrangain. The true symmetry is hidden by the arbitrary selection of a particular ground state. Spontaneous symmetry breaking $\to$ no external agent is responsible. Our example had two ground states (discrete symmetry). Now consider contineous symmetry example.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) + \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\mu^2(\phi_1^2 + \phi_2^2)^2 \quad (8.115)$$

is invariant under rotation in ϕ_1, ϕ_2 space (invariant under SO(2))

The potential is

$$\mathcal{U} = -\frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \frac{1}{4}\lambda^2(\phi_1^2 + \phi_2^2)^2 \,. \tag{8.116}$$

The minima lies on a circle or radius μ/λ: $\phi_{1min}^2 + \phi_{2min}^2 = \mu^2/\lambda^2$. We have to expand about a particular ground state (vacuum), we pick

$$\phi_{1,min} = \frac{\mu}{\lambda}, \quad \phi_{2,min} = 0 \,. \tag{8.117}$$

Introduce new fields which are fluctuations about vacuum.

$$\eta \equiv \phi_1 - \frac{\mu}{\lambda}, \quad \zeta \equiv \phi_2 \tag{8.118}$$

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu\zeta)(\partial^\mu\zeta)\right] + \left[\mu\lambda(\eta^3 + \eta\zeta^2) - \frac{\lambda^2}{4}\left(\eta^4 + \zeta^4 + 2\eta^2\zeta^2\right)\right] + \frac{\mu^4}{4\lambda^2} \tag{8.119}$$

The second term gives the mass, the last term is constant. There are five interaction terms.

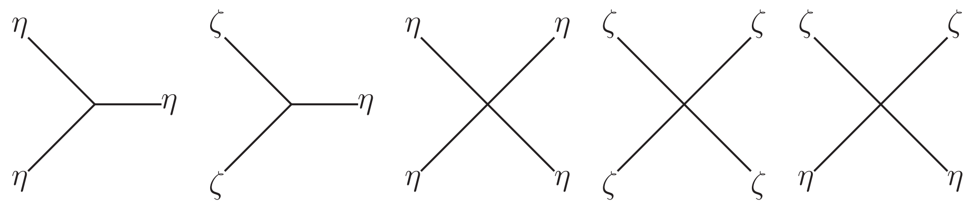


Figure 8.2: Higgs interactions.

This gives $m_\eta = \sqrt{2}\mu\hbar/c$ and $m_\zeta = 0$.

The Lagrangian does not look symmetrical. The symmetry has been broken ("hidden") by selecting a particular vacuum state. One of the fields is automatically massless. This is called Goldstone's theorm. Spontaneous breaking of a continous global symmetry is always accompanied by the appearance of one or more massless scalar (spin-0) particles (Goldstone bosons). This can not exist experimentally since any light scalar would have been seen by now.

Now apply spontaneous symmetry-breaking to local gauge invarinace

8.9 The Higgs Mechanism

Combine two real fields ϕ_1, ϕ_2 into single complex field

$$\phi \equiv \phi_1 + i\phi_2 \tag{8.120}$$

so that

$$\phi_1 = \frac{1}{2}(\phi^* + \phi), \quad \phi_2 = \frac{i}{2}(\phi^* - \phi) \tag{8.121}$$

and $\phi^*\phi = \phi_1^2 + \phi_2^2$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) + \frac{1}{2}\mu^2(\phi^*\phi) - \frac{1}{4}\lambda^2(\phi^*\phi)^2 \tag{8.122}$$

The SO(2) rotational symmetry is spontaneously broken. SO(2) symmetry becomes U(1) symmetry. Invariant under U(1) phase transformation $\phi \to e^{i\theta}\phi$. For local gauge transformation $\phi \to e^{i\theta(x)}\phi$, $\mathcal{D}_\mu = \partial_\mu + (iq/\hbar c)A_\mu$.

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\phi^*\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right)\phi\right] + \frac{1}{2}\mu^2(\phi^*\phi) - \frac{1}{4}\lambda^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} \tag{8.123}$$

Define $\eta \equiv \phi_1 - \mu/\lambda$, $\zeta \equiv \phi_2$

$$\begin{aligned} \mathcal{L} &= \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \zeta)(\partial^\mu \zeta)\right] \\ &\quad + \left[-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right] - 2i\left(\frac{\mu}{\lambda}\frac{q}{\hbar c}\right)(\partial_\mu \zeta)A^\mu \\ &\quad + \{10 \text{ interaction terms}\} + \left(\frac{\mu^2}{2\lambda}\right)^2 \end{aligned} \tag{8.124}$$

ζ is a Goldstone boson, The term in front of A^2 gives the mass $(1/8)\pi(mc/\hbar)^2 \Rightarrow m_A^2 = 4\pi\left(\frac{q\mu}{\lambda c}\right)^2$. We obtain $m_\eta = \sqrt{2}\mu\hbar/c, m_\zeta = 0$. The field A^μ has acquired a mass

$$m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda c}\right). \tag{8.125}$$

There is an extra term in Lagrangian:

$$-2i\left(\frac{\mu}{\lambda}\frac{q}{\hbar c}\right)(\partial_\mu \zeta)A^\mu. \tag{8.126}$$

Because of the i, we need to transform ζ away

$$\begin{aligned} \phi \to \phi' &= (\cos\theta + i\sin\theta)(\phi_1 + i\phi_2) \\ &= (\phi_1\cos\theta - \phi_2\sin\theta) + i(\phi_1\sin\theta + \phi_2\cos\theta). \end{aligned} \tag{8.127}$$

pick $\theta = -\tan^{-1}(\phi_2/\phi_1) \Rightarrow \phi'$ real, $\phi' = \phi_1' + i\phi_2' \Rightarrow \phi_2' = 0$. By a choice of gauge, we eliminate the Goldstone boson

$$
\begin{aligned}
\mathcal{L} &= \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2\right] + \left[-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right] \\
&\quad + \{4 \text{ interaction terms}\} + \left(\frac{\mu^2}{2\lambda}\right)^2 . \qquad (8.128)
\end{aligned}
$$

We are left with a single massive scalar η (Higgs particle) and massive gauge field A^μ. We have sacrificed manifest symmetry in favour of a notation which makes the physical content transparent for Feynman rules. A massless vector field has two degrees of freedom. When A^μ acquires mass, it picks up a 3'rd degree of freedom. The extra degree of freedom comes from the Goldstone boson, which disappeared from the theory. This is the Higgs mechanism $\rightarrow$ gauge invariance + spontanous symmetry breaking

8.9.1 Standard Model

The Higgs mechanism is responsible for the masses of weak interaction gauge bosons ($W^\pm$, Z^0). The details are still a matter of speculation (Higgs not found and potential not known). We believe all fundamental interactions - strong, EM, weak can be described by local gauge theories.