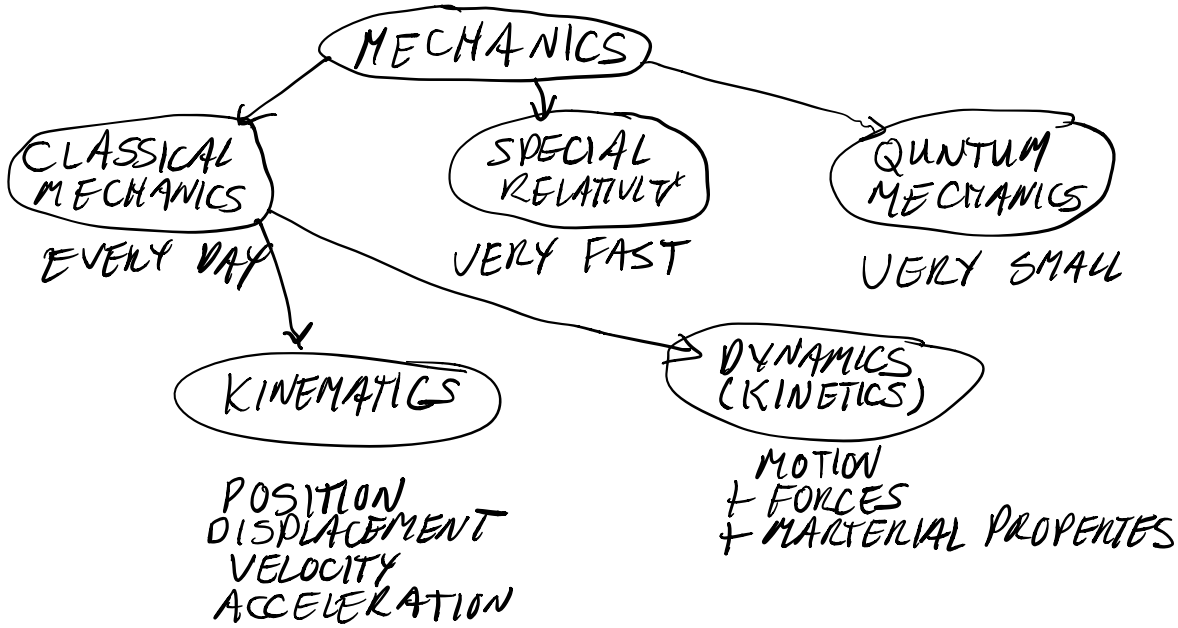
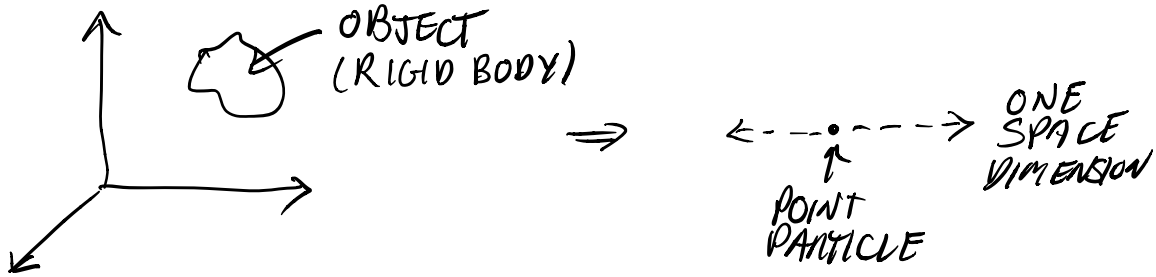
PRELIMINARIES

FIELD OF STUDY

MECHANICS - MOTION OF OBJECTS AND FORCES ON OBJECT



CLASSICAL WORLD



RIGID BODY ≡ COLLECTION OF PARTICLES THAT DO NOT CHANGE THEIR RELATIVE POSITIONS UNDER THE ACTION OF APPLIED FORCES

FUNDAMENTAL QUANTITIES

LENGTH - USED TO DESCRIBE PARTICLES POSITION

TIME - USED TO DESCRIBE AN EVENT

MASS - PROPERTY OF MATTER (AMOUNT OF MATTER AN OBJECT HAS)

FORCE - INTERACTION BETWEEN 2 OBJECTS (PUSH OR PULL)

DIMENSIONAL ANALYSIS

$$\left.\begin{array}{l}[L] = \text{LENGTH} \\ [T] = \text{TIME} \\ [M] = \text{MASS}\end{array}\right\} \text{FUNDAMENTAL DIMENSION}$$

COMBINATIONS OF FUNDAMENTAL DIMENSION

e.g. SPEED $\equiv \frac{[L]}{[T]}$

DIMENSIONAL HOMOGENEITY → CHECK CONSISTENCY OF DIMENSIONS IN A MATHEMATICAL EXPRESSION.

- SAME DIMENSIONS IN A RATIO CANCE
- NUMERICAL COEFFICICENTS HAVE NO DIMENSION (IGNORE)
- CAN NOT BE USED TO IDENTIFY CORRECT EXPRESS (RELATIONSHIP).

UNITS

LENGTH, TIME BASIC UNITS

MASS, FORE HAVE DERIVED UNIT DEPENDING ON SYSTEM OF UNITS

SI UNITS : LENGTH, meter (m) ; TIME, second (s)
MASS, kilograms (kg)
→ FORCE, newton (N) IS DERIVED.

$g = 9.81\ m/s^2$

FPS UNITS : LENGTH, feet (ft) ; TIME, second (s)
FORCE, pound (lb)
→ MASS, slug (slug) IS DERIVED

$g = 32.2\ ft/s^2$

WORKING PROBLEMS

PHYSICS BEST LEARNT BY SOLVING PROBLEMS

MULTI-STEP APPROACH

1) TURN WORDS (REAL LIFE IDEA) INTO DIAGRAM(S) AND AVAILABLE DATA.

DIAGRAM - OFTEN NEED COORDINATE SYSTEM
- CHOICE IS NOT UNIQUE
- ONCE PICKED STICK WITH IT (DO NOT MIX)

DATA - DETERMINE WHAT IS KNOWN
- DETERMINE WHAT NEEDS TO BE DETERMINED.

2) RELATING PHYSICAL SITUATION TO THEORETICAL TOPIC.
- WRITE DOWN PHYSICAL EQUATION(S) THAT WILL BE USED.

3) SOLVE THE PROLEM AS GENERAL AS POSSIBLE
- TRY TO SUBSTITUTE NUMERS ONLY AT END.

4) DOES THE ANSWER MAKE SENSE?

STEP 2) IS DIFFICULT; OTHER STEPS ARE ALGORITMIC
STEP 2) COMES FROM PRACTICE
STEP 4) COMES FROM EXPERIENCE
STEPS 2), 3): CHECK WITH DIMENSIONAL HOMOGENEITY
UNITS. - CONVERT ALL TO BASE UNITS
- CAN DROP FROM INTERMEDIATE RESULT

SIGNIFICANT FIGURES
- DO NOT OVERSAFE RESULTS
- USE EXTRA DIGITS IN INTERMEDIATE RESULTS
- USE CALCULATOR IN ONE GO

USE EXPONENTIAL NOTATION

KINEMATICS OF A PARTICLE

RECTILINEAR KINEMATICS: CONTINOUS MOTION

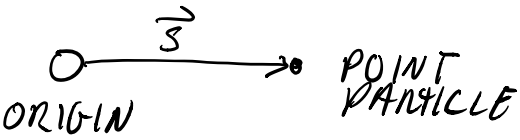
FIRST CONSIDER SIMPLEST MOTION OF SIMPLEST OBJECT.

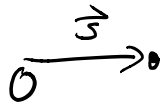
- MOTION RECTILINEAR OR ALONG STRAIGHT LINE
- OBJECT REPRESENTED BY POINT PARTICLE
  - CAN APPROXIMATE BY POINT
  - SIZE NOT IMPORTANT
  - ONLY ONE DEGREE OF FREEDOM (MASS)
  - LOCATION AT OBJECT MASS CENTER
  - ROTATION UNIMPORTANT

MOTION OF A POINT PARTICLE: POSITION $\vec{s}$
VELOCITY $\vec{v}$
ACCELERATION $\vec{a}$

THESE QUANTITIES ARE NORMALLY FUNCTIONS OF TIME $(\vec{s}, \vec{v}, \vec{a}; t)$

1 DIMENSIONAL POSITION

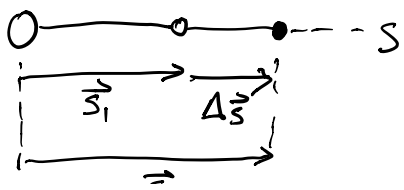




$\vec{s} = +s$

$\vec{s} = -s$

DISPLACEMENT = CHANGE IN POSITION



$$\Delta\vec{s} = \vec{s}_2 - \vec{s}_1$$

FINAL INITIAL

>2

$\Delta s = s_2 - s_1$ IN 1 DIMENSION

DISPLACEMENT IS VECTOR

DISTANCE d IS SCALAR = TOTAL LENGTH PARTICLES TRAVELS (ALWAY POSITIVE)

FOR DISPLACEMENT $\Delta\vec{s} = \vec{s}_2 - \vec{s}_1$ IN TIME INTERVAL

$\Delta t = t_2 - t_1$; Δt IS POSITIVE

DEFINE AVERAGE VELOCITY

$$\langle \vec{v} \rangle \equiv \frac{\vec{s}_2 - \vec{s}_1}{t_2 - t_1} = \frac{\Delta\vec{s}}{\Delta t}$$

INSTANTANEOUS VELOCITY $\vec{v} = \frac{d\vec{s}}{dt}$

UNITS: m/s , ft/s

Δt IS ALWAYS POSITIVE , SIGNE $\vec{v}$ SAME AS $\Delta\vec{s}$

VELOCITY SAME SIGN AS DISPLACEMENT.

MAGNITUDE OF VELOCITY IS SPEED $|\vec{v}|$

IN 1-D SPEED = UNSIGNED VELOCITY

AVERAGE SPEED $\langle u \rangle = \frac{d}{\Delta t}$ ← TOTAL DISTANCE TRAVELLED.

AVERAGE ACCELERATION $\langle \vec{a} \rangle = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}$.

INSTANTANEOUS ACCELERATION $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{s}}{dt}\right) = \frac{d^2\vec{s}}{dt^2}$

UNITS: m/s^2 ; ft/s^2

ACCELERATION IS VECTOR IN SAME DIRECTION AS CHANGE IN VELOCITY

IF VELOCITY DECREASING $\Rightarrow$ DECELERATION, $\vec{a}$ NEGATIVE

IF VELOCITY IS CONSTANT $\Rightarrow \vec{a} = 0$

FROM $\vec{v} = \frac{d\vec{s}}{dt}$ AND $\vec{a} = \frac{d\vec{v}}{dt}$ (s, v, a, t)

(s, v, t) (v, a, t)

$dt = \frac{ds}{v}$ $dt = \frac{dv}{a}$

$$\frac{ds}{v} = \frac{dv}{a}$$

$$a\,ds = v\,dv$$ NOT INDEPENDENT

(s, v, a)

THE EQUATION THAT ELIMINATES v IS $a = \frac{d^2s}{dt^2}$

IN GENERAL $\vec{a} = \vec{a}(s, t)$

CONSIDER SPECIAL CASE $\vec{a}$ = CONSTANT

TAKE $t = t_0 = 0$, $s(t=0) = s_0$, $v(t=0) = v_0$

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt$$

$$\int_{v_0}^{v} dv = \int_0^t a\,dt = a\int_0^t dt$$

$$v - v_0 = a(t - 0) \Rightarrow v = v_0 + at$$

$$v = \frac{ds}{dt} \Rightarrow ds = v\,dt$$

$$\int_{s_0}^{s} ds = \int_0^t v\,dt = \int_0^t (v_0 + at)\,dt = v_0\int_0^t dt + a\int_0^t t\,dt$$

$$s - s_0 = v_0 t + \frac{1}{2}at^2 \Rightarrow s = s_0 + v_0 t + \frac{1}{2}at^2$$

$$v\,dv = a\,ds \Rightarrow \int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a\,ds = a\int_{s_0}^{s} ds$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(s-s_0) \Rightarrow v^2 = v_0^2 + 2a(s-s_0)$$

CAN ALSO OBTAIN THIS RESULT ALGIBRACALLY BY ELININATING t IN THE FIRST TWO EQUATIONS.

IF $a = a(t)$ OR $a = a(s)$ KNOWN, CAN STILL INTEGRATE

7 VARIABLES $(t, t_0, s, s_0, v, v_0, a)$ WE TOOK $t_0 = 0$

OR 3 VARIABLES (t, s, v) + 4 INITAL CONDITIONS (t_0, s_0, v_0, a)

2 INDEPENDENT EQUATIONS IN 3 VARIABLES

$\Rightarrow$ ONLY SINGLE RELATIONSHIP (AND INITIAL CONDITIONS) TO SOLVE PROBLEM COMPLETLY

OTHER EQUATIONS CAN BE OBTAINED FROM THE 2 INDEPENDENT EQUATIONS:

ELIMINATE v_0 $\Rightarrow s = s_0 + vt - \frac{1}{2}at^2$

ELIMINATE a $\Rightarrow s = s_0 + \frac{1}{2}(v_0 + v)t$

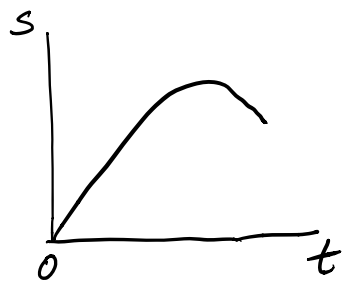
COMMON USE CASE IS OBJECT IN FREE-FALL AT EARTH'S SURFACE, $a = -g$ CONSTANT.

RECTILINEAR KINEMATICS : ERRATIC MOTION

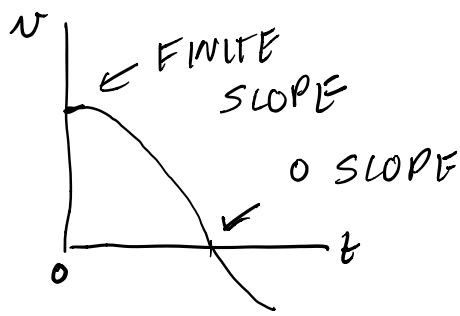
PREVIOUS ALGEBRAIC EQUATIONS ARE FOR CONSTANT ACCELERATION.

IF WE HAVE FUNCTIONAL FORMS, WE CAN INTEGRATE DIFFERENTIAL EQUATIONS

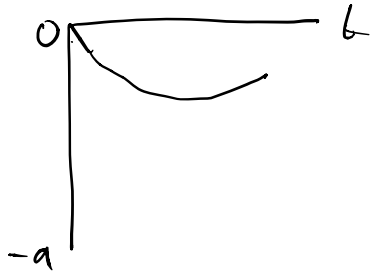
IF WE HAVE DATA, WE CAN PLOT IT.



$v = \frac{ds}{dt}$ SLOPE OF s-t CURVE IS VELOCITY



$a = \frac{dv}{dt}$ SLOPE OF v-t CURVE IS ACCELERATION



INTEGRAL IS AREA UNDER CURVE.

$\Delta v = \int a\,dt$; $v = v_0 + \Delta v$

$\Delta s = \int v\,dt$; $s = s_0 + \Delta s$

THESE ARE ALL t-AXIS, IF s-AXIS

$$v\,dv = a\,ds$$

$$\Rightarrow \frac{1}{2}(v_2^2 - v_1^2) = \int_{s_1}^{s_2} a\,ds$$

$$\Rightarrow v_2 = \sqrt{2\int a\,ds + v_1^2}$$

GENERAL CURVILINEAR MOTION

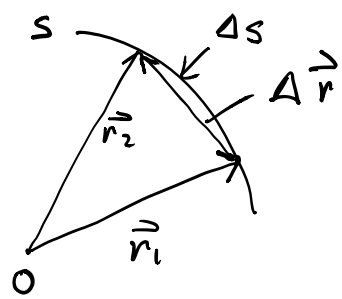
SO FAR DEAL WITH PARTICLE TRAVELING IN STRAIGHT LINES.

CONSIDER CURVED PATH IN PLANE

$\rightarrow$ CURVILINEAR MOTION

VECTOR CHANGE IN MAGNITUDE AND DIRECTION.

ORIGIN IS NOT ON THE LINE IN WHICH THE PARTICLE MOVES.



POSITION OF PARTICLE

$\vec{r} = \vec{r}(t)$

$\vec{r}$ CHANGES IN MAGNITUDE OR DIRECTION (OR BOTH)

$\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}$

FOR CIRCLE $|\vec{r}_1| = |\vec{r}_2|$

AVERAGE VELOCITY $\langle \vec{v} \rangle = \frac{\Delta \vec{r}}{\Delta t}$

INSTANTANEOUS VELOCITY $\vec{v}(t) = \frac{d\vec{r}}{dt}$ (TANGENT TO PATH)

SPEED IS MAGNITUDE OF VELOCITY $|\vec{v}|$

$\Delta r \rightarrow \Delta s$, $v = \frac{ds}{dt}$

$\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$

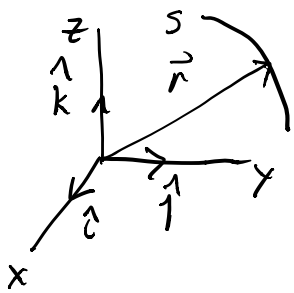
AVERAGE ACCLERATION $\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t}$

INSTANTANEOUS ACCELERATION $\vec{a}(t)$

$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\frac{d\vec{r}}{dt} = \frac{d^2\vec{r}}{dt^2}$ (NOT TANGENT TO PATH)

VELOCITY CAN CHANGE IN MAGNITUDE OR DIRECTION (OR BOTH)

CURVILINEAR MOTION: RECTANGULAR COMPONENTS



$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ UNIT VECTOR

MAGNITUDE $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

DIRECTION $\hat{r} = \frac{\vec{r}}{r}$

CONSIDER TIME DIRIVATIVE OF A COMPONENT

$$\frac{d}{dt}(x\hat{i}) = \frac{dx}{dt}\hat{i} + x\underbrace{\frac{d\hat{i}}{dt}}_{0} = \frac{dx}{dt}\hat{i}$$

0 FOR FIXED FRAME OF REFERENCE

VELOCITY $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$

$= \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}$

$= v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k}$

ACCELERATION $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \dot{v}_x(t)\hat{i} + \dot{v}_y(t)\hat{j} + \dot{v}_z(t)\hat{k}$

$= \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j} + \ddot{z}(t)\hat{k}$

$= a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k}$

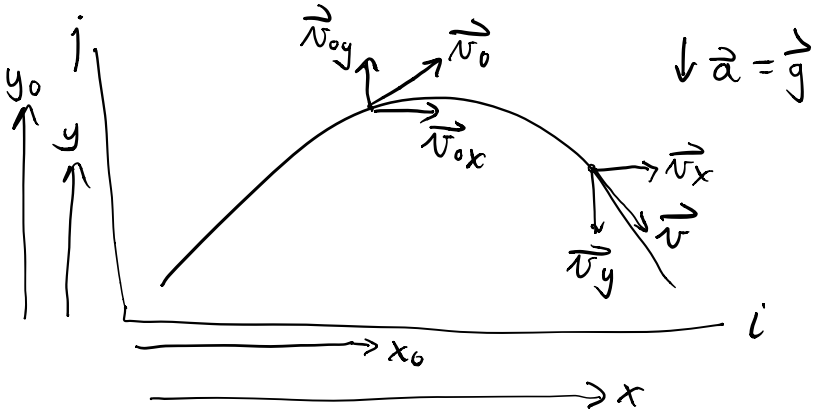
RELATIVE CHANGE IN COMPONETS GIVES CHANGE IN DIRECTION

MOTION OF A PROJECTILE

CAN CONSIDER MOTION IN x-y PLANE ONLY.

HORIZONTAL AND VERTICAL MOTION ARE INDEPENDENT.

COMMON TO TAKE $a_y = -g$; $a_x = 0$ (CONSTANT VELOCITY)



$a_x = 0$

$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \Rightarrow x = x_0 + v_{0x} t$

$v_x = v_{0x} + a_x t \Rightarrow v_x = v_{0x}$

$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \Rightarrow v_x^2 = v_{0x}^2 \Rightarrow v_x = \pm v_{0x}$

$v_x = v_{0x}$

$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \Rightarrow y = y_0 + v_{0y} t - \frac{1}{2} g t^2$

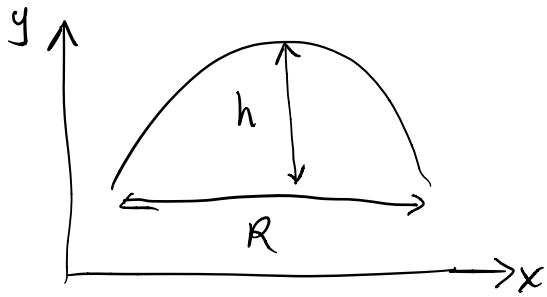
$v_y = v_{0y} + a_y t \Rightarrow v_y = v_{0y} - gt$

$v_y^2 = v_{0y}^2 + 2a_y(y - y_0) \Rightarrow v_y^2 = v_{0y}^2 - 2g(y - y_0)$

ONLY 3 OF THE EQUATIONS ARE INDEPENDENT.

OFTEN WRITE $v_{0x} = v_0 \cos\theta_0$

$v_{0y} = v_0 \sin\theta_0$



$$x = x_0 + v_{0x} t$$

$$y = y_0 + v_{0y} t - \frac{1}{2} g t^2$$

$$v_x = v_0 \cos\theta_0$$

$$v_y = v_0 \sin\theta_0$$

$$\sin\theta_0 \cos\theta_0 = \frac{\sin 2\theta_0}{2}$$

TRAJECTORY $y(x)$

$$t = \frac{x - x_0}{v_{0x}} \Rightarrow y - y_0 = v_{0y} \frac{(x - x_0)}{v_{0x}} - \frac{1}{2} g \frac{(x - x_0)^2}{v_{0x}^2}$$

$$v_{0x}^2 (y - y_0) = v_{0x} v_{0y} (x - x_0) - \frac{1}{2} g (x - x_0)^2$$

RANGE: $R \equiv (x - x_0)$ WHEN $y = y_0$

$$0 = v_{0x} v_{0y} R - \frac{1}{2} g R^2 \Rightarrow 0 = v_{0x} v_{0y} - \frac{1}{2} g R$$

$$R = \frac{2 v_{0x} v_{0y}}{g} = \frac{v_0^2}{g} \sin 2\theta_0$$

HEIGHT: $h \equiv (y - y_0)$ WHEN $\frac{dy}{dx} = 0$

$$v_{0x}^2 \frac{d(y - y_0)}{d(x - x_0)} = v_{0x} v_{0y} - g(x - x_0) = 0$$

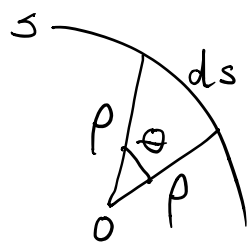
$$\therefore x - x_0 = \frac{v_{0x} v_{0y}}{g} = \frac{v_0^2}{2g} \sin 2\theta_0 = \frac{R}{2}$$

$$v_{0x}^2 h = v_{0x} v_{0y} \left(\frac{v_{0x} v_{0y}}{g} \right) - \frac{1}{2} g \left(\frac{v_{0x} v_{0y}}{g} \right)^2$$

$$h = \frac{v_{0y}^2}{g} - \frac{1}{2} \frac{v_{0y}^2}{g} = \frac{v_{0y}^2}{2g} = \frac{v_0^2}{2g} \sin^2\theta_0$$

CURVILINEAR MOTION : NORMAL AND TANGENTIAL

IF THE PATH IS KNOWN, WE CAN USE A CO-MOVING COORDINATE SYSTEM



$$\theta = \frac{s}{\rho} \quad \text{OR} \quad s = \rho\theta$$

$$v = \frac{ds}{dt} = \rho\frac{d\theta}{dt} \quad \text{FOR } \rho \text{ CONSTANT.}$$

$$\hat{u}_N = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\hat{u}_T = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d\hat{u}_N}{dt} = [\sin\theta\,\hat{i} - \cos\theta\,\hat{j}]\dot{\theta} = -\dot{\theta}\,\hat{u}_T$$

$$\frac{d\hat{u}_T}{dt} = [-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}]\dot{\theta} = \dot{\theta}\,\hat{u}_N$$

POSITION: $\vec{r} = -\rho\hat{u}_N$

VELOCITY: $\vec{v} = v\hat{u}_T = \frac{ds}{dt}\hat{u}_T = \rho\frac{d\theta}{dt}\hat{u}_T$

ACCELERATION: $\vec{a} = \dot{v}\hat{u}_T + v\dot{\hat{u}}_T = \dot{v}\hat{u}_T + v\dot{\theta}\hat{u}_N$

$$= \vec{a}_T + \vec{a}_N$$

$$\vec{a}_T = \dot{v}\hat{u}_T \; ; \; \vec{a}_N = v\dot{\theta}\hat{u}_N = \frac{v^2}{\rho}\hat{u}_N$$

VERY GENERAL: $\rho \to \infty \Rightarrow$ RECTILINEAR MOTION

$$\vec{a} \to \dot{v}\hat{u}_T \quad \text{OR} \quad a_T = \dot{v}$$

SPECIAL CASE: $\dot{v} = 0 \Rightarrow$ CONSTANT SPEED ALONG CURVE

$\vec{a} = \frac{v^2}{\rho}\hat{u}_N$ OR $a_N = \frac{v^2}{\rho}$ CENTRIPETAL ACCELERATION

ACCLERATION DUE TO CHANGE IN DIRECTION ONLY

FOR THE TANGENTIAL COMPONENT

$v = \frac{ds}{dt}$ AND $a_T = \frac{dv}{dt} \Rightarrow v\,dv = a_T\,ds$

IF a_T CONSTANT WE CAN WRITE ALGEBRAIC KINEMATIC EQUATIONS LIKE BEFORE.

IF $y = f(x)$ IS PATH, RADIUS OF CURVATURE ρ

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

$\hat{u}_T$ IS UNIQUE

$\hat{u}_N \perp \hat{u}_T$ BUT OTHERWISE FIXED BY CONVENTION

$\hat{u}_N$ ALWAYS ON CONCAVE SIDE OF CURVE POINTING INWARD

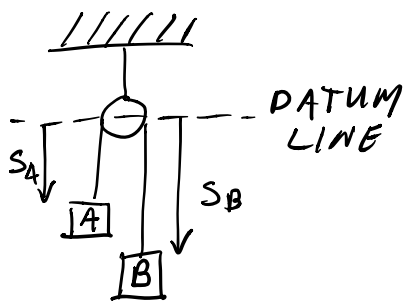
$\hat{u}_T, \hat{u}_N$ FORM OSCULATING PLANE (PLANE OF MOTION).

$\hat{u}_B$ BINORMAL AXIS $\hat{u}_B = \hat{u}_T \times \hat{u}_N$

ABSOLUTE DEPENDENT MOTION ANAYSIS OF TWO PARTICLE

IF 2 OBJECTS ARE CONNECTED
$\Rightarrow$ CONSTRAINT IN MOTION

CONSIDER ROPE OF FIXED LENGTH.



$$s_A + s_B = s_T \leftarrow \text{FIXED.}$$

$$\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0$$

$$\Rightarrow v_A = -v_B$$

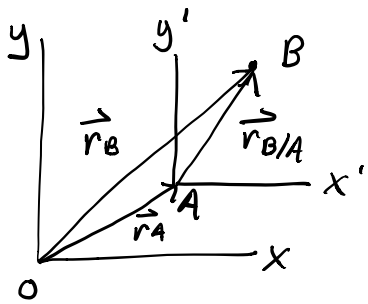
$$a_A = -a_B$$

DO NOT NEED TO INCLUDE FIXED LENGTHS OF ROPE
COORDINATE AXIS NEED TO BE ALONG
DIRECTION OF MOTION
DIRECTION FROM DATUM TO OBEJECT.

RELATIVE-MOTION OF TWO PARTICLES USING TRANSLATING AXIS

USUALLY PARTICLES MOTION DESCRIBED WITH RESPECT TO A COMMON COORDINATE SYSTEM.

IN CLASSICAL CASE → GALILEAN RELATIVITY



PRIME FRAME TRANSLATES WITH RESPECT TO UNPRIME FRAME

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

IF $\vec{a}_A = 0 \Rightarrow$ INERTIAL OR NEWTONIAN REFERENCE FRAME

KINETICS OF A PARTICLE : FORCE AND ACCELERATION

NEWTON'S SECOND LAW OF MOTION

A PARTICLE WILL ACCELERATE IN DIRECTION OF NET FORCE ACTING ON IT WITH MAGNITUDE PROPORTIONAL TO NET FORCE

$$\vec{a} \propto \vec{F} \Rightarrow \vec{F} = m\vec{a}$$

PROPORTIONALITY CONSTANT (SCALAR MASS)

m MEASURES RESISTANCE TO CHANGE. (INERTIA OF PARTICLE)

$\vec{F} = m\vec{a}$ CALLED NEWTON'S 2ND LAW

EQUATION OF MOTION

$F = ma$.

EXAMPLE FORCE : $F = G\dfrac{m_1 m_2}{r^2}$

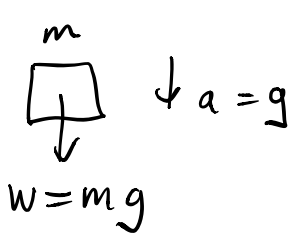
TWO PARTICLE MASS

CONSTANT

DISTANCE BETWEEN MASS CENTRES

FOR PARTICLE NEAR SURFACE OF EARTH

$r = R_E$, $m_2 = M_E$, $m_1 \equiv m \Rightarrow F = G\dfrac{M_E}{R_E^2} m$



$$W = gm$$

1 F m [kg] → F [N] (N ≡ NEWTON)

1 F F [lb] → m [slug]

EQUATION OF MOTION

$$\sum_i \vec{F} = m\vec{a}$$

NET FORCE

IF $\sum_i \vec{F} = 0 \Rightarrow$ STATIC EQUILIBRUM
(NEWTON'S FIRST LAW)

THERE CAN STILL BE FORCES ACTING ON PARTICLE
IN CASE $\vec{a} = 0$ (ASSUMING PARTICLES WITH MASS)
$\Rightarrow \vec{v}$ = CONSTANT (OR ZERO)

DRAW FREE-BODY DIAGRAM
DRAW KINEMATIC DIAGRAM

$\vec{F} = m\vec{a}$ LINK DYNAMICS TO KINEMATICS
(CAUSE GIVES EFFECT)
APPLY IN NEWTONIAN REFERENCE FRAME

EQUATION OF MOTION FOR A SYSTEM OF PARTICLES

FOR ith PARTICLE $\quad \vec{F}_i + \vec{f}_i = m_i\vec{a}_i$

EXTERNAL INTERNAL

SUM OVER PARTICLES $\quad \sum_i \vec{F}_i + \sum_i \vec{f}_i = \sum_i m_i \vec{a}_i$

$$\sum_i \vec{f}_i = 0 \Rightarrow \sum_i \vec{F}_i = \sum_i m_i \vec{a}_i$$

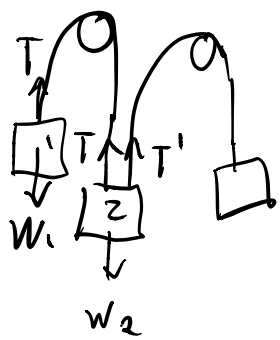
$$\left.\begin{array}{l} \text{CENTRE OF MASS } m_G = \sum_i m_i \\ \text{POSITION OF CENTRE OF MASS } \vec{r}_G \end{array}\right\} m_G \vec{r}_G = \sum_i m_i \vec{r}_i$$

$m_G \vec{a}_G = \sum_i m_i \vec{a}_i \Rightarrow \sum_i F_i = m_G \vec{a}_G$

FORCES: GRAVITY, ROPES, SPRINGS,
NORMAL FORCE, FRICTION.

GRAVITY: ALWAY THERE, ACTION AT DISTANCE

ROPES: TENSION T, CHANGE DIRECTION



SPINGS: HOOKE'S LAW: $\vec{F} = -k\Delta\vec{s}$
FORCE PROPORTIONAL TO LENGTH OF
COMPRESSION OR EXTENSION
COMPRESSED → PUSH
EXTENDED → PULL

2 BODIES IN CONTACT: NORMAL TO SURFACE, NORMAL
PARALLEL TO SURFACE, FRICTION

NORMAL DYNAMIC (CHANGE)

FRICTION AGAINST MOTION

KINETIC FRICTION - IF BODIES IN RELATIVE MOTION

$$f_K = -\mu_K N$$

STATIC FRICTION - IF BODIES ON VERGE OF RELATIVE
MOTION

$$f_s = -\mu_s N$$

EQUATION OF MOTION : RECTANGULAR COORDINATES

$$\sum F_x \hat{i} + \sum F_y \hat{j} + \sum F_z \hat{k} = m(a_x \hat{i} + a_y \hat{j} + a_z \hat{k})$$

$$\sum F_x = m a_x$$

$$\sum F_y = m a_y$$

$$\sum F_z = m a_z$$

KINETICS OF A PARTICLE : FORCE AND ACCELERATION

NEWTON'S SECOND LAW OF MOTION

A PARTICLE WILL ACCELERATE IN DIRECTION OF NET FORCE ACTING ON IT WITH MAGNITUDE PROPORTIONAL TO NET FORCE

$$\vec{a} \propto \vec{F} \Rightarrow \vec{F} = m\vec{a}$$

PROPORTIONALITY CONSTANT (SCALAR MASS)

m MEASURES RESISTANCE TO CHANGE. (INERTIA OF PARTICLE)

$\vec{F} = m\vec{a}$ CALLED NEWTON'S 2ND LAW

EQUATION OF MOTION

$F = ma$.

EXAMPLE FORCE : $F = G\dfrac{m_1 m_2}{r^2}$

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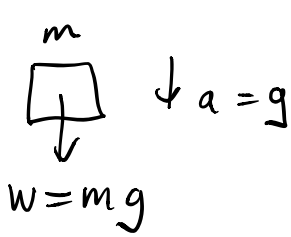

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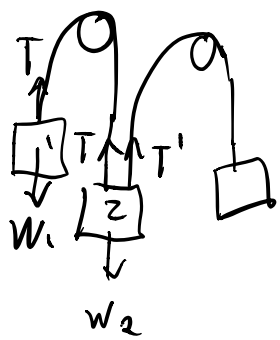

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$$\sum F_y = m a_y$$

$$\sum F_z = m a_z$$

EQUATION OF MOTION: NORMAL AND TANGENTIAL COORDINATES

USEFUL WHEN CURVED PATH KNOWN.

$$\sum F_T \hat{u}_T + \sum F_N \hat{u}_N + \sum F_B \hat{u}_B = m\vec{a}_T + m\vec{a}_N$$

$$\sum F_T \hat{u}_T = m\vec{a}_T \Rightarrow \sum F_T = m a_T$$

$$\sum F_N = m a_N$$

$$\sum F_B = 0$$

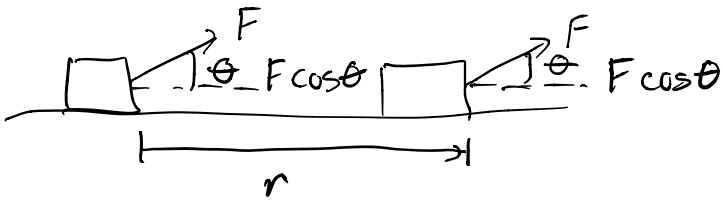
$\sum F_N$ CENTRIPETAL FORCE

KINETICS OF A PARTICLE : WORK AND ENERGY

THE WORK OF A FORCE

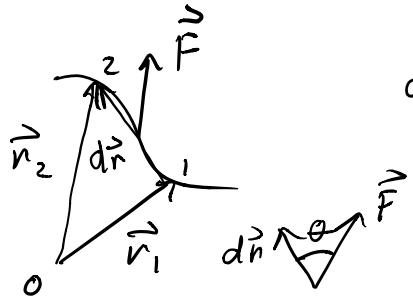
A FORCE CAN DO WORK ON A OBJECT IF IT IS DISPLACED.

WORK $dU = F\,dr\cos\theta = \vec{F}\cdot d\vec{r}$



$\vec{F}$	[N]	[lb]
$d\vec{r}$	[m]	[ft]
dU	[N-m]	[lb-ft]
	↓	
	[J]	

WORK CAN BE POSITIVE, NEGATIVE, ZERO



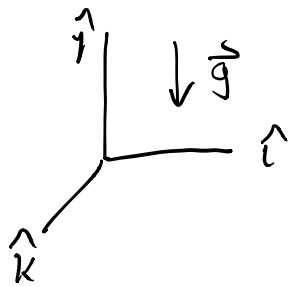
$$dU = \vec{F}\cdot d\vec{r}$$

$$U_{1-2} = \int_1^2 dU = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$$

$$= \int_{s_1}^{s_2} F\,ds\cos\theta$$

IF FORCE CONSTANT AND STRAIGHT LINE

$$U_{1-2} = F\cos\theta\int_{s_1}^{s_2} ds = F\cos\theta\,(s_2 - s_1) = F\cos\theta\,\Delta s$$



$$\vec{W} = -W\hat{j} = -mg\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$U_{1-2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (-W\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= -\int_{y_1}^{y_2} W\,dy = -W(y_2 - y_1) = -W\Delta y$$

WORK -'ve , PATH Δy POSITIVE

SPRING

$F = -ks$

$s = 0$ UNSTRETCHED

$$U_{1-2} = \int_{s_1}^{s_2} F\,ds = \int_{s_1}^{s_2} (-ks)\,ds$$

$$= -k\frac{s^2}{2}\Big|_{s_1}^{s_2} = -\frac{k}{2}(s_2^2 - s_1^2)$$

PRINCIPLE OF WORK AND ENERGY

BUNCH OF EXTERNAL FORCES

$$\sum F_T = m a_T \;;\quad a_T\,ds = v\,dv$$

$$U_{1-2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{s_1}^{s_2} F\cos\theta\,ds = \int_{s_1}^{s_2} F_T\,ds$$

$$\sum U_{1-2} = \sum \int_{s_1}^{s_2} F_T\,ds = m\int_{s_1}^{s_2} a_T\,ds = m\int_{v_1}^{v_2} v\,dv$$

$$\sum U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$ PRINCIPLE OF WORK AND ENERGY

DEFINE KINETIC ENERGY $T = \frac{1}{2}mv^2$

$$\sum U_{1-2} = T_2 - T_1$$

$$T_1 + \sum U_{1-2} = T_2$$

PRINCIPLE OF WORK AND ENERGY FOR SYSTEM OF PARTICLES

$$\sum_i (T_1)_i + \sum_i \left(\sum U_{1-2}\right)_i = \sum_i (T_2)_i$$

WHERE $\sum_i$ OF ALL PARTICLES

POWER AND EFFICIENCY

POWER, $P = \frac{dU}{dt}$ AMOUNT OF WORK PERFORMED IN TIME INTERVAL dt

SINCE $dU = \vec{F} \cdot d\vec{r}$

$$P = \frac{dU}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

AVERAGE POWER, $\langle P \rangle = \frac{\Delta U}{\Delta t}$

UNITS : SI IN WATT (W)

FPS IS HORSEPOWER (hp); 1 hp = 550 ft.lb/s

MECHANICAL EFFICINECY $\varepsilon = \frac{\text{POWER OUTPUT}}{\text{POWER INPUT}}$.

FOR THE SAMETIME INTERVAL $\varepsilon = \frac{\text{ENERGY OUTPUT}}{\text{ENERGY INPUT}}$.

CONSERVATIVE FORCES AND POTENTIAL ENERGY

CONSERVATIVE FORCE

CONSERVATIVE FORCE DOES WORK ON A PARTICLE THAT IS INDEPENDENT OF THE PATH.
THAN WORK DEPENDS ONLY ON INITIAL AND FINAL POSITIONS.
EXAMPLES : GRAVITATION, SPRING FORCE
NONCONSERVATIVE FORCE EXAMPLE : FRICTION.

ENERGY

ENERGY IS MEASURE OF CAPACITY TO DO WORK.

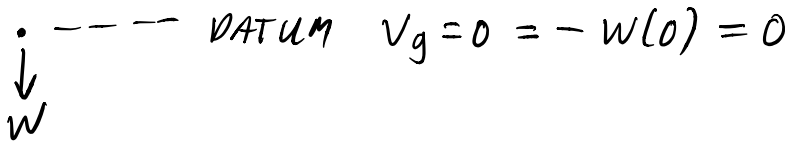
TWO CATAGORIES OF ENERGY
KINETIC ENERGY → COMES FROM MOTION OF PARTICLE
POTENTIAL ENERGY → COMES FROM POSITION OF PARTICLE

POTENTIAL ENERGY IS MEASURE OF AMOUNT OF WORK A CONSERVATIVE FORCE WILL DO WHEN IT MOVES PARTICLE FROM $\underbrace{\text{GIVEN POSITION TO DATUM}}_{\text{NOTICE ORDER.}}$

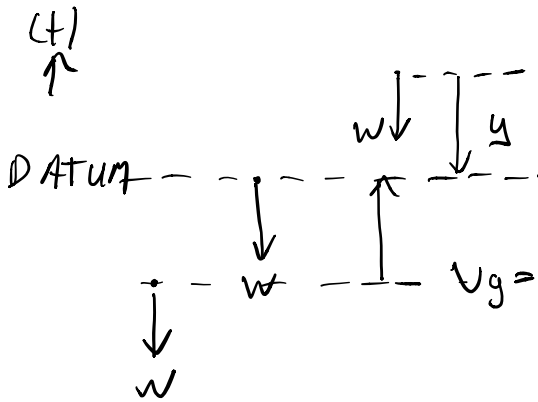
GRAVITATIONAL POTENTIAL ENERGY

$$V_g = \vec{W} \cdot \Delta\vec{y} = -W\Delta y = -W(0 - y_{INT}) = W y_{INIT}$$

$\uparrow$ (+)


DATUM $V_g = 0 = -W(0) = 0$
W

GRAVITATIONAL POTENTIAL ENERGY



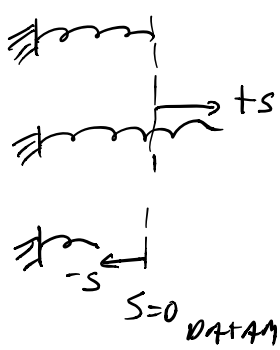
$V_g = +Wy = -W(0-y) = Wy$

$y = 0 \quad V_g = -W(0) = 0$

$V_g = -Wy = -W[0-(-y)] = -Wy$

$$V_g = \vec{W} \cdot \Delta\vec{y} = -W\Delta y = -W(0-y_0) = Wy_0$$

ELASTIC POTENTIAL ENERGY



$V_e = 0$

$V_e = +\frac{1}{2}ks^2$

$V_e = +\frac{1}{2}ks^2$

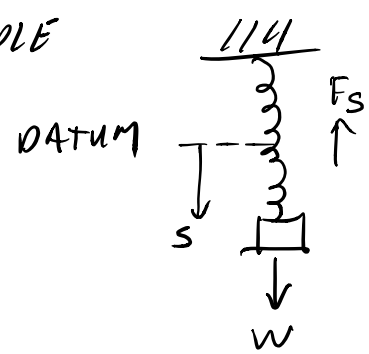
POTENTIAL ENERGY ALWAYS POSITIVE

$$V_e = -k\int_{INIT}^{DATUM} s\,ds = -\frac{k}{2}s^2\Big|_{INIT.}^{DATUM} = \frac{k}{2}s_{INT}^2 = \frac{k}{2}s_0^2$$

POTENTIAL FUNCTION

TOTAL POTENTIAL $V = \sum_i V_i$ SUM POTENTIAL ENERGIES

FOR EXAMPLE



$V = V_g + V_s$

$= -Ws + \frac{1}{2}ks^2$

WORK DONE $U_{1-2} = V_1 - V_2$ CHANGE IN POTENTIAL.

NOTICE ORDER.

OUR EXAMPLE, $U_{1-2} = (V_0 - V_2) - (V_0 - V_1) = V_1 - V_2$

$$U_{1-2} = V_1 - V_2 = \left(-ws_1 + \frac{1}{2}ks_1^2\right) - \left(-ws_2 + \frac{1}{2}ks_2^2\right)$$
$$= w(s_2 - s_1) - \frac{1}{2}k(s_2^2 - s_1^2)$$

WE NOW SHOW THE GRADIENT OF POTENTIAL IS APPLIED FORCE.

INFINITESIMAL DISPLACEMENT

$(x, y, z) \to (x + dx, y + dy, z + dz)$

$$dU = V(x, y, y) - V(x + dx, y + dy, z + dz) \quad [V_1 - V_2]$$
$$= -dV(x, y, z)$$

$$dU = \vec{F} \circ d\vec{r} = (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \circ (dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= F_x dx + F_y dy + F_z dz$$

$$\therefore F_x dx + F_y dy + F_z dz = -dV = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

$$\vec{F} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V$$

$\vec{F} = -\vec{\nabla} V \to$ CRITERION FOR CONSERVATIVE FORCE

EXAMPLE : $V_g = wy \Rightarrow \frac{\partial V_g}{\partial y} = w$; $-\frac{\partial V_g}{\partial y} = -w$

$F = -w$

CONSERVATION OF ENERGY

$$\sum_i U_{1-2} = \sum_i (U_{1-2})_{cons} + \sum_i (U_{1-2})_{noncons}$$

WORK-ENERGY THEORM $\sum_i U_{1-2} = T_2 - T_1$

WORK DUE TO CONSERVATIVE FORCES

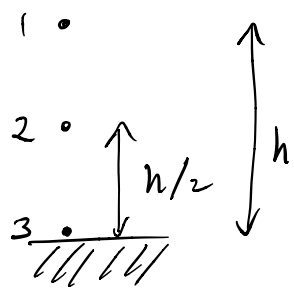
$$(\sum_i U_{1-2})_{cons} = V_1 - V_2$$

$$T_2 - T_1 = V_1 - V_2 + \sum_i (U_{1-2})_{noncons}$$

$$T_1 + V_1 + \sum_i (U_{1-2})_{noncons} = T_2 + V_2$$

IF ONLY CONSERVATIVE FORCES : $T_1 + V_1 = T_2 + V_2$

IF DEFINE $E = T + V$, THE ENERGY IS CONSTANT



$$E_1 = T_1 + V_1 = 0 + mgh$$

$$E_2 = T_2 + V_2 = \frac{1}{2} m v_2^2 + mg(h/2)$$

$$E_3 = T_3 + V_3 = \frac{1}{2} m v_3^2 + 0$$

$$v^2 = v_0^2 + 2gy \Rightarrow v_2^2 = 2g(h/2) = gh$$

$$v_3^2 = 2gh$$

$$\therefore \quad E_1 = mgh$$

$$E_2 = mgh$$

$$E_3 = mgh$$

SYSTEM OF PARTICLES

$$\sum_i (T_1)_i + \sum_i (V_1)_i = \sum_i (T_2)_i + \sum_i (V_2)_i$$

ONLY CONSERVATIVE FORCES.

IMPULSE AND MOMENTUM

PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

INTEGRATE OVER TIME, $\sum \int_{t_1}^{t_2} \vec{F}\,dt = m\int_{\vec{v}_1}^{\vec{v}_2} d\vec{v} = m\vec{v}_2 - m\vec{v}_1$

PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM

$$\sum \int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2 - m\vec{v}_1$$

GOOD IF $\vec{F} = \text{CONSTANT}$

$\vec{F} = \vec{F}(t)$

LINEAR MOMENTUM $\vec{p} = m\vec{v}$

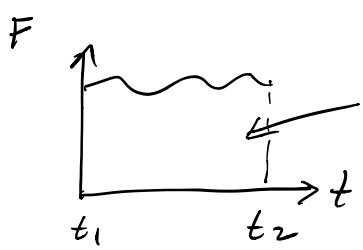
LINEAR IMPULSE $\vec{I} = \int \vec{F}\,dt$

FOR CONSTANT FORCE, $\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt = \vec{F}(t_2 - t_1)$

PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM

$$m\vec{v}_1 + \sum \int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2$$

$$m v_{x1} + \sum \int_{t_1}^{t_2} F_x\,dt = m v_{x2} \quad ; \text{etc FOR } y, z$$



AREA $= I = \int_{t_1}^{t_2} F(t)\,dt$

$m\vec{v}_1$ $\quad$ $\vec{I}$

$$\bigcirc \quad + \quad \bigcirc \quad = \quad \bigcirc$$

$m\vec{v}_2$

INITIAL MOMENTUM $\quad$ IMPULSE $\quad$ FINAL MOMENTUM

PRINCIPLE OF LINEAR IMPULSE AND MOMENTUM OF A SYSTEM OF PARTICLES

FOR THE i'TH PARTICLE $\sum_i F_i = \sum m_i \frac{d\vec{v}_i}{dt}$

INTEGRATE $\sum_i m_i \vec{v}_{i1} + \sum_i \int_{t_1}^{t_2} \vec{F}_i \, dt = \sum_i m_i \vec{v}_{i2}$

REMEMBER $m\vec{r}_G = \sum m_i \vec{r}_i$, WHERE $m = \sum m_i$

TIME DERIVATIVE $m\vec{v}_G = \sum m_i \vec{v}_i$

$\therefore m\vec{v}_{G1} + \sum \int_{t_1}^{t_2} \vec{F}_i \, dt = m\vec{v}_{G2}$

CONSERVATION OF LINEAR MOMENTUM FOR A SYSTEM OF PARTICLES

PRINCIPLE OF IMPULSE AND MOMENTUM APPLIED TO A SYSTEM OF PARTICLES

$$\sum_i m_i \vec{v}_{i1} + \sum_i \int_{t_1}^{t_2} \vec{F}_i \, dt = \sum_i m_i \vec{v}_{i2}$$

THE INTERNAL IMPULSES CANCEL (MAYBE UNKNOWN)

IF THE SUM OF THE EXTERNAL IMPULSE VANISH

$\sum_i m_i \vec{v}_{i1} = \sum m_i \vec{v}_{i2}$ CONSERVATION OF LINEAR MOMENTUM

USING $m_G \vec{v}_{G_1} = m_G \vec{v}_{G_2} \Rightarrow \vec{v}_{G1} = \vec{v}_{G2}$

USED $m_G \vec{v}_G = \sum m_i \vec{v}_i$

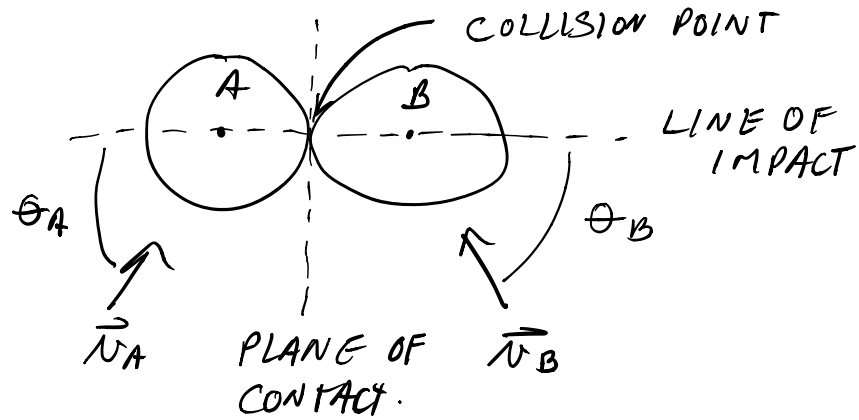
MOMENTUM IS CONSERVED IN ORTHOGONAL DIRECTION TO IMPULSE FORCE

IF Δt SMALL $\Rightarrow \vec{I}$ NEGLIGIBLE FOR SOME FORCES RELATIVE TO OTHER.

IMPULSIVE FORCES VS NON-IMPULSIVE FORCE

IMPACT

2 BODIES COLLIDE + IN CONTACT SHORT TIME



COLLISION POINT – POINT OF FIRST CONTACT
(MAY NOT BE UNIQUE)

LINE OF IMPACT – LINE BETWEEN CENTRES OF EACH BODY AT MOMENT OF CONTACT

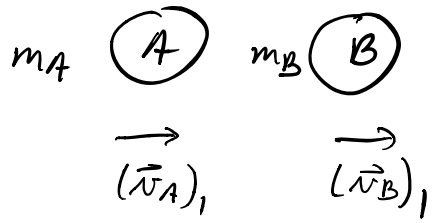
PLANE OF CONTACT – PERPENDICULAR TO LINE OF IMPACT
(LOCATE AT COLLISION POINT)

CENTRAL IMPACT – TWO OBJECTS COLLIDE ALONG LINE OF IMPACT.

OBLIQUE IMPACT – IF OBJECT MOTION HAS ANGLE WITH LINE OF IMPACT

CENTRAL IMPACT

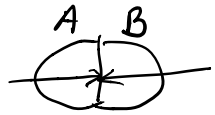
BEFORE IMPACT



REQUIRE

$(v_A)_1 > (v_B)_1$

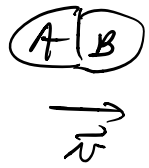
DEFORMATION IMPACT



$\int \vec{P}\,dt \qquad -\int \vec{P}\,dt$

$\int \vec{P}\,dt$ INTERNAL IMPULSE OF DEFORMATION.

MAXIMUM DEFORATION

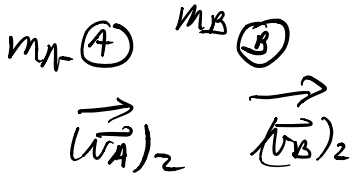


RESTITUTION IMPULSE

$\int \vec{R}\,dt \qquad -\int \vec{R}\,dt$

$\int \vec{R}\,dt$ INTERNAL IMPUSE OF RESTITUTION.

AFTER IMPACT



REQURE

$(v_B)_2 > (v_A)_2$

IF INTERNAL IMPULSES $\int \vec{P}\,dt$ AND $\int \vec{R}\,dt$ CANCEL DURING COLLISION, MOMENTUM IS CONSERVED.

$$m_A v_{A1} + m_B v_{B1} = m_A v_{A2} + m_B v_{B2}$$

IN REALITY $\int P\,dt > \int R\,dt$

COMMON VELOCITY

DURING A DEFORMATION $m_A v_{A1} - \int P\,dt = m_A v$

DURING A RESTITUTION $m_A v - \int R\,dt = m_A v_{A2}$

$$e \equiv \frac{\int R\,dt}{\int P\,dt} = \frac{v - v_{A2}}{v_{A1} - v}$$ COEFFICIENT OF RESTITUTION

SIMILARLY FOR OBJECT B

$$e \equiv \frac{\int R\,dt}{\int P\,dt} = \frac{v_{B2} - v}{v - v_{B1}}$$

ELIMINATE $v \Rightarrow e = \dfrac{v_{B2} - v_{A2}}{v_{A1} - v_{B1}}$

COEFFICIENT OF RESTITUTION

$0 \le e \le 1$; CONSTANT.

IN REALITY, $0 < e < 1$; $e = e(v)$, SIZE, SHAPE ETC.

ELASTIC IMPACT $(e=1)$

IF PERFECTLY ELASTIC, $\int \vec{P}\,dt = \int \vec{R}\,dt$, AND $e=1$

PLASTIC IMPACT $(e=0)$

IF INELASTIC OR PLASTIC, $\int \vec{R}\,dt = 0$, AND $e=0$

OBJECTS STICK TOGETHER AND MOVE WITH COMMON VELOCITY

IF $e = 1$, NO ENERGY LOST DURING COLLISION

IF $e = 0$, ENERGY LOSS IS MAXIMUM

$\Rightarrow$ ENERGY LOSS IN FORM OF HEAT, SOUND.

$\Rightarrow$ PRINCIPLE OF WORK AND ENERGY CAN NOT BE APPLIED.

OBLIQUE IMPACT

$\int \vec{P}\,dt$ AND $\int \vec{R}\,dt$ ACT ALONG LINE OF IMPACT

ALONG LINE OF IMPACT, MOMENTUM OF SYSTEM IS CONSERVED
(SINCE FOR SYSTEM IMPULSES ARE INTERNAL)

ALONG PLANE OF CONTACT, MOMENTUM OF EACH OBJECT IS CONSERVED

WORKING METHOD:

1) PUT x-AXIS ALONG LINE OF IMPACT
2) PUT y-AXIS IN PLANE OF CONTACT
3) APPLY MOMENTUM CONSERVATION OF SYSTEM ALONG x-AXIS
4) APPLY RESTITUTION FORMULA ALONG x-AXIS
5) APPLY MOMENTUM CONSERVATION OF INDIVIDUAL OBJECT ALONG y-AXIS

ROTATION

PREVIOUSLY : MOTION ALONG STRAIGHT OR CURVED PATH

NOW : ROTATION OF OBJECT ABOUT AXIS

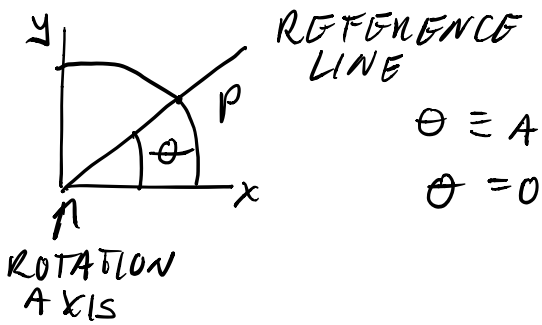
RIGID BODY ≡ ALL PARTS ARE FIXED RELATIVE TO EACH OTHER, AND BODY DOES NOT CHANGE SHAPE.

ROTATION USUALLY ABOUT FIXED AXIS THAT DOES NOT MOVE RELATIVE TO OBJECT.

PURE ROTATION ≡ EVER POINT IN OBJECT MOVES IN CIRCLE ABOUT ROTATION AXIS (AXIS OF ROTATION).

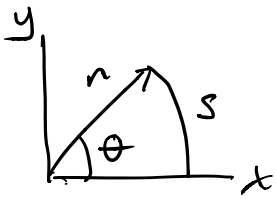
EVERY POINT MOVES THRU SAME ANGLE IN SAME TIME.

ANGULAR POSITION



θ ≡ ANGULAR POSITION

$\theta = 0$ AT x-AXIS



$$\theta = \frac{s}{r}$$

MEASURE IN RADIANS (rad)

INTERESTED IN $\theta = \theta(t)$

ANGULAR DISPLACEMENT

$$\Delta\theta = \theta_2 - \theta_1$$

ANGULAR DISPLACEMENT

FINAL ANGULAR POSITION

INITIAL ANGULAR POINT.

ANGULAR VELOCITY

AVERAGE ANGULAR VELOCITY $\langle \omega \rangle = \dfrac{\theta_2 - \theta_1}{t_2 - t_1} = \dfrac{\Delta\theta}{\Delta t}$

INSTANTANEOUS ANGULAR VELOCITY $\omega = \dfrac{d\theta}{dt}$

ω +'VE $\Rightarrow$ COUNTER-CLOCKWISE ROTATION

$|\omega|$ ANGULAR SPEED

ω UNIT, RADIANS PER SECOND (rad/s)

ANGULAR ACCELERATION

AVERAGE ANGULAR ACCELERACTION $\langle \alpha \rangle = \dfrac{\omega_2 - \omega_1}{t_2 - t_2} = \dfrac{\Delta\omega}{\Delta t}$

INSTANTANEOUS ANGULAR ACCELERATION $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

α UNITS, rad/s^2

WE HAVE $\omega = \dfrac{d\theta}{dt}$, $\alpha = \dfrac{d\omega}{dt}$, ELIMINATE dt

$\rightarrow \; r\,d\theta = w\,dw$

WE USE SIGN TO KEEP TRACK OF VECTOR NATURE

r IS OUT OF PLANE OF ROTATION

(GIVEN BY RIGHT-HAND RULE)

DISPLACEMENT NOT TRUE VECTOR (UNDER MATHS)

ROTATION WITH CONSTANT ANGULAR ACCELERATION

FROM DEFINITIONS

$$\omega = \frac{d\theta}{dt} \; ; \; \alpha = \frac{d\omega}{dt} \; : \; \alpha \, d\theta = \omega \, d\omega$$

BY ANALOGY :

$$s \rightarrow \theta$$
$$v \rightarrow \omega$$
$$a \rightarrow \alpha$$
$$t \rightarrow t$$

FOR CONSTANT ACCELERATION α

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$ [NOT INDEPENDENT]

ALSO :

$$\theta = \theta_0 + \omega t - \frac{1}{2}\alpha t^2$$

$$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t$$

RELATING LINEAR AND ANGULAR VARIABLES

EACH PARTICLE MOVES IN A CIRCLE WITH SAME θ AND ω

POSTION

$$\theta = \frac{s}{r} \Rightarrow s = \theta r$$ [θ IN RADIANS]

SPEED

FOR CONSTANT r, $\dot{s} = \dot{\theta} r \Rightarrow v = \omega r$

$(\vec{v} = \vec{\omega} \times \vec{r})$

VELOCITY IS TANGENT TO PATH.

PERIOD $T = \frac{2\pi}{\omega}$

ACCELERATION.

$\dot{v} = \dot{\omega} r \Rightarrow a_T = \alpha r$ $\quad (\vec{a}_T = \vec{\alpha} \times \vec{r})$ TANGENTAL COMPONENT

NORMAL COMPONENT

$a_N = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$ $\quad [\vec{a}_N = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 \vec{r}]$

NOTICE : a_N HAS NO α DEPENDENCE

KINETIC ENERGY OF ROTAION

$$T = \sum_i' \frac{1}{2} m_i v_i^2 = \sum_i' \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \sum_i' (m_i r_i)^2 \omega^2 = \frac{1}{2} I \omega^2$$

ALL PARTICLES OF RIDGED BODY

$I = \sum_i' m_i r_i^2 \equiv$ ROTATIONAL INERTIA OR MOMENT OF INERTIA

REPRESENTS HOW MASS IS DISTRIBUTED ABOUT AXIS OF ROTATION

MOMENT OF INERTIA DEPENDS ON ROTATING BODY + AXIS OF ROTATION

UNITS $[kg][m^2]$

CALCULATING ROTATIONAL INERTIA

FOR A FEW PARTICLES, USE $I = \sum_i' m_i r_i^2$

FOR A CONTINOUS BODY, INTEGRATE $I = \int r^2 dm$

MASS DENSITY $\rho = \frac{m}{V}$. IF ρ IS CONSTANT $\rho = \frac{dm}{dV}$

$\therefore I = \rho \int r^2 dV$

USUALLY MOMENTS OF INERTIA ARE PRE-TABULATED ABOUT AXIS THRU CENTRE OF OBJECT.

PARALLEL-AXIS THEOREM

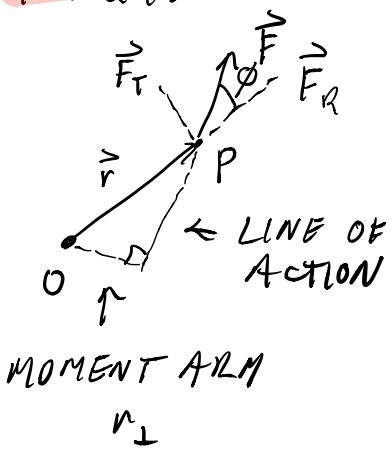
$$I = I_{com} + Mh^2$$

$M \equiv$ MASS OF BODY

$I_{com} \equiv$ MOMENT OF INERTIA ABOUT AXIS THRU CENTER OF MASS

$h \equiv$ PERPENDICULAR DISTANCE TO ANOTHER AXIS

TORQUE



ROTATION DEPENDS ON TANGENTIAL COMPONENT OF FORCE + WHERE FORCE APPLIED RELATIVE TO AXIS

TORQUE $\tau = rF\sin\phi$

$(\vec{\tau} = \vec{r} \times \vec{F})$

NOTE: $F\sin\phi = F_T$ OR $r\sin\phi = r_\perp$

$\therefore \tau = rF_T$ OR $\tau = r_\perp F$

+'VE SIGN FOR TORQUE THE CASE ANTI-CLOCKWISE ROTATION.

UNITS [N][m]

NEWTON'S SECOND LAW FOR ROTATION

IN ANALOGY $\sum \tau = \tau_{NET} = I\alpha$

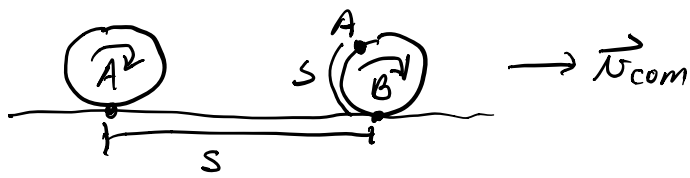
FOR A SINGLE PARTICLE

$$\left.\begin{aligned} a_T &= r\alpha \\ F_T &= m a_T \end{aligned}\right\} \quad \begin{aligned} \tau &= rF_T \\ &= r m a_T \\ &= r m (r\alpha) \\ &= m r^2 \alpha \\ &= I\alpha \end{aligned}$$

ROLLING, TORQUE, AND ANGULAR MOMETUM

ROLLING SMOOTHLY ≡ WITHOUT SLIPPING OR BOUNCING.

ROLLING AS TRANSLATION AND ROTATION COMBINED



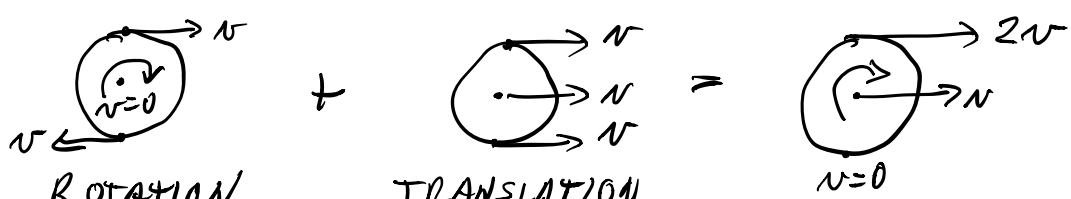
MOTION = TRANSLATION OF CENTRE OF MASS
+ ROTATION AROUND CENTRE OF MASS

$s = \theta R$; $v_{com} = \frac{ds}{dt}$; $\omega = \frac{d\theta}{dt}$

$v_{com} = \frac{d\theta}{dt} R = \omega R \Rightarrow v_{com} = \omega R$ FOR SMOOTH ROLLING MOTION.

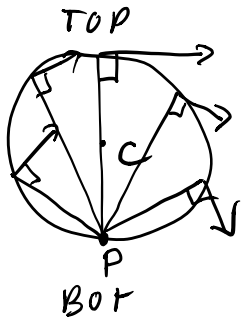
v_{com}, ω ARE SPEEDS NOT VELOCITIES

CONSIDER 3 POINTS ON WHEEL



ROTATION ONLY | TRANSLATION ONLY | TRANSLATION AND ROTATION.

ALTERNATIVE VIEW - PURE ROTATION ABOUT AXIS THRU POINT OF CONTACT



$v_{top} = \omega(2R) = 2v_{com}$

$v_c = \omega R = v_{com}$

$v_{Bot} = v_p = 0$

FORCES AND KINETIC ENERGY ROLLING

KINETIC ENERGY OF ROLLING

FOR PURE ROTATION $T = \frac{1}{2} I_p \omega^2$

AXIS AT A POINT OF CONTACT.

$I_p = I_{com} + MR^2$ PARALLEL AXIS THEOREM.

$$\therefore T = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \underbrace{\frac{1}{2} I_{com}\omega^2}_{\text{ROTATIONAL}} + \underbrace{\frac{1}{2} M v_{com}^2}_{\text{TRANSLATIONAL.}}$$

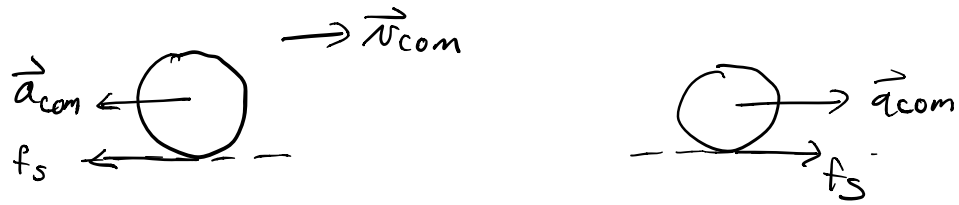
FORCES OF ROLLING.

PREVIOUSLY NO ACCELERATION $\Rightarrow$ NO FORCE (NO FRICTION)

FRICTION AND ROLLING.

IF ACCELERATING, WHEEL MOVES FASTER OR SLOWER THAN NORMAL.

TO MAINTAIN NO SLIDING $\Rightarrow$ NEED FRICTION.



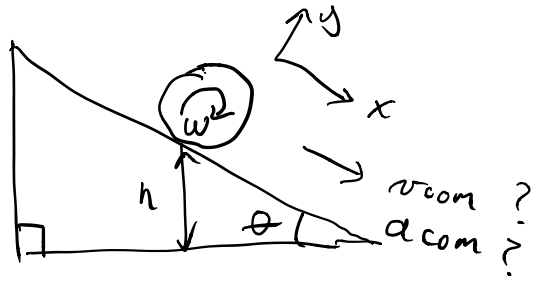
SLOW DOWN SPEED UP

IF WHEEL DOES NOT SLIDE, FRICTION IS <u>STATIC</u>
$\Rightarrow$ SMOOTH ROLLING.

DIFFERENTIATE $v_{com} = \omega R \Rightarrow a_{com} = \alpha R$
MAGNITUDES ONLY

KINETIC FRICTION IF WHEEL SLIDES $\Rightarrow$ EQUATION NOT VALID.

ROLLING DOWN A RAMP



USE NEWTON'S LAW IN TWO FORMS

$\sum F = M a_{com}$; $\tau_{NET} = I_{com}\alpha$

GIVES a_{com}, INTERATE

USE $v^2 = v_0^2 + 2as \Rightarrow 2as$

$$T_i + V_i = T_f + V_f$$

$$0 + Mgh = \frac{1}{2}Mv_{com}^2 + \frac{1}{2}I_{com}\omega^2 + 0$$

$$Mgx\sin\theta = \frac{1}{2}Mv_{com}^2 + \frac{1}{2}I_{com}\left(\frac{v_{com}}{R}\right)^2$$

$$v_{com}^2 = \frac{Mg\sin\theta}{\frac{1}{2}M + \frac{1}{2}\frac{I_{com}}{R^2}}x = \frac{2g\sin\theta}{1 + \frac{I_{com}}{MR^2}}x$$

$$v_{com} = \left[\frac{2g\sin\theta}{1 + \frac{I_{com}}{MR^2}}\right]^{1/2} x^{1/2}$$

$$a_{com} = \frac{dv_{com}}{dt} = \left[\frac{2g\sin\theta}{1 + I_{com}/MR^2}\right]^{1/2} \frac{1}{2x^{1/2}}\frac{dx}{dt}$$

$$= \left[\frac{2g\sin\theta}{1 + I_{com}/MR^2}\right]^{1/2} \frac{1}{2v_{com}} \left[\frac{2g\sin\theta}{1 + \frac{I_{com}}{MR^2}}\right]^{1/2} v_{com}$$

$$= \frac{g\sin\theta}{1 + I_{com}/MR^2}$$

FOR NO ROTATION, $a_{com} = g\sin\theta$

FOR $I_{com} = MR^2 \Rightarrow a_{com} = \frac{g}{2}\sin\theta$

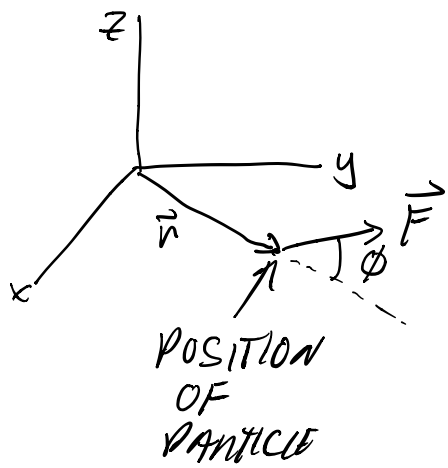
$I_{com} = \eta MR^2$; $0 < \eta < 1 \Rightarrow a_{com} = \frac{g}{1+\eta}\sin\theta$

TORQUE REVISITED

CONSIDER A PARTICLE.

AXIS OF RATATION → POINT OF ROTATION

CIRCLE → CURVED PATH



TORQUE FOR PARTICLE

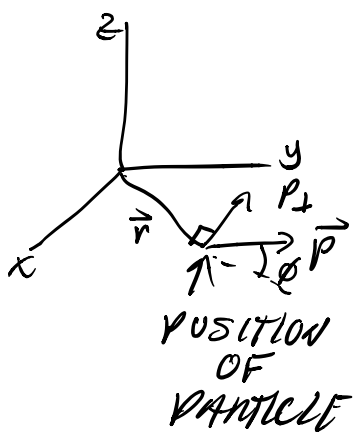
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\phi$$

ALSO $\tau = rF_{\perp}$

$$\tau = r_{\perp}F$$

ANGULAR MOMENTUM



$$\vec{\ell} = \vec{r} \times \vec{p} \; ; \quad \vec{p} = m\vec{v}$$

LINEAR MOMENTUM

$$\therefore \vec{\ell} = m(\vec{r} \times \vec{v})$$

ANGULAR MOMENTUM

A PARTICLE CAN HAVE ANGULAR MOMENTUM ABOUT ORIGIN WITHOUT ROTATING ABOUT ORIGIN

UNITS: $[kg][m][m/s] = [kg\, m^2/s] = [J\, s]$

DIRECTION: GIVEN BY RIGHT-HAND RULE.

MAGNITUDE : $\ell = mrv\sin\phi$ OR $\ell = mr_\perp v = r_\perp p$

$\ell = mrv_\perp = rp_\perp$

IN CARTESIAN COORDINATES

$$\vec{\ell} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

NENTON'S SECOND LAW OF MOTION IN ANULAR FORM

NENTON'S SECOND LAW $\sum_i \vec{F} = \frac{d\vec{p}}{dt}$

ASIDE : $\vec{p} = m\vec{v} \Rightarrow \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$

BY ANALOGY: $\sum_i \vec{\tau} = \frac{d\vec{\ell}}{dt}$

PROOF: $\vec{\ell} = m(\vec{r} \times \vec{v})$

$$\frac{d\vec{\ell}}{dt} = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) = m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a})$$

$$= m\vec{r} \times \vec{a} = \vec{r} \times (m\vec{a})$$

$$\vec{F} = m\vec{a} \Rightarrow \frac{d\vec{\ell}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

ANGULAR MOMENTUM OF A RIGID BODY

FOR SYSTE OF n PARTICLES

$$\vec{L} = \sum_{i=1}^{n} \vec{\ell}_i$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l}_i}{dt} = = \sum_{i=1}^{n} \sum_{j} (\vec{\tau}_i)_j$$

SUM OVER PARTICLES

SUM OVER FORCES ON i'TH PARTICLE

THE SUM INCLUDES INTERNAL TORQUES. BY NEWTON'S THIRD LAW THE INTERNAL TORQUES CANCEL.

$$\therefore \underset{\substack{\text{NET} \\ \text{EXTERNAL} \\ \text{TORQUE}}}{\sum \tau} = \frac{d\vec{L}}{dt}.$$

PAY ATTENTION TO ORIGIN

- SAME ORIGIN USED TO DEFINE ALL QUANTITIES
- IF com NOT ACCELERATING $\Rightarrow$ USE ANY ORIGIN
- IF com ACCELERATING $\Rightarrow$ USE com AS ORIGIN.

ANGULAR MOMENTUM OF RIGED BODY ROTATING ABOUT FIXED AXIS

$$L = I\omega$$

CONSERVATION OF ANGULAR MOMENTUM

NEWTON'S SECOND LAW IN ANGULAR FORM

$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$, FOR NO EXTERNAL TORQUES $\frac{d\vec{L}}{dt} = 0$

$\Rightarrow \vec{L} = \text{CONSTANT.}$; $\vec{L}_i = \vec{L}_f$ FOR ISOLATED SYSTEM.

$I_i \omega_i = I_f \omega_f$

IF I CHANGES INTERNALLY, ω CAN CHANGE

$\vec{L}$ ALSO CONSERVED IN RELATIVITY & QUANTUM MECHANICS

THIS IS VECTOR EQUATION

IF FORCE $\Rightarrow$ ANGULAR MOMENTUM CONSERVED IN TWO ORTHOGONAL DIRECTIONS