PHYS 530: Problem Set 7

Due: 4:30 pm, 26 March 2013

If the answer is shown, all the marks will be given for the derivation not for writing down the answer. In your solutions, you may need to make some assumptions. Make sure that you formulate all of them clearly.

1. [7] Consider the general expression for the entropy

$$S = -k \sum_s P_s \ln P_s \,, \tag{1}$$

where P_s is the probability for the entire system to be in quantum state s. In a system of noninteracting particles, it was shown that the probability for single particle quantum state i to be occupied by n particles is

$$p_i(n) = \frac{(\langle n_i \rangle)^n}{(\langle n_i \rangle + 1)^{n+1}} \tag{2}$$

for Bose-Einstein statistics and

$$p_i(n) = \begin{cases} 1 - \langle n_i \rangle & \text{for } n = 0 \\ \langle n_i \rangle & \text{for } n = 1 \end{cases} \tag{3}$$

for Fermi-Dirac statistics. In these expressions, n_i is the average occupancy of the single particle quantum state i, i.e., it is the average number of particles in that state. Using these expressions for the probabilities $p_i(n)$ for having n particles in single particle quantum state i, show that

$$S = k \sum_i \left[\langle n_i + 1 \rangle \ln \langle n_i + 1 \rangle - \langle n_i \rangle \ln \langle n_i \rangle \right] \tag{4}$$

for bosons and

$$S = k \sum_i \left[-\langle 1 - n_i \rangle \ln \langle 1 - n_i \rangle - \langle n_i \rangle \ln \langle n_i \rangle \right] \tag{5}$$

for fermions.
This is similar to problem 6.1 in Pathria, except that you start from Eq. (1), instead of showing that the derived expressions are consistent with Eq. (1).

Hint: Make use of the fact that the probability of finding the entire system in quantum state s characterised by a distribution of n_1 particles in the single particle state 1, n_2 particles in state 2, etc. is given by $P_s = p_1(n_1) p_2(n_2) \cdots$.

2. [5] Solve problem 6.7 in Pathria.

Through a small window in a furnace, which contains a gas at a high temperature T, the spectral lines emitted by the gas molecules are observed. Because of molecular motions, each spectral line exhibits Doppler broadening. Show that the variation of the relative intensity $I(\lambda)$ with wavelength λ in a line is given by

$$I(\lambda) \propto \exp\left[-\frac{mc^2(\lambda-\lambda_0)^2}{2\lambda_0^2 kT}\right], \tag{6}$$

where m is the molecular mass, c the speed of light, and λ_0 the mean wavelength of the line.

3. [10] Solve problem 6.13 in Pathria.

   (a) Determine the number of impacts made by gas molecules on a unit area of the wall in a unit time for which the angle of incidence lies between θ and $\theta + d\theta$.

   (b) Determine the number of impacts made by gas molecules on a unit area of the wall in a unit time for which the speed of the molecules lies between u and $u + du$.

   (c) A molecule AB dissociates if it hits the surface of a solid catalyst with a normal translational energy greater than 10^{-19} J. Show that the rate of dissociative reactions $AB \to A + B$ is more than doubled by raising the temperature of the gas from 300 K to 310 K.

4. [12] Solve problem 6.17 in Pathria.

   A small sphere, with initial temperature T, is immersed in an ideal Boltzmannian gas at temperature T_0. Assuming that the molecules incident on the sphere are first absorbed and then reemitted with the temperature of the sphere, determine the variation of the temperature of the sphere with time. [*Note*: The radius of the sphere may be assumed to be much smaller than the mean free path of the molecules.]

   Clearly list all the necessary and independent assumptions that you make in your solution, including the ones given in Pathria. List these assumptions in point form at the beginning of your solution. Your solution should consist of a differential equation for the sphere temperature, involving the ideal Boltzmannian gas parameters, the sphere radius, mass, specific heat capacity, and any other relevant parameter.

5. [8] Solve problem 6.19 in Pathria.

   What is the probability that two molecules picked at random from a Maxwellian gas will have a total energy between E and $E + dE$? Verify that $\langle E \rangle = 3kT$.