PHYS 530: Problem Set 4

Due: 4:30 pm, 26 February 2013

If the answer is shown, all the marks will be given for the derivation not for writing down the answer. In your solutions, you may need to make some assumptions. Make sure that you formulate all of them clearly.

1. [20] The Tsallis entropy is defined by

$$S_q = \sum_r \frac{p_r - p_r^q}{q-1}\,, \tag{1}$$

where p_r is the probability of state r, and the summation is carried over all possible states.

(a) Prove that in the limit where q approaches one, the Tsallis entropy reduces to the usual form

$$\lim_{q\to 1} S_q = -\sum_r p_r \ln p_r. \tag{2}$$

The latter expression, similar to the Boltzmann-Gibbs entropy seen in class, is called the Shannon entropy.

(b) Consider two systems A and B with respective Tsallis entropies S_A and S_B. Derive an expression for the total entropy $S_q(A \cup B)$ of the combined system in terms of S_A and S_B (and possibly q).

(c) It is said that the Tsallis entropy leads to energy distribution functions that are described by power laws, as opposed to a normal (or Maxwellian) distribution. By maximizing the Tsallis entropy subject to appropriate constraints, obtain such a power law.

2. [20] Consider two independent macroscopic systems 1 and 2 at equilibrium, for which the functions $\Sigma_1(E_1)$ and $\Sigma_2(E_2)$ are known.

(a) Show that for the combined system, the function Σ is given by

$$\Sigma(E) = \int_0^E dE' \frac{d\Sigma_1(E')}{dE'} \Sigma_2(E - E') = \int_0^E dE' \Sigma_1(E - E') \frac{d\Sigma_2(E')}{dE'}, \tag{3}$$

where E is the total energy of the combined systems.

(b) In proving the equation above, you need to make some assumptions. Enumerate all of them clearly.

(c) Using the result derived in part (a), prove that the Γ function of the combined systems can be calculated from Γ_1 and Γ_2 of the individual systems as

$$\Gamma(E) = \frac{1}{\Delta} \int_0^E dE' \Gamma_1(E') \Gamma_2(E - E')\,. \tag{4}$$

(d) Using this result, prove that the partition function for the combined system can be written as

$$Q(\beta) = Q_1(\beta)Q_2(\beta). \tag{5}$$

(e) Using the result above, explain how this property implies that that $A = -kT \ln Q$ is an extensive quantity.

3. [10] Consider a system of stars constituting a small galaxy located in a remote location of the Universe. This small galaxy is sufficiently far from anything else in the Universe for any interaction energy with the rest of the Universe to be negligible. For simplicity, we may also assume that the star velocities are small compared to that the speed of light, and relativity (special or general) effects are altogether negligible. The gravitational force between any two stars of masses m_i and m_j is given by

$$\vec{F}_{ij} = -\frac{m_i m_j G}{r_{ij}^{2+s}} \left(\vec{r}_i - \vec{r}_j\right), \tag{6}$$

where s is a small number, $|s\,| < 1$ Assuming equilibrium:

(a) Use the virial theorem and the equipartition theorem to find a relation between the average kinetic energy of the system of stars and its average gravitational potential.

(b) In part (a), did you have to account for the effect of any confining wall (or that of an effective wall) as with the virial of an ideal gas)? Explain your answer.

4. [20] Consider a homogeneous system made of N identical particles. The N-particle probability density function in phase space is $f_N(1, 2, ..., N, t)$, where $1, 2, ...$ stand for the single particle phase space coordinates $\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2, ...$ respectively. Assuming a normalisation for f_N such that

$$\int \frac{d1}{V}\frac{d2}{V}\cdots\frac{dN}{V} f_N = 1, \tag{7}$$

we define the reduced n-particle distribution function as

$$f_n(1, 2, \ldots, n, t) = \int \frac{d(n+1)}{V}\frac{d(n+2)}{V}\cdots\frac{dN}{V} f_N. \tag{8}$$

Use the equation of evolution for f_N

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N}\left(\vec{v}_i \cdot \frac{\partial f_N}{\partial \vec{r}_i} + \vec{F}_i \cdot \frac{\partial f_N}{\partial \vec{p}_i}\right) = 0, \tag{9}$$

where

$$\vec{F}_i = \vec{F}_i^{\text{ext}} - \sum_{j\neq i}^{N} \frac{\partial u}{\partial \vec{r}_{ij}}, \tag{10}$$

with $u = u(r_{ij})$ being the two-particle interaction potential and $r_{ij} = |\vec{r}_i - \vec{r}_j|$ is the distance between points i and j.

(a) Construct an equation describing the evolution in time of f_1 and f_2. From these two cases, explain in words how an equation of evolution for f_n will involve an expression involving f_{n+1}.

(b) Write an expression for the pair distribution function $g(r)$ in terms of f_1 and f_2.

(c) In the mean field approximation

$$f_2(1,2) = f_1(1)f_1(2). \tag{11}$$

Derive an equation for the evolution of f_1 in that approximation.

5. [5] One centimeter cube of an ideal gas at 300 K and pressure 10^5 Pa is in equilibrium with a large heat reservoir.

   (a) Calculate the most probable internal energy.

   (b) Calculate the standard deviation in that energy.

6. [5] The derivation of the viral of an ideal gas involves a certain number of assumptions, some of which were stated in class. Enumerate all the assumptions (as many applicable and independent assumptions as you can think of, including the ones mentioned in class) that are necessary in order for this derivation to be valid.

7. [10] Using expression $S = -k\langle \ln P_r \rangle = -k \sum_r P_r \ln P_r$, prove that the entropy is extensive. That is, in your solution, you may need to make some assumptions. Make sure that you formulate all of them clearly.

8. [5] Solve problem 3.5 in Pathria's book
   Making use of the fact that the Helmholtz free energy $A(N, V, T)$ of a thermodynamic system is an extensive property of the system, show that

$$N\left(\frac{\partial A}{\partial N}\right)_{V,T} + V\left(\frac{\partial A}{\partial V}\right)_{N,T} = A\,. \tag{12}$$

   [Note that this result implies the well-known relationship: $N\mu = A + PV (\equiv G)$.]

9. [15] Solve problem 3.6 in Pathria's book.

   (a) Assuming that the total number of microstates accessible to a given statistical system is Ω, show that the entropy of the system, as given by $S = -k\langle \ln P_r \rangle = -k \sum_r P_r \ln P_r$, is maximum when all Ω states are equally likely to occur.

   (b) If, on the other hand, we have an ensemble of systems sharing energy (with mean value $\bar{E}$), then show that the entropy, as given by the same formal expression, is maximum when $P_r \propto \exp(\beta E_r)$, β begin a constant to be determined by the given value of $\bar{E}$.

(c) Further, if we have an ensemble of systems sharing energy (with mean value $\bar{E}$) and also sharing particles (with mean value $\bar{N}$), then show that the entropy, given by a similar expression, is maximum when $P_{r,s} \propto \exp(-\alpha N_r - \beta E_s)$, α and β being constants to be determined by the given values of $\bar{N}$ and $\bar{E}$.

10. [10] Solve problem 3.11 in Pathria's book
Determine the work done on a gas and the amount of heat absorbed by it during a compression from volume V_1 to volume V_2, following the law $PV^\gamma =$ constant.

11. [5] Solve problem 3.39 in Pathria's book.
Atoms of silver vapour, each having a magnetic moment $\mu_B(g = 2, J = 1/2)$, align themselves either parallel or antiparallel to the direction of an applied magnetic field. Determine the respective fractions of atoms aligned parallel and antiparallel to a field of flux density 0.1 weber/m^2 at a temperature of 1,000 K.

12. [10] Solve problem 3.40 in Pathria's book.

(a) Show that, for any magnetisable material, the heat capacities at constant field H and at constant magnetisation M are connected by the relation

$$C_H - C_M = -T \left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial M}{\partial T}\right)_H . \tag{13}$$

(b) Show that for a paramagnetic material obeying Curie's law

$$C_H - C_M = C \frac{H^2}{T^2} , \tag{14}$$

where C on the right side of this equation denotes the Curie constant of the given sample.

13. [5] Solve problem 3.43 in Pathria's book.
Consider a system of charge particle (not dipoles), obeying classical mechanics and classical statistics. Show that the magnetic susceptibility of this system is identically zero (Bohr-van Leeuwen theorem). [Note that the Hamiltonian of this system in the presence of a magnetic field $\vec{H}$ $(= \vec{\nabla} \times \vec{A})$ will be a partition function of the quantities $\vec{p}_j + (e_j/c)\vec{A}(\vec{r}_j)$, and not of the $\vec{p}_j$ as such. One has now to show that the partition function of the system is independent of the applied field.]

14. [10] Consider an ideal gas made of electrons in a fixed and uniform neutralising background (so that the volume charge density vanishes). Assume that the electron gas has a volume density n and temperature T, and that it is immersed in a constant and uniform magnetic B.

(a) Calculate the average magnetic moment per unit volume.

(b) If the plasma is in a cylinder with circular cross section of radius a and length L, and if the magnetic field is oriented along the axis of the cylinder, calculate the total magnetic moment of the plasma cylinder.

(c) Is there any contradiction between this result and what you found in problem 13? If so, how can it be reconciled?