Universal Extensions

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Abstract

In the category of left modules over a unital ring we show that a left exact reflector determines, for each $n \geq 1$, a torsion theoretic setting in which universal extensions of length n exist. Combined with recent work of Rodelo-Van der Linden [9] this establishes the existence of universal central extensions of groups and Lie algebras. Interpreted in the homotopy category of topological spaces, it provides a new perspective on existing results about Quillen's plus construction and its effect on homotopy groups.

Résumé

Dans la catégorie des modules à gauche sur un anneau unitaire, nous démontrons qu'un réflecteur exact à gauche détermine pour chaque $n \geq 1$, un cadre conforme à la théorie de la torsion dans lequel existent des extensions universelles de longueur n. Combiné avec les travaux récents de Rodelo - Van der Linden [9], ce résultat établit l'existence d'extensions centrales universelles de groupes et d'algèbres de Lie. Interprété dans la catégorie d'homotopie des espaces topologiques, elle offre une nouvelle perspective sur les résultats existants sur la construction plus de Quillen et ses effets sur les groupes d'homotopie.

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1. Introduction

In the category of left modules over a unital ring Λ every right exact reflector L determines a class of L-local objects $\mathcal{R}$, together with a nested family of subcategories $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \cdots \supseteq \mathcal{T}_n \supseteq \cdots$ of Λ-$\mathcal{M}od$. A module M is in $\mathcal{T}_n$ if the left derived functors L_i of L vanish on M for $0 \leq i \leq n$. We identify $\mathcal{T}_0$ as the torsion class of a torsion theory $(\mathcal{T}_0, \mathcal{F})$. Accordingly, for $i \geq 1$, the $\mathcal{T}_i$ could be called higher torsion classes.

Our key result (2.2) establishes for M in $\mathcal{T}_{n-1}$ a natural equivalence $Hom_\Lambda(L_nM, -)|_{\mathcal{R}} \rightarrow Ext^n_\Lambda(M, -)|_{\mathcal{R}}$. Via Yoneda's interpretation of Ext, there exists a universal n-step extension $L_nM \rightarrowtail X_n \longrightarrow \cdots \longrightarrow X_1 \twoheadrightarrow M$; see (2.3).

Specializing to commutative Λ yields a bit of a surprise (2.5): if a finitely generated M belongs to $\mathcal{T}_0$, then it belongs $\mathcal{T}_i$ for all $i \geq 0$. Hence only non-finitely generated Λ-modules can have an interesting universal extension.

Specializing to the case where $\Lambda = RG$ is a group ring yields the existence of universal central group extensions via the classical equivalence between Ext and group cohomology; see (3.1). For 1-step extensions this has long been known. For $n \geq 2$, this connection hinges upon a suitable interpretation of the concept 'central n-extension' over a group. Building upon work of Janelidze [7, 2], Rodelo and Van der Linden [9] have just achieved a cohomological interpretation of central n-extensions. - Underlying universal central n-extensions of groups are noncommutative torsion theories paralleling the development described above for universal extensions of modules.

Specializing to 1-step module extensions, we observe an aspect of higher torsion theories (2.8): there is a coreflector $\tau_1 \colon \mathcal{T}_0 \rightarrow \mathcal{T}_1$ if and only if the universal extension of any M in $\mathcal{T}_0$ is of the form $A \rightarrowtail \tau_1 M \twoheadrightarrow M$. This happens whenever $L = R \otimes_\Lambda -$ comes from a ring augmentation $I \rightarrowtail \Lambda \twoheadrightarrow R$ in which $I \otimes_\Lambda I \rightarrow I$ is an isomorphism.

Specializing further to universal 1-step module extensions over a group ring, we obtain a new vista of the treatment of such extensions in earlier works; compare for example [3, 1, 6, 8].

The torsion theoretic framework for universal extensions also adapts nicely to provide a new perspective on existing developments in algebraic topology. In Section 4 we outline how homological localization of spaces yields a homotopy torsion theory, to which there are associated a torsion theory of groups and a torsion theory of modules over a group ring. We identify Quillen's plus construction as the 'torsion-free' reflector of the homotopical torsion-theory, and the associated group and module theoretic torsion theories yield insight into its effect on the homotopy groups of a space.

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2. Universal Module Extensions

An epimorphism $f\colon \Lambda \to R$ in the category of unital rings yields the right exact reflector $L := R \otimes_\Lambda -$ on the category Λ-$\mathcal{M}od$ of left modules over Λ. The L-local objects form a full subcategory $\mathcal{R}$ of Λ-$\mathcal{M}od$. It is the image of the embedding of R-$\mathcal{M}od$ in Λ-$\mathcal{M}od$ via f.

We then obtain a torsion pair $(\mathcal{T}, \mathcal{F})$, with $\mathcal{T}$, the left orthogonal complement of $\mathcal{R}$, and $\mathcal{F} \supseteq \mathcal{R}$ the right orthogonal complement of $\mathcal{T}$. Thus for every X in Λ-$\mathcal{M}od$ there is the functorial torsion-torsion free short exact sequence $\tau X \rightarrowtail X \twoheadrightarrow \varphi X$ in which τX is maximal amongst those submodules S of X with $LS = 0$.

The left derived functors $L_i \cong Tor_i^\Lambda(R, -)\colon \Lambda\text{-}\mathcal{M}od \to \mathcal{R}$ determine subcategories

$$\cdots \subseteq \mathcal{T}_{n+1} \subseteq \mathcal{T}_n \subseteq \cdots \subseteq \mathcal{T}_0 = \mathcal{T}$$

where $\mathcal{T}_n$ consists of all those Λ-modules X with $L_0X = \cdots = L_nX = 0$.

Lemma 2.1. *A Λ-module X belongs to $\mathcal{T}_n$ if and only if $Ext_\Lambda^0(X, A) = \cdots = Ext_\Lambda^n(X, A) = 0$ for each $A \in \mathcal{R}$.*

Thus Yoneda's interpretation of Ext tells us that a Λ-module M belongs to $\mathcal{T}_{n-1}$ if and only if, $LM = 0$ and, for A in $\mathcal{R}$ and $1 \le k \le n-1$, every k-step extension $A \rightarrowtail X_k \to \cdots \to X_1 \twoheadrightarrow M$ is congruent to the trivial k-step extension. In this situation, L_nM is a universal object because of the

Theorem 2.2. **$(\mathcal{T}_{n-1}, \mathcal{R})$-representation theorem**
For $M \in \mathcal{T}_{n-1}$, there is a transformation $\rho\colon Hom_\Lambda(L_nM, -) \to Ext_\Lambda^n(M, -)$ which restricts to an equivalence

$$\rho|_{\mathcal{R}}\colon Hom_\Lambda(L_nM, -)|_{\mathcal{R}} \longrightarrow Ext_\Lambda^n(M, -)|_{\mathcal{R}}.$$

Corollary 2.3. **Existence of universal $(\mathcal{T}_{n-1}, \mathcal{R})$-module extensions**
In the setting of (2.2) the element $[E] := \rho(Id_{L_nM})$ is a universal $(\mathcal{T}_{n-1}, \mathcal{R})$-extension; i.e. for an arbitrary n-step extension $[E']$ of B in $\mathcal{R}$ and over M there exists a unique Λ-map $u\colon L_nM \to B$ such that $u_*[E] = [E']$.

Corollary 2.4. **Strong property of universal $(\mathcal{T}_0, \mathcal{R})$-extensions**
A Λ-module M with $LM = 0$ has a short exact sequence $L_1M \rightarrowtail \tilde{M} \twoheadrightarrow M$ such that every diagram

$$\begin{array}{ccccc} L_1M & \rightarrowtail & \tilde{M} & \twoheadrightarrow & M \\ \downarrow & & \downarrow & & \| \\ B & \rightarrowtail & X & \twoheadrightarrow & M \end{array}$$

with B in $\mathcal{R}$ can be filled uniquely with dashed arrows as shown.

Thus our existence result for universal extensions constitutes a simultaneous generalization of classical work in two independent directions: a) from modules over group rings to modules over arbitrary unital rings, and b) extensions of arbitrary length. We mention some features of these universal extensions.

Universal extensions over commutative rings

If Λ is commutative (2.3) says that each M in $\mathcal{T}_{n-1}$ has a universal n-step extension. How is it possible that, in an extensively researched subject like commutative algebra, we have not encountered such extensions already? - It appears they have been able to hide, perhaps because of the following somewhat unexpected phenomenon.

Theorem 2.5. *Let $\pi\colon \Lambda \twoheadrightarrow R$ be a map of commutative rings whose underlying set map is surjective, and let $L := R \otimes_\Lambda -$ be the associated reflector. If the Λ-module M satisfies $LM = 0$, then $L_nM = 0$ for all $n \ge 0$.*

This follows by adapting the argument given for Λ a commutative group ring by W.G. Dwyer in [5, Thm 1]. Thus only a non-finitely generated Λ-module can have an interesting universal extension. Such extensions exist.

Universal 1-step extensions

Theorem 2.6. **Recognizing universal 1-step extensions**
With respect to an arbitrary categorical ring epimorphism $\Lambda \twoheadrightarrow R$, *a* $(\mathcal{T}_0, \mathcal{R})$*-extension* $A \rightarrowtail X \twoheadrightarrow M$ *is universal if and only if the connecting homomorphism* $\partial\colon L_1M \to A$ *is an isomorphism.*

Corollary 2.7. *For an arbitrary* $(\mathcal{T}_0, \mathcal{R})$*-extension* $A \rightarrowtail X \twoheadrightarrow M$ *the following hold:*

(i) *If E is universal, then* $X \in \mathcal{T}_0$.
(ii) *if X is in* $\mathcal{T}_1$, *then E is universal.*

Examples show that (2.7.ii) is not necessary. However,

Theorem 2.8. $\mathcal{T}_1$ *is coreflective in* $\mathcal{T}$ *with coreflector* $\tau_1\colon \mathcal{T}_0 \to \mathcal{T}_1$ *if and only if all universal* 1*-step* $(\mathcal{T}_0, \mathcal{R})$*-extensions are of the form* $L_1M \rightarrowtail \tau_1 M \twoheadrightarrow M$.

Situations where (2.8) applies include the following:

Theorem 2.9. *Let* $I \rightarrowtail \Lambda \twoheadrightarrow R$ *describe an augmented ring whose augmentation ideal I is idempotent. If the multiplication map* $\mu\colon I \otimes_\Lambda I \twoheadrightarrow I$ *is an isomorphism, then* $\mathcal{T}_1$ *is coreflective in* $\mathcal{T}_0$ *with coreflector the transformation* $(I \otimes_\Lambda -) \twoheadrightarrow (\Lambda \otimes_\Lambda -)$.

Examples of (2.9) arise for group rings: given a group G and a commutative ring R, consider the torsion theory associated to the augmentation $I_RG \rightarrowtail RG \twoheadrightarrow R$.

Theorem 2.10. *Compare [1]. If* $H_1(G; R) = 0 = H_2(G; R)$, *then* $\mathcal{T}_1$ *is coreflective in* $\mathcal{T}_0$ *with coreflector* $I_RG \otimes_{RG} -$. *If an RG-module M satisfies* $H_0(G; M) = 0$, *then its universal extension is* $H_1(G; M) \rightarrowtail I_RG \otimes_{RG} M \twoheadrightarrow M$.

3. Universal central group extensions

The existence of universal central group extensions follows from the existence of universal central module extensions; see Section (2). Given a group G and a commutative unital ring R, we have the classical equivalences of functors on the category of R-modules

$$Ext^n_{RG}(I_RG, -) \cong H^{n+1}(G; -) \cong Centr^n(G, -).$$

The latter term denotes equivalence classes of central n-extensions over G, developed within the framework of categorical Galois theory. Their classification in cohomological terms has just been achieved by Rodelo and Van der Linden [9]. For $n = 1$ it agrees with the classical interpretation $H^2(G; A)$ as equivalence classes of extensions over G with central kernel.

We sketch an approach to discuss universal central group extensions in a torsion theoretic framework: Let $\hat{L}$ be a right exact reflector from $\mathcal{G}rp$ onto a subcategory $\mathcal{R}$ of $\mathcal{A}b$. Then $R := L\mathbb{Z}$ is a commutative unital ring for which the characteristic map $\chi\colon \mathbb{Z} \to R$ is a ring theoretic epimorphism, and L is abelianization followed by $(R \otimes_{\mathbb{Z}} -)$. The group theoretic torsion theory associated to $\hat{L}$ yields for a given group X the 'torsion-torsion free' sequence $G \rightarrowtail X \twoheadrightarrow Q$, in which G is the unique maximal R-perfect subgroup of X; so $R \otimes G_{ab} = 0$.

The functor $L := R \otimes_{RG} -$ reflects $RG\text{-}\mathcal{M}od$ onto the target $\mathcal{R}$ of $\hat{L}$. Whenever G is R-perfect, the ideal in the augmentation sequence $I_RG \rightarrowtail RG \twoheadrightarrow R$, is idempotent. Therefore $\mathcal{R}$ equals the torsion-free category of the torsion pair $(\mathcal{T}_0, \mathcal{F})$; i.e. this torsion theory is a TTF-theory.

Theorem 3.1. **Universal group vs. module extensions**
Compare [8]. Given a ring epimorphism $\mathbb{Z} \twoheadrightarrow R$ *and a group G with* $R \otimes_{\mathbb{Z}} G_{ab} = 0$, *the following hold. The module* I_RG *has a universal* $(\mathcal{T}_0, \mathcal{R})$*-extension*

$$H_2(G; R) \cong H_1(G; I_RG) \rightarrowtail \widetilde{I_RG} \twoheadrightarrow I_RG.$$

It determines a universal central extension of G via the pullback

$$\begin{array}{ccccccc} H_2(G;R) & \rightarrowtail & \widetilde{I_RG} \rtimes G & \twoheadrightarrow & I_RG \rtimes G & & (g-1,g) \\ \| & & \uparrow & \text{pull} & \uparrow & & \uparrow \\ H_2(G;R) & \rightarrowtail & \hat{G} & \twoheadrightarrow & G & & g \end{array}$$

Conversely, from the universal central extension G, the universal module extension of I_RG *may be recovered as*

$$H_2(G;R) \rightarrowtail RG \otimes_{R\hat{G}} I_R\hat{G} \twoheadrightarrow I_RG.$$

4. Relevance to Quillen's plus construction

We sketch an example to explain how the torsion theoretic framework outlined above encompasses known developments in algebraic topology. A homotopical reflector on topological spaces is Bousfield's localization with respect to singular homology $h := H(-;R)$, where R is a subring of the rational numbers. The associated homotopical torsion-theory of pointed path connected spaces yields for a space Y the torsion-torsion free sequence $A_hY \to Y \to Y^{+h}$. The coreflector A_hY is Dror's acyclization [4], while the reflector is Quillen's plus construction.

The group theoretic torsion theory associated to h comes from the reflector $\hat{L} := (R \otimes -) \circ ab$ of Section (3). It conceptualizes the exact sequence

$$\begin{array}{ccccccc} H_2(G;R) & \rightarrowtail & \tilde{G} := \pi_1 A_hY & \longrightarrow & \pi_1 Y & \twoheadrightarrow & \pi_1 Y^{+h} \\ & & & \searrow\!\!\!\!\searrow \quad G \quad \nearrow\!\!\!\!\!\!\rightarrowtail & & & \end{array}$$

as being spliced together from the group theoretic torsion-torsion free sequence of $X := \pi_1 Y$ on the right, and the universal R-central extension over the maximal R-perfect subgroup G of $\pi_1 Y$.

The module theoretic torsion theory associated to h and G comes from the reflector L induced by the ring augmentation $R\tilde{G} \twoheadrightarrow R$ as in Section (2). As in [1], it provides information about the exact sequence $\pi_nA_hY \to \pi_nY \to \pi_nY^{+h}$ via

$$\begin{array}{ccccccc} H_1(\tilde{G};\tau\pi_nY) & \rightarrowtail & \widetilde{\tau\pi_nY} & \longrightarrow & \pi_nY & \twoheadrightarrow & \pi_nY/\tau\pi_nY \\ & & & \searrow\!\!\!\!\searrow \quad \tau\pi_nY \quad \nearrow\!\!\!\!\!\!\rightarrowtail & & & \end{array}$$

This exact sequence is spliced together from, on the right, the torsion-torsion free sequence of π_nY. While the left hand terms form the universal extension over $\tau\pi_nY$ as in (2.10).

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