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The Theory of Ends

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This is a retyped copy of the original article. Some typos that existed in the original publication have been corrected. Unfortunately, new typos might inadvertently have been introduced.

The Theory of Ends

Dedicated to Hans Freudenthal

Georg Peschke

Introduction It lies in the nature of compact spaces that many of their properties and invariants are completely determined by a finite collection of local data. For non-compact spaces this reduction to finiteness does not usually work. Therefore, we might wish to supplement compactly supported information by invariants concerning the complements of compact sets. This is possible. Indeed, the key step towards this goal is the notion of an 'end' of a space which was introduced by H. Freudenthal in his Ph.D. thesis and submitted for publication 60 years ago in March 1930; see H. Freudenthal [10].

If X belongs to a certain class of spaces, Freudenthal achieves a universal compactification $\overline{X}$ of X by adjoining to X its end points. More precisely:

1 Theorem **(Freudenthal [10])** *There is a compact Hausdorff space* $\overline{X}$ *and a map* $i\colon X \to \overline{X}$ *such that*

(i) X *is homeomorphically imbedded as a dense open subset of* $\overline{X}$*;*

(ii) $\overline{X} - X$ *is totally disconnected;*

(iii) every map $j\colon X \to \hat{X}$ *satisfying (i) and (ii) factors uniquely through* $\overline{X}$.

The space $\overline{X}$ is called the Freudenthal compactification of X and the complement of X in $\overline{X}$ is the end space of X, denoted by $\mathcal{E}(X)$. Both, $\overline{X}$ and $\mathcal{E}(X)$, are uniquely determined, up to homeomorphism, by the universal property (1.iii).

Thus the end-space of a compact Hausdorff space is empty. The real line $\mathbb{R}$ has two ends, one at $+\infty$ and one at $-\infty$. So the Freudenthal compactification $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1, 1]$. If $n \geq 2$, $\mathbb{R}^n$ has one end. In this case the Freudenthal compactification coincides with the 1-point compactification of Alexandroff.

It appears that Freudenthal was led to his notion of 'ends' by the following observation. On a space X consider any path connected family of homeomorphisms $\mathcal{F}$ containing the identity map of X. Pick any compact subset of X and let U denote a connected component of its complement. Then U is an almost invariant subset of X with respect to any $f \in \mathcal{F}$, i.e. $f(U) - U$ is contained in some compact subset of X. After passing to the Freudenthal compactification $\overline{X}$ of X, this observation translates into: $\mathcal{F}$ extends to a family of homeomorphisms of $\overline{X}$ and each of the extended homeomorphisms is the identity map on $\mathcal{E}(X)$. If, in addition, $\mathcal{F}$ acts transitively on X, this places severe constraints on the end space $\mathcal{E}(X)$. in particular:

2 Theorem **(Freudenthal [10])** *A path connected topological group has at most two ends.*

For example, remove two points from $\mathbb{R}^3$. The resulting space has three ends, hence cannot carry the structure of a topological group.

Some properties typical of a transitive group action are still shared by groups acting with compact fundamental domain. Covering spaces with compact base provide examples of such actions. H. Hopf extended Freudenthal's investigations in this direction and discovered:

3 Theorem **(Hopf [15])** *Let $p: X \to B$ be a covering map with compact base. Then X has 0, 1 or 2 (discrete) ends or $\mathcal{E}(X)$ is a Cantor space. Moreover, if $p': X' \to B'$ is another covering map with compact base and p, p' have isomorphic groups of covering transformations G, then the end spaces $\mathcal{E}(X)$ and $\mathcal{E}(X')$ are homeomorphic.*

The first conclusion is in line with Freudenthal's work. The second conclusion says that the end space of such a covering space is an invariant of the groups G of covering transformations. Thus we define the end space of G by $\mathcal{E}(G) := \mathcal{E}(X)$. For example, $\mathcal{E}(\mathbb{Z})$ is the (discrete) 2-point space because the covering map $p: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ has $\mathbb{Z}$ as its group of covering transformations.

In Section 1, we shall explain the central aspects of the theory of ends of topological spaces and discrete groups. In honour of the originators H. Freudenthal and H. Hopf we shall comment on some of the verifications of their work.

There is a maximal class of spaces to which Freudenthal's end space construction applies; see Freudenthal [11]. This class contains all connected countable and locally finite CW-complexes. For simplicity, we assume throughout that spaces whose ends we consider are CW-complexes of this type. Together with proper maps (preimages of compact sets are compact) these spaces form a good category to develop end-sensitive invariants. Often such invariants are obtained by suitably modifying functors from ordinary Algebraic Topology. We provide some flavour of the resulting proper homotopy theory in Section 2.

Ends of spaces and of groups turn out to be a cohomological invariant. This transition was forged by Specker [24] in the late 1940's. It lead to fruitful interplay between algebraic and topological methods when dealing with ends. In Section 3, we shall sample some results that have been obtained in this fashion.

When compactifying a manifold it is desirable that the result be a manifold. This leads to the following refinement, called completion, of the Freudenthal compactification within the category of manifolds: The completion of a manifold M is an imbedding of M in a compact manifold W such that $W - M$ is contained in the boundary of W. For example the completion of the open unit ball in $\mathbb{R}^n$ is the compact unit ball and the compactifying boundary is the unit sphere of $\mathbb{R}^n$. This and related problems were first answered by the work of Browder, Levine, Livesay [2] and Siebenmann [23] and have since then received persistent attention. In Section 4, we shall outline the key aspects of this development.

These are the main strands in the theory of ends that we chose to follow. The presentation is not comprehensive. However, it is hoped that most of the germinating ideas and phenomena in the theory of ends are exposed.

It must be a pleasure for any researcher to observe how his own ample contributions are eventually picked up by others and developed further in numerous directions. Hans Freudenthal enjoyed this privilege over a period of, now, 60 years. Congratulations!

1 Ends of spaces and groups

A space permitting a free and transitive action of a topological group has at most two ends; Freudenthal [10]. In this section we sketch the underlying facts as well as directly related further developments.

Let us begin by explaining the Freudenthal-compactification of a space X; see [10]. We remind the reader of the stipulation of the introduction: X is a connected countable locally finite CW-complex.

Then X is the union of an ascending sequence of compact sets

$$C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots .$$

The complementary sets $C_n{}' := X - C_n$ form an inverse sequence

$$C_1{}' \supset C_2{}' \supset \cdots \supset C_n{}' \supset \cdots .$$

As a point set take $\mathcal{E}(X)$ to be the collection of sequences (U_n) of nonempty sets

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$$

where U_n is a connected component of $C_n{}'$. The Freudenthal-compactification of X is the space $\overline{X} := X \cup \mathcal{E}(X)$ whose topology is generated by the open sets of X together with all sets

$$U_n \cup \left\{(V_i) \in \mathcal{E}(X) : V_{n+j} \subset U_n \text{ for all } j \geq 0\right\}.$$

Thus the sequence elements of an end $\varepsilon = (U_n) \in \mathcal{E}(X)$ also define a neighbourhood basis of ε in $\overline{X}$. A different choice of an ascending sequence (C_n) only changes $\overline{X}$ up to homeomorphism.

From this construction we see that a proper map $f\colon X \to Y$ (preimages of compact sets are compact) has a continuous extension $\overline{f}\colon \overline{X} \to \overline{Y}$ over the Freudenthal-compactifications of X and Y.

We shall now sketch the key issues behind

2 Theorem *[10] A path connected topological group X has at most two ends.*

The argument involves two key steps.

Firstly, left translating X by any $x \in X$ is a homeomorphism f_x and, hence, has a continuous extension over X. As X is connected, f_x is homotopic to the identity through homeomorphisms. Such a homotopy also extends over $\overline{X}$. As the end space of X is totally disconnected, $\overline{f_x}$ must leave all end points of X fixed.

Secondly, suppose X has distinct ends ε and ε' with neighbourhoods U and U'. Then Freudenthal shows the existence of an element $x \in X$ which translates the complement of U in X into U'. Therefore, if X has three distinct ends $\varepsilon, \varepsilon', \varepsilon''$, choose disjoint open neighbourhoods U, U', U''. The complement of U contains $U' \cup U''$, yet can be translated into U''. So, there is a translation of X sending ε' to ε'', contradicting the fact that translations leave ends fixed.

It is clear that these arguments also apply to certain actions of a topological group on X. The interplay between the two end-spaces involved is also explained in Freudenthal's paper [10]. The desire to extend such insight as far as possible was the motivation for Freudenthal's work [11], where the feasibility of the end space construction under weakest possible hypotheses is studied and previous applications to actions of topological groups are accordingly generalized.

Countable discrete groups cannot act transitively on spaces of positive dimension but may act with compact fundamental domain. Such actions still share some essential behavioral phenomena with transitive actions. So we may ask to which extent Freudenthal's results carry over to this more general situation.

The answer was provided by H. Hopf in the early 1940's. For convenience, we restate

3 Theorem **(H. Hopf [15])** *Let G be a finitely generated discrete group acting on a space X by covering transformations. Suppose the orbit space $B := X/G$ is compact. Then (i) and (ii), below, hold.*

(i) The end space of X has 0, 1 or 2 (discrete) elements or is a Cantor space.

(ii) If G also acts on Y satisfying the hypotheses above, then X and Y have homeomorphic end spaces.

Conclusion (i) is in line with Freudenthal's work. That $\mathcal{E}(X)$ may have more than two ends comes from the fact that the action of a discrete group on X need not fix the end space; compare the explanation given below Theorem 2.

Conclusion (ii) suggests to regard the end space of X as an invariant of the group G itself:

4 Definition *Let $p\colon X \to B$ be a covering map with compact base B and group of covering transformations G. The end space of G is*

$$\mathcal{E}(G) := \mathcal{E}(X).$$

It should be possible to determine an invariant of G by looking at G itself. Indeed, a discrete group is a space in which a fixed subset S determines a combinatorial notion of connectedness: $U \subset G$ is S-connected, if for all $u, u' \in U$ there is some $(u_0, \ldots, u_r) \in U^{r+1}$ with $u = u_0, u' = u_r$ and $u_{i-1}^{-1} u_i \in S \cup S^{-1} \cup \{1\}$, for all $1 \leq i \leq r$. If G is finitely generated and S is a finite set of generators, then the complement of any finite set C in G has finitely many S-connected components $(G - C)^*$. Finitely generated groups are countable. Hence, we may proceed as with spaces and define

$$\mathcal{E}(G) \ := \ \operatorname{inv}\lim (G - C_0)^* \leftarrow (G - C_1)^* \leftarrow \cdots$$

where G is the union of the ascending sequence of finite subsets (C_n).

This approach can be paralleled topologically in the corresponding Cayley graph of G. So, the topological and combinatorial end spaces of G are homeomorphic. This combinatorial approach to $\mathcal{E}(G)$ is a very special case of a general theory of ends of a discrete space with a combinatorial relation of connectedness. Freudenthal developed this theory in response to Hopf's work; see [12].

It is possible to carry the work of Freudenthal and Hopf further. Even if a group G acts on a space X by covering maps without compact fundamental domain, the end spaces of X, G and the orbit space $B := X/G$ are interrelated. This has been shown by the present author [19].

Here G need no longer be finitely generated. However, Hopf's work renders legitimate the construction

$$\mathcal{E}(G) := \text{dir lim}\,\mathcal{E}(H)$$

where H runs through the system of finitely generated subgroups of G. The action of G on X determines uniquely a continuous map

$$\lambda\colon \mathcal{E}(G) \longrightarrow \mathcal{E}(X).$$

If B is compact, λ is a homeomorphism and we have restated Hopf's result. In general, the deviation of λ from being a homeomorphism is completely explained by end data of B.

2 Proper homotopy theory

The opening paragraph of this article refers to the development of end-sensitive invariants of non-compact spaces which supplement the usual compactly supported ones. Many of these invariants are directly related to familiar ones in Algebraic Topology. We proceed to give some flavour of this development.

The appropriate setting for our discussion is the category of connected locally finite countable CW-complexes and proper maps, proper homotopies, etc.

First of all, let us recall a description of ends due to Hopf [15].

5 Theorem *(i) For every end ε of a space X, there exists a proper map $a\colon [0,\infty) \to X$ with $\overline{a}(\infty) = \varepsilon$.*

(ii) Two proper maps $a_i\colon \{i\} \times [0,\infty) \to X$, $i = 1$ or 2, determine the same end ε if and only if they have a proper extension $f\colon L \to X$, where L is the 'infinite ladder' $\{0,1\} \times [0,\infty) \cup [0,1] \times \{0,1,2,\dots\}$.

In order to define proper homotopy groups at the end ε of X, replace the role of the base point in the ordinary theory by a proper base ray $\underline{*}\colon [0,\infty) \to X$, representing ε. Moreover, replace the n-sphere S^n by some 1-ended analogue $\underline{S}^n$ containing $[0,\infty)$ as a subspace. Then consider proper maps $(\underline{S}^n,[0,\infty)) \to (X,\underline{*})$ and suitable proper homotopy relations.

One meaningful choice for $\underline{S}^n$ is

$$\underline{S}^n \;:=\; [0,\infty)\times S^n/\{0\}\times S^n.$$

In this case $\underline{S}^n$ is homeomorphic to $\mathbb{R}^{n+1}$. This illustrates that $\mathbb{R}^{n+1}$, from the proper homotopy point of view is seen like an n-dimensional sphere - not like a point as in ordinary homotopy theory.

There are various meaningful candidates for $\underline{S}^n$ and also for proper homotopy relations. They are all interrelated; see e.g. E.M. Brown [3], Z. Cerin [4], J.I. Extremiana, L.J. Hernández and M.T. Rivas [8] and the references there.

Proper homotopy groups permit an analogue of the Whitehead-theorem for proper homotopy equivalences:

6 Theorem **(E.M. Brown [3])** *A proper map $f: X \to Y$ is a proper homotopy equivalence if and only if*

(i) f induces a homeomorphism of end spaces

(ii) f is a weak homotopy equivalence in the ordinary sense

(iii) for each end ε of X, f induces isomorphism between all proper homotopy groups at ε and those at $\overline{f}(\varepsilon)$.

Similarly, proper homology groups are defined by replacing the standard domain of singular maps by a suitable non-compact analogue. For example, the unit cube I^n is replaced by $K^n := K_1\times\cdots\times K_n$, where K_i is either I or $[0,\infty)$. Singular maps $K^n \to X$ are now required to be proper.

There is a Hurewicz homomorphism from proper homotopy groups to proper homology groups and a proper version of the Hurewicz-isomorphism theorem holds; see Extremiana, Hernández and Rivas [8].

On manifolds, proper homotopy theory offers a proper version of the h-cobordism theorem for non-compact manifolds; see Freedman [9] and Quinn [21]. Sometimes proper homotopy equivalent manifolds are homeomorphic. We state explicitly two theorems of this type in the 'tough' dimension 4. The corresponding result in higher dimensions had been obtained earlier; cf. Theorem 16, below.

7 Theorem **(M.H. Freedman [9])** *A topological 4-manifold, which is proper homotopy equivalent to $\mathbb{R}^4$, is homeomorphic to $\mathbb{R}^4$.*

8 Theorem **(M.H. Freedman [9])** *A smooth* 4*-manifold, which is proper homotopy equivalent to* $S^3 \times \mathbb{R}$ *is homeomorphic to* $S^3 \times \mathbb{R}$.

3 Ends of groups revisited

Given a space X, the cardinality $e(X)$ if its end-space $\mathcal{E}(X)$ is a cohomological invariant (we do not distinguish between various infinite cardinalities). The transition to this algebraic interpretation of the end invariant was forged by Specker [24]. We shall proceed to outline the development in this direction. Part of this material was already compiled by Epstein [7].

9 Theorem **(E. Specker [24])** $e(X) = dim_{\mathbb{Z}/2} H_c^0(X; \mathbb{Z}/2)$.

Here, H_c^* denotes cohomology with cellular cochains module compactly supported cochains. Applying this result to the Cayley graph of a finitely generated group G yields

10 Theorem **(E. Specker [24])** $e(G) = 1 + dim\ H^1(G; \mathbb{Z}/2[G])$.

Interpreting a 0-cochain with $\mathbb{Z}/2$-coefficients in the Cayley-graph of G as a subset of G yields yet another description of the number of ends of G:

11 Theorem **(D.E. Cohen [5])** $e(G) = dim_{\mathbb{Z}/2} \frac{a-inv(G)}{f(G)}$.

Here the collection of subsets of G, denoted by $S(G)$, is treated as a Boolean algebra with 'symmetric difference' and 'intersection' as operations. Then $a - \mathrm{inv}(G)$ is the collection of subsets of G which are almost invariant under left-translation by all elements of G (compare the explanation given with Theorem (2) in the introduction). This makes $a - \mathrm{inv}(G)$ a subalgebra of $S(G)$. Moreover, $f(G)$, the collection of finite subsets of G, is an ideal of $S(G)$.

This description of $E(G)$ does no longer require the hypothesis 'finitely generated', hence produces an invariant for all groups. Classifying groups by the number of their ends has been begun by Hopf [15], was completed for finitely generated groups by Stallings [26] and has only recently been completed for all groups by Dicks and Dunwoody [6].

12 Theorem *Let G by any group. Then* $e(G)$ *is 0,1,2 or* ∞.

(i) $e(G) = 0$ *if and only if G is finite.*

(ii) $e(G) = 2$ *if and only if* G *has an infinite cyclic subgroup of finite index.*

(iii) $e(G) = \infty$ *if and only if one of the following holds*

(a) $G \cong B *_C D$ *with* C *finite and* $B \neq C \neq D$ *and* G *not of type (ii).*

(b) $G \cong B *_C x$ *with* C *finite and* G *is not of type (ii).*

(c) G *is countably-infinite and locally-finite.*

(iv) $e(G) = 1$ *if and only if* G *is not of type (i),(ii), or (iii).*

We also remark that it is possible and meaningful to define the notion of a relative end of a group G with respect to a given subgroup $S \subset G$; see Kropholler and Roller [16] and the references there.

Let us now sample some applications, first of algebraic nature: Stallings obtained a criterion which detects non trivial free products among finitely generated groups with infinitely many ends. This was an essential ingredient in the proof of a conjecture of Serre (Theorem (13)) and a conjecture of Eilenberg-Ganea (Theorem (14)). Theorem (14) was first proven by Stallings for finitely generated groups and then for arbitrary groups by Swan; compare also Cohen [5].

13 Theorem **(Stallings [25][26])** *Suppose* G *is a finitely generated torsion-free group with a free subgroup of finite index. Then* G *is free.*

14 Theorem **(Stallings [25][26], Swan [27])** *Groups of cohomological dimension 1 are free.*

The topological application of ends of groups below had already been foreseen by H. Hopf and was completed by Specker [24]. By combining his cohomological interpretation of the end-invariant with Poincar'e duality, he saw that the $(n-1)$-st homology group of an open orientable n-manifold is free abelian of rank $e(M) - 1$. Applying this result to the universal cover of a closed 3-manifold M yields information on $\pi_2 M$ via the Hurewicz-isomorphism theorem.

15 Theorem **(Specker [24])** *Suppose* M *is a closed 3-manifold. Then* $\pi_2 M$ *is free abelian of rank*

- *0* *if* $\pi_1 M$ *is finite or* $e(\pi_1 M) = 1$
- *1* *if* $e(\pi_1 M) = 2$
- ∞ *if* $e(\pi_1 M) = \infty$

Moreover, M is aspherical if and only if $e(\pi_1 M) = 1$.

The invariant $e(G)$ can be enriched by additional end-structure (in the spirit of Section 2). Thus there are the notions of a fundamental group of a finitely presented group, Lee and Raymond [17], semistability at ∞, Mihalik [18], as well as higher dimensional invariants; the article of Geoghegan [13] provides excellent perspective.

Such additional structure components sharpen considerably the use of the invariant $e(G)$. To illustrate this let us consider the question of which closed manifolds M have Euclidean space $\mathbb{R}^n$ as their universal cover.

In dimensions 1 and 2, the answer follows from well known classification theorems. So, we may assume $n \geq 3$. As $\mathbb{R}^n$ is contractible, M must be an Eilenberg-Mac Lane space of type $(\pi, 1)$ and, by Hopf's work, π must have precisely one end. Now our problem splits into two parts:

(a) among the contractible n-manifolds identify those which are homeomorphic to $\mathbb{R}^n$

(b) impose conditions on $\pi_1 M$ which imply those found in (a).

A satisfactory response to (a) is the following result of Siebenmann:

16 Theorem (Siebenmann [23]) *Let M be a contractible topological manifold without boundary which is 1-locally connected at ∞. If $n :=$ dim $M \geq 5$, then M is homeomorphic to $\mathbb{R}^n$.*

A space X is 1-locally connected at ∞ if every neighbourhood U of ∞ contains a neighbourhood V of ∞ such that every loop in V is contractible in U.

Part (b) of the above program caught the attention of Lee and Raymond who showed that the fundamental group of the finitely presented group π is trivial if it contains a finitely generated non-trivial abelian normal subgroup. Consequently:

17 Theorem (Lee, Raymond [17]) *Let M be a closed manifold of type $K(\pi, 1)$ whose dimension n is greater or equal 5. If π contains a finitely generated non-trivial abelian normal subgroup, then the universal cover of M is homeomorphic to $\mathbb{R}^n$.*

This discussion applies to all closed connected manifolds which admit a Riemannian metric with non-positive sectional curvature: at any point of such a manifold the exponential map is a covering projection.

In the context of Riemannian Geometry, Siebenmann's theorem (16) can be sharpened considerably. This result illustrates perfectly the combined power of information on compact sets together with end data.

18 Theorem **(Greene, Wu [14])** *Let M be a complete Riemannian manifold, of dimension ≥ 3, with non-negative sectional curvature. If M is simply connected at ∞ and the sectional curvature of M is 0 outside some compact set, then M is isometric to $\mathbb{R}^n$.*

4 Completion of manifolds

When compactifying a manifold it is desirable that the result again be a manifold. This process is called completion and constitutes a non-trivial adaptation of Freudenthal's compactification to the category of manifolds: The completion of a manifold M is an imbedding of M in a compact manifold W such that the complement of M is contained in the boundary of W. A priori it is not at all clear if a given manifold admits a completion.

This challenge was taken up by Browder, Levine, Livesay [2] and Siebenmann [22] in the first half of the 1960's. We follow Siebenmann to get some idea of what is involved in the case where we wish to complete an open C^∞-manifold M whose boundary is empty; i.e. we ask for an imbedding $i\colon M \to W$ such that W is compact and $\partial W = W - i(M)$.

If such a completion exists and N is a closed connected component of the boundary of W, then N has a collar diffeomorphic to $N \times [0, 1)$ in W. Removing $N \cong B \times \{0\}$ from W creates an end in the sense of Freudenthal which has a collar neighbourhood of the form $N \times (0, 1)$ in M. Consequently,

19 Lemma *An open manifold without boundary can be completed by an embedding into a compact manifold if and only if M has finitely many ends each of which has a collar.*

Thus the task is clear: If the end space of M is finite, pick an end ε and test if it has a collar in M. Siebenmann's approach to 'testing' is this: start with some neighbourhood V of ε and try to modify it until it looks like a collar from the homotopy point of view; i.e. $\mathrm{Bd}V$, the boundary of V in M, is a closed manifold such that the inclusion $\mathrm{Bd}V \to V$ is a homotopy equivalence.

Obstructions to this program can occur and are of two possible types:

(1) The fundamental groups of the open sets forming a neighbourhood basis of ε fluctuate too much.

(2) Neighbourhoods of ε do not have a finite CW-complex in their homotopy type. The first obstruction involves simple technicalities with fundamental groups. The second obstruction is measured by the 'Wall-obstruction' to finite domination of a CW-complex; see [28]. If both obstructions vanish and $\dim M \geq 6$, then M has a completion.

Completion of manifolds has since received persistent attention. Subsequent research was dedicated to the case where $\mathrm{Bd}\, M$ is not empty and/or was specialized to low dimensions; see [1] for an overview as well as for more references.

In fact, these authors characterize orientable 3-manifolds with 'missing' boundary pieces as those 3-manifolds which are end 1-movable, have finitely many summands and finitely generated first homology. 'End 1-movable' means that every compact $K \subset M$ is contained in some compact $L \subset M$ such that every loop in $M - L$ can be homotoped to an end of M by a homotopy in $M - K$.

Quinn generalizes further the problem of completing a manifold to that of completing a map: Suppose M is a manifold and $\varepsilon\colon M \to X$ is a map. A completion of ε is an imbedding $M \hookrightarrow W$ such that $W - M$ is contained in the boundary of W, together with a proper extension $e'\colon W \to X$ of ε over W. In his paper [20], Quinn develops conditions under which a map ε admits a completion.

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Erratum R. Geoghegan for pointed out that Theorem 6 is false as stated. – It is true under the additional hypothesis that X and Y are finite dimensional.