Localization of epimorphisms and monomorphisms in Homotopy theory II

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Abstract:

In this note, we prove that localization of spaces preserves epimorphisms and monomorphisms in homotopy theory in some conditions.

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1. Introduction

Recall that f **:** $X \to Y \in$ HCW^*, the homotopy category of pointed path-connected CW-spaces, is a homotopy epimorphism (monomorphism) if given u,v **:** $Y \to Z \in HCW^*$ (u,v **:** $Z \to X \in$ HCW^*), $u \circ f = v \circ f$ implies $u = v$ ($f \circ u = f \circ v$ implies $u = v$).

In [4], we proved the following theorem, answering a guestion posed by Hilton and Roitberg in [3] whether localization of nilpotent spaces preserves homotopy epimorphisms and homotopy monomorphisms.

Theorem 1. If f **:** $X \to Y$ is a homotopy epimorphism (monomorphism) of nilpotent spaces, then the p-localized map f_p **:** $X_p \to Y_p$ is also a homotopy epimorphism (monomorphism). Here p is a prime or 0.

In this note, we shall show that Theorem1 is true for some non-nilpotent spaces. It is interesting and difficult to study whether localization of spaces in Casacuberta-Peschke's sence preserves epimorphisms and monomorphisms in homotopy theory. The main result of the note is the followings.

Theorem 2. Let f **:** $X \to Y$ be a homotopy epimorphism such that f_* **:** $\pi_1(X) \to \pi_1(Y)$ is an isomorphism. If it satisfies one of the following conditions,

(1) $\pi_1(X)$ is a finite group,

(2) $\pi_1(X)$ is a nilpotent torsion group,

(3) $\pi_1(X)$ is a p- torsion group,

then the p-localized map f_p **:** $X_p \to Y_p$ is also a homotopy epimorphism.

Theorem 3. Let f **:** $X \to Y$ be a homotopy monomorphism such that f_* **:** $\pi_1(X) \to \pi_1(Y)$ is an isomorphism. If $\pi_1(X)$ is a p- torsion group, then the p-localized map f_p **:** $X_p \to Y_p$ is also a homotopy monomorphism.

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2. Lemmas

Lemma 1[4]**.** (1) Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "j_1"] \\ Y \arrow[r, "j_2"] & C \end{tikzcd} \qquad \text{...........} \quad (2.1)$$

be a homotopy pushout. Then $f: X \to Y$ is a homotopy epimorphism if and only if $j_1 = j_2$.

(2) Let

$$\begin{tikzcd} E \arrow[r, "i_2"] \arrow[d, "i_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \text{...........} \quad (2.2)$$

be a homotopy pullback. Then $f: X \to Y$ is a homotopy monomorphism if and only if $i_1 = i_2$.

Lemma 2. Let $f: X \to Y$ be a map such that $f_*: \ \pi_1(X) \to \pi_1(Y)$ is an isomorphism. If X_p and Y_p are 1-connected, then the p-localization of the square (2.1)

$$\begin{tikzcd} X_p \arrow[r, "f_p"] \arrow[d, "f_p"'] & Y_p \arrow[d, "j_{1p}"] \\ Y_p \arrow[r, "j_{2p}"] & C_p \end{tikzcd} \qquad \text{...........} \quad (2.3)$$

is also a homotopy pushout.

Proof. Recall that if $l: W \to W_p$ is the p-localization of space W, then $l_*: H_n(W) \to H_n(W_p)$ is a p-equivalent, and $l_*: \ \pi_1(W) \to \pi_1(W_p)$ is the p-localization of $\pi_1(W)$ (see[1]). If W_p is 1-connected, then $H_n(W_p)$ is p-local for all $n \geq 1$(see[1]). Hence $l_*: H_n(W) \to H_n(W_p)$ is the p-localization of $H_n(W)$, i.e., $H_n(W_p) \cong H_n(W) \otimes Z_p$, where Z_p is the p-localization of interger abelian group.

Suppose $f: X \to Y$ is a map such that $f_*: \ \pi_1(X) \to \pi_1(Y)$ is an isomorphism. since $f_*: \ \pi_1(X) \to \pi_1(Y)$ is an isomorphism, and the square (2.1) is a homotopy pushout, then $j_{1p} = j_{2p}: \ \pi_1(Y) \to \pi_1(C)$ is also an isomorphism. This implies that $j_{1p*} = j_{2p*}: \pi_1(Y_p) \to \pi_1(C_p)$ is also an isomorphism. Thus, if X_p and Y_p are 1-connected, then C_p is also 1-connected.

Let

$$\begin{tikzcd} X_p \arrow[r, "f_p"] \arrow[d, "f_p"'] & Y_p \arrow[d, "j'_1"] \\ Y_p \arrow[r, "j'_2"] & C' \end{tikzcd} \qquad \text{...........} \quad (2.4)$$

be a homotopy pushout. Then C' is 1-connected, and there is a map $\varphi: C' \to C_p$ yielding a homotopy commutative diagram

$$\begin{tikzcd}
X_p \arrow[r, "f_p"] \arrow[d, "f_p"'] & Y_p \arrow[d, "j'_1"] \arrow[ddr, bend left, "j_{1p}"] & \\
Y_p \arrow[r] \arrow[drr, bend right, "j_{2p}"'] & C' \arrow[dr, "\varphi"] & \\
 & & C_p
\end{tikzcd}$$

and hence a map of the Mayer-Vietoris sequence of the square (2.4) to the p-localization of Mayer-Vietoris sequence of the square (2.1), Since $H_n(X)\otimes Z_p \cong H_n(X_p)$, $H_n(Y)\otimes Z_p \cong H_n(X_p)$, $H_n(C)\otimes Z_p \cong H_n(C_p)$. In this map of Mayer-Vietor sequences all groups except $H_n(C')$ are mapped by the identity. This φ induces an isomorphism of homology groups. Since C' and C_p is 1-connected, this follows that φ is a homotopy equivalence. Then the square (2.3) is a homotopy pushout.

Lemma 3[4]. If X and Y are nilpotent, then the p-localization of the square (2.2)

$$\begin{tikzcd}
E_p \arrow[r, "i2p"] \arrow[d, "i1p"'] & X_p \arrow[d, "f_p"] \\
X_p \arrow[r, "f_p"] & Y_p
\end{tikzcd}$$

is also a homotopy pullback.

Lemma 4[2]. Let $F\to E\to K(G,1)$ be the fibration sequene. If G is a p-torsion group, then $F_p \to E_p \to K(G,1)$ is also a fibration sequence.

Lemma 5. Let

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, "\varphi"] \\
F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

be homotopy commutative diagram of fibration sequences. If φ is a homotopy equivalence, then its left square is a homotopy pullback.

Proof. Consider the following homotopy commutative diagram

$$\begin{tikzcd}
 & F \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & E \arrow[dl] \arrow[dd] \\
F' \arrow[rr, crossing over] \arrow[dd] & & E' & \\
 & * \arrow[rr, dashed] \arrow[dl, dashed] & & B \arrow[dl, "\varphi"] \\
* \arrow[rr] & & B' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the front and back faces are homotopy pullbacks. If ϕ is a homotopy equivalence, then the bottom face is homotopy pullback. This follows that the top face is a homotopy pullback.

Lemma 6[5]**.** Suppose that we have a homotopy commutative cube

$$\begin{tikzcd}
& A' \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & B' \arrow[dl] \arrow[dd] \\
C' \arrow[rr] \arrow[dd] & & D' \arrow[dd] & \\
& A \arrow[rr, dashed] \arrow[dl, dashed] & & B \arrow[dl] \\
C \arrow[rr] & & D &
\end{tikzcd}$$

in which all vertical faces are homotopy pullbacks. If one of the top or bottom faces is a homotopy pushout, then so is the other.

3. Proofs

Theorem 4. Let $f: X \to Y$ be a homotopy epimorphism and $f_*: \pi_1(X) \to \pi_1(Y)$ be an isomorphism. If there is a normal subgroup N of $\pi_1(X)$, such that $\pi_1(X)/N$ is a p-torsion group and N_p is trivial, then the p-localized map $f_p: X_p \to Y_p$ is also a homotopy epimorphism.

Proof. Let $f: X \to Y$ be a homotopy epimorphisms such that $f_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism. Denote $\hat{X}$ and $\hat{Y}$ are the covering spaces of X, Y, respectively, for the subgroup N of $\pi_1(X) \cong \pi_1(Y)$. Then we have the following homotopy commutative diagram of covering fibration sequences:

$$\begin{tikzcd}
\hat{X} \arrow[r] \arrow[d, "\hat{f}"] & X \arrow[r] \arrow[d, "f"] & K(\pi_1(X)/N, 1) \arrow[d, "f_*"] \\
\hat{Y} \arrow[r] & Y \arrow[r] & K(\pi_1(Y)/N, 1)
\end{tikzcd} \qquad \text{...........} \quad (2.5).$$

If $\pi_1(X)/N \cong \pi_1(Y)/N$ is the p-torsion group, by Lemma 4 and the diagram (2.5), then the following the fibration sequences diagram is homotopy commutative:

$$\begin{tikzcd}
\hat{X}_p \arrow[r] \arrow[d, "\hat{f}_p"] & X_p \arrow[r] \arrow[d, "f_p"] & K(\pi_1(X)/N, 1) \arrow[d, "f_*"] \\
\hat{Y}_p \arrow[r] & Y_p \arrow[r] & K(\pi_1(Y)/N, 1)
\end{tikzcd} \qquad \text{...........} \quad (2.6).$$

Let

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "j_1"] \\
Y \arrow[r, "j_2"] & C
\end{tikzcd}$$

be a homotopy pushout. Then $j_1 = j_2$ by Lemma 1(1). Note that $j_{1*} = j_{2*}: \pi_1(Y) \to \pi_1(C)$ is an isomorphism. Denote $\hat{C}$ is the covering space of C for subgroup $N \subset \pi_1(C)$.

Consider the following homotopy commutative cube:

$$
\begin{tikzcd}[row sep=small, column sep=small]
\hat{X} \arrow[rr] \arrow[dr] \arrow[dd] & & \hat{Y} \arrow[dr] \arrow[dd] & \\
& \hat{Y} \arrow[rr, crossing over] & & \hat{C} \arrow[dd] \\
X \arrow[rr, dashed] \arrow[dr, dashed] & & Y \arrow[dr] & \\
& Y \arrow[rr] \arrow[from=uu, crossing over] & & C
\end{tikzcd}
\qquad \cdots\cdots\cdots\cdots \ (2.7)
$$

in which all vertical faces are homotopy pullbacks by Lemma 5 and the diagram (2.5). Hence, the top face is a homotopy pushout by Lemma 6.

Consider the p-localization of the diagram (2.7), that is, the following homotopy commutative cube:

$$
\begin{tikzcd}[row sep=small, column sep=small]
\hat{X}_p \arrow[rr] \arrow[dr] \arrow[dd, dashed] & & \hat{Y}_p \arrow[dr] \arrow[dd] & \\
& \hat{Y}_p \arrow[rr, crossing over] & & \hat{C}_p \arrow[dd] \\
X_p \arrow[rr, dashed, "f_p"] \arrow[dr, dashed, "f_p"'] & & Y_p \arrow[dr, "j_{1p}"] & \\
& Y_p \arrow[rr, "j_{2p}"'] \arrow[from=uu, crossing over] & & C_p
\end{tikzcd}
\qquad \cdots\cdots\cdots\cdots \ (2.8)
$$

in which all vertical faces are homotopy pullbacks by Lemma 5 and the diagram (2.6). Since $\pi_1(\hat{X}) \cong \pi_1(\hat{Y}) = N$, so $\pi_1(\hat{X}_p) \cong \pi_1(\hat{Y}_p) = N_p$, but N_p is trivial, then $\hat{X}_p$ and $\hat{Y}_p$ is 1-connected. By Lemma 2, the top face of the diagram (2.8) is a homotopy pushout. This follows from Lemma 6 that the bottom face of the diagram (2.8) is a homotopy pushout. Since $j_1=j_2$ implies $j_{1p}=j_{2p}$, then $f_p: X_p \to Y_p$ is a homotopy epimorphism by Lemma 1(1).

Proof of Theorem 2. Obvious, if $\pi_1(X)$ satisfies one of the conditions of Theorem 2, then there must be a normal subgroup N of $\pi_1(X)$, such that $\pi_1(X)/N$ is a p-torsion group and N_p is trivial. Hence, Theorem 4 follows from Theorem 2.

Proof of Theorem 3. Let $f: X \to Y$ be a homotopy monomorphism such that $f_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism. Suppose $\pi_1(X)$ is a p- torsion group. In the proof of Theorem 4, let N = 0, and then $\hat{X}$ and $\hat{Y}$ are the universal covering spaces of *X*, *Y*, respectively. Hence, we have the following homotopy commutative diagram of fibration sequences:

$$
\begin{tikzcd}
\hat{X} \arrow[r] \arrow[d, "\hat{f}"] & X \arrow[r] \arrow[d, "f"] & K(\pi_1(X),1) \arrow[d, "f_*"] \\
\hat{Y} \arrow[r] & Y \arrow[r] & K(\pi_1(Y),1)
\end{tikzcd}
\qquad \cdots\cdots\cdots\cdots \ (2.9).
$$

Note that $\pi_1(X)$ is a p- torsion group , by lemma 4, the following the fibration sequences diagram is homotopy commutative:

$$\begin{tikzcd} \hat{X}_p \arrow[r] \arrow[d, "\hat{f}_p"] & X_p \arrow[r] \arrow[d, "f_p"] & K(\pi_1(X),1) \arrow[d, "f_*"] \\ \hat{Y}_p \arrow[r] & Y_p \arrow[r] & K(\pi_1(Y),1) \end{tikzcd} \qquad \text{...........} \quad (2.10).$$

Let

$$\begin{tikzcd} E \arrow[r, "i_2"] \arrow[d, "i_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a homotopy pullback. Then i_1=i_2 by Lemma 1(2). Denote H = Ker $\{i_{1*}=i_{2*} : \pi_1(E) \to \pi_1(X)\}$, and $\hat{E}$ is the covering space of E for the subgroup H of $\pi_1(E)$. Thus we have the following homotopy commutative cube, in which all vertical faces of the diagram (2.11) are homotopy pullbacks by Lemma 5 and the diagram (2.9)

$$\begin{tikzcd} & \hat{E} \arrow[rr] \arrow[dl] \arrow[dd] & & \hat{X} \arrow[dl] \arrow[dd] \\ \hat{X} \arrow[rr] \arrow[dd] & & \hat{Y} \arrow[dd] & \\ & E \arrow[rr, dashed] \arrow[dl, dashed] & & X \arrow[dl] \\ X \arrow[rr] & & Y & \end{tikzcd} \qquad \text{...........} \quad (2.11).$$

Consider the p-localization of the diagram (2.11), that is, the following homotopy commutative cube:

$$\begin{tikzcd} & \hat{E}_p \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & \hat{X}_p \arrow[dl] \arrow[dd] \\ \hat{X}_p \arrow[rr] \arrow[dd] & & \hat{Y}_p \arrow[dd] & \\ & E_p \arrow[rr, dashed, "i_{2p}"] \arrow[dl, dashed, "i_{1p}"] & & X_p \arrow[dl, "f_p"] \\ X_p \arrow[rr, "f_p"] & & Y_p & \end{tikzcd} \qquad \text{...........} \quad (2.12)$$

in which all vertical faces are homotopy pullbacks by Lemma 5 and the diagram (2.10). Note that $\hat{X}$ and $\hat{Y}$ are 1-connected. By lemma 3, the top face of the diagram (2.12) is a homotopy pullbck. This implies that the bottom face of the diagram (2.12) is a homotopy pullbck. Since i_1=i_2 implies i_{1p}=i_{2p}, then $f_p : X_p \to Y_p$ is a homotopy monomorphism by Lemma 1(2).

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