ON ORTHOGONAL PAIRS IN CATEGORIES AND LOCALISATION*†

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In memory of Frank Adams

0 Introduction

Special forms of the following situation are often encountered in the literature: Given a class of Morphisms $\mathcal{M}$ in a category $\mathcal{C}$, consider the full subcategory $\mathcal{D}$ of objects $X \in \mathcal{C}$ such that, for each diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ X & & \end{array}$$

with $f \in \mathcal{M}$, there is a unique morphism $h\colon B \to X$ with $hf = g$. The *orthogonal subcategory problem* [13] asks whether $\mathcal{D}$ is reflective in $\mathcal{C}$, i.e., under which conditions the inclusion functor $\mathcal{D} \to \mathcal{C}$ admits a left adjoint $E\colon \mathcal{C} \to \mathcal{D}$; see [17]. Many authors have given conditions on the category $\mathcal{C}$ and the class of morphisms $\mathcal{M}$ ensuring the reflectivity of $\mathcal{D}$, sometimes even providing an explicit construction of the left adjoint $E\colon \mathcal{C} \to \mathcal{D}$; see for example Adams[1], Bousfield [3],[4], Deleanu-Frei-Hilton [9][10], Heller [15], Yosimura [22], Dror-Farjoun [11], Kelly [12]. The functor E is often referred to as a *localisation functor* of $\mathcal{C}$ at the subcategory $\mathcal{D}$. Most of the known existence results of left adjoints work well when the category $\mathcal{C}$ is cocomplete [12] or complete [19]. Unfortunately, these methods cannot be directly applied to the homotopy category of CW-complexes, as it

*London Mathematical Society Lecture Note Series. 175, Adams Memorial Symposium on Algebraic Topology: 1, Manchester 1990, Edited by N. Ray and G. Walker

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is neither complete nor cocomplete. This difficulty is often circumvented by resorting to semi-simplicial techniques.

In this paper we offer a construction of localisation functors depending only on the availability of certain weak colimits in the category $\mathcal{C}$. From a technical point of view, the existence of such weak colimits reduces our arguments essentially to the situation in cocomplete categories. From a practical point of view, however, our result is a simple recipe for the explicit construction of localisation functors. It unifies a number of constructions created for specific purposes; cf. [4][18],[20]. In fact, its scope goes beyond these applications: For example, it can be used to show that there is a whole family of functors extending P-localisation of nilpotent homotopy types to the homotopy category of all CW-complexes. We deal with this issue in [7], where we discuss the geometric significance of these functors as well as their interdependence. Section 1 of the present paper contains background followed by the statement and proof of our main result: the affirmative solution of the orthogonal subcategory problem in a wide range of cases. In Section 2 we discuss extensions of a localisation functor in a category $\mathcal{C}$ to localisation functors in supercategories of $\mathcal{C}$. Our results allow us to give, in Section 3, a uniform existence proof for various localisation functors and also to explain their interrelation. The basic features of our project have been outlined in [8].

Acknowledgements. We are indebted to Emmanuel Dror-Farjoun, discussions with whom significantly helped the present development. We are also grateful to the CRM of Barcelona for the hospitality extended to the authors.

1 Orthogonal pairs and localisation functors

We begin by explaining the basic categorical notions we shall use. Our main sources are [1],[3],[4],[13].

A morphism $f\colon A \to B$ and an object X in a category $\mathcal{C}$ are said to be *orthogonal* if the function

$$f^*\colon \mathcal{C}(\mathcal{B}, \mathcal{X}) \longrightarrow \mathcal{C}(\mathcal{A}, \mathcal{X})$$

is bijective, where $\mathcal{C}(\ ,\ )$ denotes the set of morphisms between two given objects of $\mathcal{C}$. For a class of morphisms $\mathcal{M}$, we denote by $\mathcal{M}^\perp$ the class of objects orthogonal to each $f \in \mathcal{M}$. Similarly, for a class of objects $\mathcal{O}$, we denote by $\mathcal{O}^\perp$ the class of morphisms orthogonal to each $X \in \mathcal{O}$.

1.1 Definition An *orthogonal pair* in $\mathcal{C}$ is a pair $(\mathcal{S}, \mathcal{D})$ consisting of a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{D}^\perp = \mathcal{S}$ and $\mathcal{S}^\perp = \mathcal{D}$.

If (E, η) is an idempotent monad [1] in $\mathcal{C}$, then the classes

$$\begin{aligned} \mathcal{S} &= \{f\colon A \to B | Ef\colon EA \cong EB\} \\ \mathcal{D} &= \{X | \eta_X\colon S \cong EX\} \end{aligned}$$

form an orthogonal pair (note that these could easily be proper classes). The morphisms in $\mathcal{S}$ are then called *E-equivalences* and the objects in $\mathcal{D}$ are said to be E-local. Not every orthogonal pair $(\mathcal{S}, \mathcal{D})$ arises from an idempotent monad in this way; cf. [19]. If so, we call E the *localisation functor* associated with $(\mathcal{S}, \mathcal{D})$. Then the full subcategory of objects in $\mathcal{D}$ is reflective and E is left adjoint to the inclusion $\mathcal{D} \to \mathcal{C}$. The following proposition enables us to detect localisation functors.

1.2 Proposition Let $\mathcal{C}$ be a category and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. If for each object X there exists a morphism $\eta_X\colon X \to EX$ in $\mathcal{S}$ with EX in $\mathcal{D}$, then

(i) η_X is terminal among the morphisms in $\mathcal{S}$ with domain X;

(ii) η_X is initial among the morphisms of $\mathcal{C}$ from X to an object of $\mathcal{D}$;

(iii) The assignment $X \mapsto EX$ defines a localisation functor on $\mathcal{C}$ associated with $\mathcal{S}, \mathcal{D})$.

For each class of morphisms $\mathcal{M}$, the pair $(\mathcal{M}^{\perp\perp}, \mathcal{M}^{\perp})$ is orthogonal. We say that this pair is *generated* by $\mathcal{M}$ and call $\mathcal{M}^{\perp\perp}$ the *saturation* of $\mathcal{M}$. If $\mathcal{M}^{\perp\perp} = \mathcal{M}$, then $\mathcal{M}$ is said to be *saturated*. This terminology applies to objects as well. Note that if $\mathcal{S}, \mathcal{D})$ is an orthogonal pair then both $\mathcal{S}$ and $\mathcal{D}$ are saturated. The next properties of saturated classes are easily checked and well-known in a slightly more general context [3][13].

1.3 Lemma If a class of morphisms $\mathcal{S}$ is saturated, then

(i) $\mathcal{S}$ contains all isomorphisms of $\mathcal{C}$.

(ii) If the composition gf of two morphisms is defined and any two of f, g, gf are in $\mathcal{S}$, then the third is also in $\mathcal{S}$.

(iii) Whenever the coproduct of a family of morphisms of $\mathcal{S}$ exists, it is in the class $\mathcal{S}$.

(iv) If the diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[t]{} & D \end{array}$$

is a push-out in which $s \in \mathcal{S}$, then $t \in \mathcal{S}$.

(v) If α is an ordinal and $F\colon \alpha \to \mathcal{C}$ is a directed system with direct limit T, such that for each $i < \alpha$ the morphism $s_i\colon F(0) \to F(i)$ is in $\mathcal{S}$, then $s_\alpha\colon F(0) \to T$ is in $\mathcal{S}$.

We call a class of morphisms $\mathcal{S}$ *closed* in $\mathcal{C}$ if it satisfies (i), (ii) and (iii) in Lemma 1.3 above. We restrict attention to closed classes from now on.

We proceed with the statement of our main result. Recall that a *weak colimit* of a diagram is defined by requiring only existence, without insisting on uniqueness, in the defining universal property [17].

1.4 Theorem Let $\mathcal{C}$ be a category with coproducts and let $\mathcal{S}$ be a closed class of morphisms in $\mathcal{C}$. Suppose that:

(C1) There is a set $\mathcal{S}_0 \subseteq \mathcal{S}$ with $\mathcal{S}_0^{\perp} = \mathcal{S}^{\perp}$.

(C2) For every diagram $C \overset{f}{\leftarrow} A \overset{s}{\rightarrow} B$ with $s \in \mathcal{S}$ there exists a weak push-out

$$\begin{array}{ccc} A & \xrightarrow{\;s\;} & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ C & \xrightarrow[\;t\;]{} & Z \end{array}$$

with $t \in \mathcal{S}$.

(C3) There is an ordinal α such that, for every $\beta \le \alpha$, every directed system $f\colon \beta \to \mathcal{C}$ in which the morphisms $s_i\colon F(0) \to F(i)$ are in $\mathcal{S}$ for $i < \beta$ admits a weak direct limit T satisfying

(a) the morphism $s_\beta\colon F(0) \to T$ is in $\mathcal{S}$;

(b) for each $s\colon A \to B$ in $\mathcal{S}_0$, every morphism $f\colon A \to T$ factors through $f'\colon A \to F(i)$ for some $i < \alpha$;

(c) if two morphisms $g_1, g_2\colon B \to T$ satisfy $g_1 s = g_2 s$ with $s\colon A \to B$ in $\mathcal{S}_0$, then they factor through $g_1', g_2'\colon B \to F(i)$ for some $i < \alpha$, in such a way that $g_1' s = g_2' s$.

Then the class $\mathcal{S}$ is saturated and the orthogonal pair $(\mathcal{S}, \mathcal{S}^{\perp})$ admits a localisation functor E. Furthermore, for each object X, the localizing morphism $\eta_X\colon X \to EX$ can be constructed by means of a weak direct limit indexed by α.

Proof For each morphism $s\colon A \to B$ in $\mathcal{S}_0$ fix a weak push-out

$$\begin{array}{ccc} A & \xrightarrow{\;s\;} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t_2} \\ B & \xrightarrow[\;t_1\;]{} & Z_s \end{array}$$

in which $t_1 \in \mathcal{S}$. Then also $t_2 \in \mathcal{S}$ because $\mathcal{S}$ is closed.

1.5 Remark With applications in mind, it is worth observing that part (c) of hypothesis (C3) in Theorem 1.4 is satisfied if each map $f\colon Z_s \to T$ factors through $f'\colon Z_s \to F(i)$ for some $i < \alpha$.

Choose next a morphism $u_s\colon Z_s \to B$ rendering commutative the diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & B & & \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t_2} & \searrow{\scriptstyle \mathrm{Id}} & \\ B & \xrightarrow[t_1]{} & Z_s & & \\ & \searrow{\scriptstyle \mathrm{Id}} & & \overset{u_s}{\searrow} & \\ & & & & B \end{array}$$

and note that $u_s \in \mathcal{S}$. Write $\mathcal{D}$ for $\mathcal{S}^\perp$. We shall construct, for each object $X \in \mathcal{C}$, a morphism $\eta_X\colon X \to EX$ with $EX \in \mathcal{D}$ and $\eta_X \in \mathcal{S}$. Set $X_0 = X$. Given $i < \alpha$, assume that X_i has been constructed, together with a morphism $X \to X_i$ belonging to $\mathcal{S}$. Define a morphism $\sigma_i\colon X_i \to X_{i+1}$ as follows: For each $s\colon A \to B$ in the set $\mathcal{S}_\prime$, consider all morphisms $\varphi\colon A \to X_i$ and $\psi\colon Z_s \to X_i$ for which no factorisation through $s\colon A \to B$, resp. $u_s\colon Z_s \to B$, exists (if there are no such morphisms, then $X_i \in \mathcal{D}$ and we may set $EX = X_i$). Choose a weak push-out

$$\begin{array}{ccc} \coprod_{s\in\mathcal{S}_0}\Big((\coprod_\varphi A)\coprod(\coprod_\psi Z_s)\Big) & \xrightarrow{\phi} & \coprod_{s\in\mathcal{S}_0}\Big((\coprod_\varphi B)\coprod(\coprod_\psi B)\Big) \\ {\scriptstyle f}\downarrow & & \downarrow \\ X_i & \xrightarrow[\sigma_i]{} & X_{i+1} \end{array}$$

with $\sigma_i \in \mathcal{S}$, in which f is the coproduct morphism and ϕ is the corresponding coproduct of copies of $s\colon A \to B$ and $u_s\colon Z_s \to B$ (which is therefore a morphism in $\mathcal{S}$). Iterate this procedure until reaching the ordinal α. If $\beta \le \alpha$ is a limit ordinal, define X_β by choosing a weak direct limit of the system $\{X_i, i < \beta\}$, according to (C3). Set $EX = X_\alpha$. The construction guarantees that the composite morphism $\eta_X\colon X \to EX$ is in $\mathcal{S}$. We claim that $EX \in \mathcal{D}$. Since $\mathcal{D} = \mathcal{S}_0^\perp$, it suffices to check that EX is orthogonal to each morphism in $\mathcal{S}_0$. Take a diagram

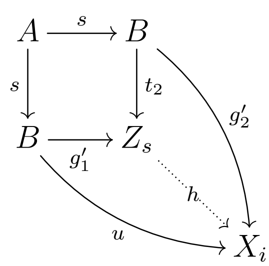
$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle f}\downarrow & & \\ EX & & \end{array}$$

with $s \in \mathcal{S}_0$. Then f factors through $f'\colon A \to X_i$ for some $i < \alpha$ and hence, either f'

factors through $s\colon A\to B$, or there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ s\ } & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ X_i & \xrightarrow[\ \sigma_i\ ]{} & X_{i+1} \end{array}$$

which provides a morphism $g\colon B\to EX$ such that $gs=f$. Now suppose that there are two maps $g_1,g_2\colon B\to EX$ with $g_1s=g_2s=f$. Then we can choose an object X_i with $i<\alpha$, and morphisms $g_1',g_2'\colon B\to X_i$ such that $g_1's=g_2's$. Using the weak push-out property of Z_s, we obtain a morphism $h\colon Z_s\to X_i$ rendering commutative the diagram



Then, either h factors through $u_s\colon Z_s\to B$ and $g_1'=g_2'$, or there is a commutative diagram

$$\begin{array}{ccc} Z_s & \xrightarrow{\ u_s\ } & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ X_i & \xrightarrow[\ \sigma_i\ ]{} & X_{i+1} \end{array}$$

which yields

$$\sigma_i g_1' = \sigma_i h t_1 = k u_s t_1 = k = k u_s t_2 = \sigma_i h t_2 = \sigma_i g_2'$$

and hence $g_1=g_2$. This shows that $EX\in\mathcal{D}$.

To complete the proof it remains to show that $\mathcal{S}^{\perp\perp}=\mathcal{S}$. The inclusion $\mathcal{S}\subseteq\mathcal{S}^{\perp\perp}$ is trivial. For the converse, let $f\colon A\to B$ be orthogonal to all objects in $\mathcal{D}$. Since $\eta_A\colon A\to EA$ is in $\mathcal{S}$ and $EB\in\mathcal{D}$, there is a unique morphism Ef rendering commutative the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ EA & \xrightarrow[\ Ef\ ]{} & EB. \end{array}$$

But $\eta_B f$ is orthogonal to EA and this provides a morphism $g\colon EB\to EA$ which is two-sided inverse to Ef. Hence Ef is an isomorphism and $f\in\mathcal{S}$ because $\mathcal{S}$ is closed. $\square$

Given an orthogonal pair $(\mathcal{S},\mathcal{D})$, the class $\mathcal{S}$ is saturated and, a fortiori, closed. Therefore

1.6 Corollary Let $\mathcal{C}$ be a category with coproducts and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. Suppose that some set $\mathcal{S}_0 \subseteq \mathcal{S}$ generates the pair $(\mathcal{S}, \mathcal{D})$ and that the class $\mathcal{S}$ satisfies conditions (C2) and (C3) in Theorem 1.4. Then the pair $(\mathcal{S}, \mathcal{D})$ admits a localisation functor E.

Moreover, if the category $\mathcal{C}$ is cocomplete, then it follows from Lemma 1.3 that for each orthogonal pair $(\mathcal{S}, \mathcal{D})$ condition (C2) and part (a) of condition (C3) are automatically satisfied. This leads to Corollary 1.7 below. An object X has been called *presentable* [14] or *s-definite* [3] if, for some sufficiently large ordinal α, the functor $\mathcal{C}(\mathcal{X}, \ )$ preserves direct limits of directed systems $F \colon \alpha \to \mathcal{C}$. For example, all groups are presentable [3]. For finitely presented groups it suffices to take α to be the first infinite ordinal.

1.7 Corollary [3] Let $\mathcal{C}$ be a cocomplete category. Let $(\mathcal{S}, \mathcal{D})$ be the orthogonal pair generated by an arbitrary set $\mathcal{S}_{/}$ of morphisms of $\mathcal{C}$. Suppose that the domains and codomains of morphisms in $\mathcal{S}_0$ are presentable. Then $(\mathcal{S}, \mathcal{D})$ admits a localisation functor. □

Since any colimit of presentable objects is again presentable, the following definition together with the results of [19] imply Corollary 1.9 below.

1.8 Definition A set $\{E_\alpha\}$ of objects of a category $\mathcal{C}$ is a cogenerator set of $\mathcal{C}$ if any morphism $f \colon X \to Y$ of $\mathcal{C}$ inducing bijections $f_* \colon \mathcal{C}(E_\alpha, X) \cong \mathcal{C}(E_\alpha, Y)$ for each α, is an isomorphism.

1.9 Corollary Let $\mathcal{C}$ be a cocomplete category. Suppose that $\mathcal{C}$ has a cogenerator set whose elements are presentable. Then any orthogonal pair generated by an arbitrary set of morphisms of $\mathcal{C}$ admits a localisation functor. □

2 Extending localisation functors

Let E be a localisation functor on the subcategory $\mathcal{C}'$ of $\mathcal{C}$. We wish to discuss extensions of E over $\mathcal{C}$. Familiar examples include the extension of P-localisation of abelian groups to nilpotent groups and further to all groups. Two problems arise here: existence – for which we often refer to Theorem 1.4 – and uniqueness. An appropriate setting for discussing the latter is obtained by partially ordering the collection of all orthogonal pairs in $\mathcal{C}$ as follows: For two given orthogonal pairs $(\mathcal{S}_1, \mathcal{D}_1)$, $(\mathcal{S}_2, \mathcal{D}_2)$ in $\mathcal{C}$ we write $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$ if $\mathcal{D}_1 \supseteq \mathcal{D}_2$ (or, equivalently, if $\mathcal{S}_1 \subseteq \mathcal{S}_2$).

2.1 Remark If E_1, E_2 are localisation functors associated to $(\mathcal{S}_1, \mathcal{D}_1)$ and $(\mathcal{S}_2, \mathcal{D}_2)$ respectively, and if $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$, then there is a natural transformation of functors $E_1 \to E_2$. In fact, the restriction of E_2 to $\mathcal{D}_1$ is left adjoint to the inclusion $\mathcal{D}_2 \to \mathcal{D}_1$. □

An orthogonal pair $(\mathcal{S}, \mathcal{D})$ of $\mathcal{C}$ is said to *extend* the orthogonal pair $(\mathcal{S}', \mathcal{D}')$ of the subcategory $\mathcal{C}'$ if both $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{D}' \subseteq \mathcal{D}$. The collection of all extensions of $(\mathcal{S}', \mathcal{D}')$ is partially ordered. Moreover we have

2.2 Proposition Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ and $(\mathcal{S}', \mathcal{D}')$ an orthogonal pair in $\mathcal{C}'$. If $(\mathcal{S}, \mathcal{D})$ is an extension of $(\mathcal{S}', \mathcal{D}')$ to $\mathcal{C}$, then

$$((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}) \geq (\mathcal{S}, \mathcal{D}) \geq ((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}),$$

where orthogonality is meant in $\mathcal{C}$.

In this situation, we call the orthogonal pair in $\mathcal{C}$ generated by the class $\mathcal{S}'$ the *maximal extension* of $(\mathcal{S}', \mathcal{D}')$, and the one generated by $\mathcal{D}'$ the *minimal extension*. A convenient tool for recognising such extremal extensions is given in the next proposition.

2.3 Proposition Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$, $(\mathcal{S}', \mathcal{D}')$ an extension of $(\mathcal{S}', \mathcal{D}')$ to $\mathcal{C}$. Then

(i) $(\mathcal{S}, \mathcal{D})$ is the maximal extension of $(\mathcal{S}', \mathcal{D}')$ if and only if there is a subclass $\mathcal{S}_0 \subseteq \mathcal{S}'$ such that $\mathcal{S}_0^{\perp} \subseteq \mathcal{D}$.

(ii) $(\mathcal{S}, \mathcal{D})$ is the minimal extension of $(\mathcal{S}', \mathcal{D}')$ if and only if there is a subclass $\mathcal{D}_0 \subseteq \mathcal{D}'$ such that $\mathcal{D}_0^{\perp} \subseteq \mathcal{S}$.

Of course $(\mathcal{S}', \mathcal{D}')$ admits a unique extension to $\mathcal{C}$ if and only if the minimal and the maximal extensions coincide.

2.4 Example Let $\mathcal{C}$ be the category of finite groups and $\mathcal{C}'$ the subcategory of finite nilpotent groups. Fix a prime p and consider the orthogonal pair $(\mathcal{S}', \mathcal{D}')$ in $\mathcal{C}'$ associated to p-localisation [16]. The class $\mathcal{D}'$ consists of all p-groups, and the orthogonal pair $(\mathcal{S}, \mathcal{D}) = ((\mathcal{D}')^{\perp}, \mathcal{D}')$ in $\mathcal{C}$ is both the maximal and the minimal extension of $(\mathcal{S}', \mathcal{D}')$ to $\mathcal{C}$. The pair $(\mathcal{S}, \mathcal{D})$ admits a localisation functor – namely, mapping each finite group G onto its maximal $p$quotient, – which is therefore the unique extension to all finite groups of the p-localisation of finite nilpotent groups.

3 Applications of the basic existence result

Examples 3.1, 3.2 and 3.3 below discuss well-known functors, each of whose constructions may be viewed as particular cases of Theorem 1.4. Examples 3.4 and 3.7 are new.

3.1 Example Let $\mathcal{H}_\infty$ be the pointed homotopy category of simply-connected CW-complexes, and P a set of primes. The P-localisation functor described by Sullivan [21] is associated to the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 \; = \; \{\rho_n^k \colon S^k \to S^k | \deg \rho_n^k = n, \quad k \geq 2, \quad n \in P'\},$$

where P' denotes the set of primes not in P. Objects in $\mathcal{D}$ are simply connected CW-complexes whose homotopy groups are $\mathbb{Z}_P$-modules. Morphisms in $\mathcal{S}$ are $H_*(\ ;\mathbb{Z}_P)$-equivalences. The hypotheses of Corollary 1.6 are fulfilled by taking α to be the first infinite ordinal and using homotopy colimits.

3.2 Example Let $\mathcal{H}$ denote the pointed homotopy category of connected CW-complexes and h_* an additive homology theory. Take $\mathcal{S}$ to be the class of morphisms $f \colon X \to Y$ inducing an isomorphism $f_* \colon h_*(X) \cong h_*(Y)$. We know from [4] that $\mathcal{S}$ satisfies the hypotheses of Theorem 1.4: Choose α to be the smallest infinite ordinal whose cardinality is bigger than the cardinality of $h_*(\mathrm{pt})$; the collection of all CW-inclusions $\varphi \colon A \to B$ with $h_*(\varphi) = 0$ and $\mathrm{card}(B) < \mathrm{card}(\alpha)$ represents a set $\mathcal{S}_0$ with $\mathcal{S}_0^\perp = \mathcal{S}^\perp$.

In the case $h_* = H_*(\ ;\mathcal{Z}_\mathcal{P})$, the corresponding orthogonal pair $(\mathcal{S}, \mathcal{D})$ extends the pair $(\mathcal{S}', \mathcal{D}')$ associated with P-localisation of nilpotent spaces (see [4]). It is indeed the minimal extension of $(\mathcal{S}', \mathcal{D}')$, because he spaces $K(\mathbb{Z}_P, n)$, $n \geq 1$, belong to $\mathcal{D}'$ (cf. Proposition 2.2).

3.3 Example Let $\mathcal{G}$ be the category of groups and P a set of primes. The P-localisation functor described by Ribenboim [20] is associated to the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 \; = \; \{\rho_n \colon \mathbb{Z} \to \mathbb{Z} | \rho_n(1) = n, \quad n \in P'\}.$$

Groups in $\mathcal{D}$ are those in which P'-roots exist and are unique. Such groups have been studied for several decades (see [2][20] and the references there). The hypotheses of Theorem 1.4 are readily checked (use Corollary 1.7). We may choose α to be first infinite ordinal. We denote by $l \colon G \to G_P$ the P-localisation homomorphism.

If $(\mathcal{S}', \mathcal{D}')$ is the orthogonal pair corresponding to P-localisation of nilpotent groups, then, since $\mathcal{S}_0 \subset \mathcal{S}'$, Proposition 2.2 implies that $(\mathcal{S}, \mathcal{D})$ is the *maximal* extension of $(\mathcal{S}', \mathcal{D}')$. In particular, for each group G there is a natural homomorphism from G_P to the Bousfield $H\mathbb{Z}_P$-localisation of G (cf. [5].

3.4 Example Example 3.3 can be generalised to the category $\mathcal{C}$ of π-groups for a fixed group π; that is, objects are groups with a π-action and morphisms are action-preserving group homomorphisms. Let $F(\xi)$ be the free π-group on one generator (it can be explicitly described as the free group on the symbols $\xi^x, x \in \pi$, with the obvious left π-action; cf. [18]. Define a π-homomorphism $\rho_{n,x}\colon F(\xi) \to F(\xi)$ for each $x \in \pi$, $n \in \mathbb{Z}$, by the rule

$$\rho_{n,x}(\xi) \;=\; \xi(x\cdot\xi)(x^2\cdot\xi)\ldots(x^{n-1}\cdot\xi)$$

and consider the set morphisms

$$\mathcal{S}_0 \;=\; \{\rho_{n,x}\colon F(\xi) \to F(\xi) | x \in \pi, \;\; n \in P'\}.$$

By Corollary 1.7, the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by $\mathcal{S}_0$ admits a localisation functor. It again suffices to take the first infinite ordinal as α in the construction. Example 3.3 is the special case $\pi = \{1\}$.

We extend the term P-*local* to the π-groups in $\mathcal{D}$ and term P-*equivalences* to the morphisms in $\mathcal{S}$. They are particularly relevant to the next example.

3.5 Example This example is extracted from [7]. Let $\mathcal{H}$ be the pointed homotopy category of connected CW-complexes and P a set of primes. We consider the class $\mathcal{D}$ of those spaces X in $\mathcal{H}$ for which the power map $\rho_n\colon \Omega X \to \Omega X$, $\rho_n(\omega) = \omega^n$ is a homotopy equivalence for all $n \in P'$. Then there exists a set of morphisms $\mathcal{S}_0$ such that $\mathcal{S}_0^{\perp} = \mathcal{D}$, namely

$$\mathcal{S}_0 \;=\; \{\rho_n^k\colon S^1 \wedge (S^k \cup \mathrm{pt}) \to S^1 \wedge (S^k \cup \mathrm{pt}) | K \geq 0, \; n \in P'\},$$

where $\rho_n^k = \rho_n \wedge \mathrm{Id}$, $\rho_n\colon S^1 \to S^1$ denotes the standard map of degree n, and pt denotes a disjoint basepoint. Morphisms in $\mathcal{S} = \mathcal{D}^{\perp}$ turn out to be those $f\colon X \to Y$ for which $f_*\colon \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and $f_*\colon H_*(X;A) \to H_*(Y;A)$ is an isomorphism for each abelian $\pi_1(Y)_P$-group A which is P-local in the sense of Example 3.4. The conditions of Corollary 1.6 are satisfied. One can take α to be the first infinite ordinal. Spaces in $\mathcal{D}$ will be called P-*local* and maps in $\mathcal{S}$ will be called P-*equivalences*. We denote the P-localisation map by $l\colon X \to X_P$. The pair $(\mathcal{S}, \mathcal{D})$ extends the pair $(\mathcal{S}', \mathcal{D}')$ corresponding to P-localisation of nilpotent spaces.

Since the orthogonal pair corresponding to $H_*(\;\;;\mathbb{Z}_P)$-localisation is minimal among those pairs extending P-localisation of nilpotent spaces (see Example 3.2), for each space X there is a natural map from X_P to the $H_*(\;\;;\mathbb{Z}_P)$-localisation of X.

3.6 Example Let $\mathcal{H}$ denote the pointed homotopy category of connected CW-complexes and P a set of primes. Consider the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 \;=\; \{\rho_n^k\colon S^k \to S^k | \deg\rho_n^k = n, \; k \geq 1, \; n \in P'\}.$$

The class $\mathcal{D}$ consists of spaces whose homotopy groups are P-local, and one finds, with the same methods as in [7],[9], that $\mathcal{S}$ consists of morphisms $f\colon X \to Y$ such that $f_*\colon \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and $f^*\colon H^k(Y;A) \to H^k(X;A)$ is an isomorphism for $k \geq 2$ and every $\mathbb{Z}_P[\pi_1(Y)_P]$-module A. This class $\mathcal{S}$ is not closed under homotopy colimits, because the natural map from $\mathcal{S}^1$ to $K(\mathbb{Z}_P, 1)$, which is the homotopy colimit of a certain direct system of maps ρ_n^1, $n \in P'$, fails to induce an isomorphism in H^2 with coefficients in the group ring $\mathbb{Z}_P[\mathbb{Z}_P]$, and hence does not belong to $\mathcal{S}$. Thus, Corollary 1.6 does not apply in this case. In fact, the orthogonal pair $(\mathcal{S}, \mathcal{D})$ does not admit a localisation functor [7].

On the other hand, if we delete from $\mathcal{S}_0$ the maps ρ_n^1, $n \in P'$, then the resulting class $\mathcal{D}$ consists of spaces whose higher homotopy groups are P-local, and $\mathcal{S}$ consists of morphisms $f\colon X \to Y$ inducing an isomorphism of fundamental groups and such that $f^*\colon H^k(Y;A) \to H^k(X;A)$ is an isomorphism for all k and every $\mathbb{Z}_P[\pi_1(Y)]$-module A. This orthogonal pair $(\mathcal{S}, \mathcal{D})$ is the maximal extension to $\mathcal{H}$ of the pair described in Example 3.1. Now Corollary 1.6 provides a localisation functor associated to $(\mathcal{S}, \mathcal{D})$. This functor induces an isomorphism of fundamental groups and P-localises the higher homotopy groups, i.e., corresponds to fibrewise localisation with respect to the universal covering fibration $\tilde{X} \to X \to K(\pi_1(X), 1)$.

3.7 Example Fix a group G and let $\mathcal{H}(\mathcal{G})$ be the category whose objects are maps $X \to K(G,1)$ in $\mathcal{H}$ and whose morphisms are homotopy commutative triangles. Given an abelian G-group A, let $\mathcal{S}(A)$ be the class of morphisms f such that $f_*\colon H_*(X;A) \to H_*(H;A)$ is an isomorphism. Then $\mathcal{S}(A)$ satisfies the conditions of Theorem 1.4. Example 3.2 corresponds to the particular case $G = \{1\}$. In [7] we show that several idempotent functors on $\mathcal{H}$ extending P-localisation of nilpotent spaces can be obtained by splicing localisation functors with respect to twisted homology in a suitable way. In fact, Example 3.5 can be alternatively obtained as a special case of this procedure.

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