THE FIBER OF FUNCTORS BETWEEN CATEGORIES OF ALGEBRAS

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Abstract. We investigate the fiber of a functor $F : \mathcal{C} \to \mathcal{D}$ between sketchable categories of algebras over an object $D \in \mathcal{D}$ from two points of view: characterizing its classifying space as a universal $\operatorname{Aut}(D)$-space, and parametrizing its objects in cohomological terms.

Introduction

In many mathematical situations we study objects through their image under 'structure-reducing' functors F of various kinds – such as a forgetful functor, or various forms of categorical localization, including abelianization. We therefore need mechanisms for recovering information about X from data related to FX. At least implicitly, any such mechanism involves the 'fiber' (i.e., preimage) of $F : \mathcal{C} \to \mathcal{D}$ over a given object or morphism of $\mathcal{D}$.

We study this fiber from two points of view: first, we show that the classifying space of the full subcategory of $\mathcal{C}$ with objects $F^{-1}(D)$ is a universal $\operatorname{Aut}(D)$-space (see §6). Secondly, under suitable assumption we obtain a parametrization of the isomorphism classes of objects in the pre-image of F (or of two related functors), by elements of relative André-Quillen cohomology groups.

For this purpose, we work with a finite-product sketchable category $\mathcal{C} = \Theta\text{-}\mathcal{M}dl(\mathcal{W})$ of product-preserving functors from a theory Θ, which encodes the given structure (such as groups or rings) to a category $\mathcal{W}$ (often $\mathcal{S}et_*$). Such categories $\Theta\text{-}\mathcal{M}dl(\mathcal{W})$ are well adapted to the methods of homotopical algebra, so one can try to parametrize the fiber of functors between such categories in cohomological terms.

We apply these tools to study two instances of our original question:

I. When $\mathcal{C} = \Theta\text{-}\mathcal{M}dl$, we consider the common fiber of two structure-reducing functors: one a forgetful functor $U : \Theta\text{-}\mathcal{M}dl \to \Xi\text{-}\mathcal{M}dl$, induced by the inclusion of a subcategory $\Xi \hookrightarrow \Theta$, and the other the abelianization functor $T : \mathcal{C} \to \mathcal{C}_{\mathrm{ab}}$.

   Here the category Ξ retains precisely that part of the structure on $\mathcal{C}$ – typically, some kind of product or (in the group case) commutator – which vanishes under abelianization.

II. In the second instance, we assume that Θ has a positive grading. In this case, Θ-objects may be decomposed into central extensions, and one can classify such extensions in cohomological terms – as in the familiar examples of group and module extensions. Under mild assumptions, this leads to a parametrization of the isomorphism classes of objects in the fiber of a structure-forgetting functor on $\Theta\text{-}\mathcal{M}dl(\mathcal{W})$ in cohomological terms.

0.1. **Notation**.
$\mathcal{S}et$ denotes the category of sets, and $\mathcal{S}et_*$ that of pointed sets. $\mathcal{T}$ denotes the category of topological spaces, and $\mathcal{T}_*$ that of pointed connected topological spaces

with base point preserving maps; their homotopy categories will be denoted by $\operatorname{ho}\mathcal{T}$ and $\operatorname{ho}\mathcal{T}_*$, respectively.

For any category $\mathcal{C}$ and set K, we denote by $\operatorname{gr}_K\mathcal{C}$ the category of K-*graded* objects over $\mathcal{C}$ – that is, the functor category $\mathcal{C}^K$ (where K is discrete). If κ is an object of K, the inclusion $\kappa \hookrightarrow K$ induces the projection functor $\operatorname{pr}_\kappa : \operatorname{gr}_K\mathcal{C} \to \mathcal{C}$. In particular, if $K = \mathbb{N}$ (the non-negative integers), we write simply $\operatorname{gr}\mathcal{C}$ for the category of non-negatively graded objects $T_* = (T_n)_{n=0}^\infty$ over $\mathcal{C}$, and $|x| = n \Leftrightarrow x \in X_n$.

0.2. **Organization**.

The first three sections of the paper set up the necessary background material on Θ-models, group objects and abelianization in Θ-$\mathcal{M}dl$, and modules over Θ-models. Section 4 sets up model categories of (simplicial) Θ-models, and their cohomology is described in section 5.

The second part of the paper, devoted to fibers of functors of algebraic theories, begins with a general discussion of the full fiber of a functor in section 6. Section 7 defines and discusses complementary subcategories for a theory Θ. We then study the fiber of the abelianization in section 8. Section 9 deals with positively-graded categories.

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1. Categories Modeled on a Finite Product Sketch

The idea of describing algebraic objects by means of functors $\Theta \to \mathcal{S}et$ appears first in Lawvere [L]. Here Θ is a fixed category, called a 'theory', whose objects and morphisms correspond to the structural axioms underlying the algebraic object. Initially Lawvere considered only Θ whose objects are finite products of a single object. Thus product preserving functors $\Theta \to \mathcal{S}et$ corresponded directly to algebras as characterized, for example, in [Bor, §3.2] or [Mc, p. 120]. Subsequently, Ehresmann introduced the notion of a 'sketch', [E1, E2] thereby allowing more general Θ's to act as structure-encoding categories; see [BE, CL] for further evolutions of this concept.

For our purposes, it suffices to consider a particular class of sketches Θ, called finite product sketches (see below). For the convenience of the reader, we collect here relevant concepts and facts from the literature, primarily from [Bor, Ch. 3-5] and [AR].

1.1. **Definition.** A *finite product sketch*, FP-sketch for short, is a small category Θ together with a designated set $\mathcal{P}$ of finite discrete limit cones. A morphism of FP-sketches $(\Theta, \mathcal{P})$ and $(\Theta', \mathcal{P}')$ is a covariant functor $f : \Theta \to \Theta'$ which turns every limit cone in $\mathcal{P}$ into one in $\mathcal{P}'$. A *finite product theory*, FP-theory for short, is an FP-sketch Θ, with these two additional properties: all finite products (including the empty product) exist in Θ; $\mathcal{P}$ consists of all finite product cones.

1.2. **Definition.** A *model* of an FP-sketch Θ in a category $\mathcal{W}$ is a covariant functor $X : \Theta \to \mathcal{W}$ which preserves the products in $\mathcal{P}$. A morphism of models is a natural transformation of functors.

The category of models of an FP-sketch Θ in $\mathcal{W}$ is denoted by Θ-$\mathcal{M}dl(\mathcal{W})$. We say that Θ *sketches* or *corepresents* Θ-$\mathcal{M}dl(\mathcal{W})$, and we refer to any category equivalent to Θ-$\mathcal{M}dl(\mathcal{W})$ as Θ-sketchable. We reserve the term Θ-*model* for a model of Θ in

$\mathcal{S}et$, the category of sets, and we write $\Theta\text{-}\mathcal{M}dl$ for $\Theta\text{-}\mathcal{M}dl(\mathcal{S}et)$. Similarly, pointed Θ-models form the objects of $\Theta\text{-}\mathcal{M}dl_* := \Theta\text{-}\mathcal{M}dl(\mathcal{S}et_*)$.

1.3. **Definition.** An FP-sketch $(\Theta, \mathcal{P})$ is called *K-sorted* if every object in Θ is isomorphic to the point of a cone in $\mathcal{P}$ whose constituents are in K. In this situation we also refer to the subset K of $\mathrm{Obj}\,\Theta$ as a set of *generators* or of *sorts* for Θ.

Whenever an FP-sketch Θ is K-sorted, the values of a Θ-model $X : \Theta \to \mathcal{W}$ on the objects of Θ are uniquely determined by the composition $K \hookrightarrow \Theta \xrightarrow{X} \mathcal{W}$. So X can be thought of as a K-graded algebra $(X_\kappa)_{\kappa\in K}$ in $\mathcal{W}$, equipped with an action of n-ary operations corresponding to the morphisms from the $\mathcal{P}$-products. Here are some examples.

1.4. **Simplicial Objects**. Let Δ denote the category of finite ordinals and order-preserving maps, and let $\Theta := \Delta^{\mathrm{op}}$. Setting $\mathcal{P} := \emptyset$, we obtain an FP-sketch. It has exactly one set of sorts, namely the entire object set. Its models in a category $\mathcal{W}$ are usually called simplicial objects in $\mathcal{W}$.

1.5. **Monoids**. A 'minimal' single sorted product sketch whose models in $\mathcal{S}et_*$ are monoids is the category $\mathfrak{m}$, with:

(i) null-object 0, object m, products $m^0 = 0$, m^2 and m^3 in $\mathcal{P}$;
(ii) morphisms generated by $e : 0 \to m$, $\nu : m \to 0$, and $\mu : m^2 \to m$ with properties represented by the commuting diagrams below:

$$\begin{tikzcd}
m \arrow[r, "{(e\nu,\mathrm{Id})}"] \arrow[d, "{(\mathrm{Id},e\nu)}"'] \arrow[dr, "\mathrm{Id}"] & m\times m \arrow[d, "\mu"] \\
m\times m \arrow[r, "\mu"'] & m
\end{tikzcd}
\qquad
\begin{tikzcd}
m\times m\times m \arrow[r, "\mathrm{Id}\times\mu"] \arrow[d, "\mu\times\mathrm{Id}"'] & m\times m \arrow[d, "\mu"] \\
m\times m \arrow[r, "\mu"'] & m
\end{tikzcd}$$

Similarly, we have a minimal FP-sketch $\mathfrak{g}$ whose models in $\mathcal{S}et_*$ give groups, $\mathfrak{a}$ for abelian groups, and so on.

The finite product completions of many such sketches can be 'geometrically realized' by selecting the appropriate FP-theories of the category Π^{op} in the following example.

1.6. **Π-algebras**. Let $\Theta = \Pi^{\mathrm{op}}$, where Π denotes the full subcategory of $\mathrm{ho}\,\mathcal{T}_*$ whose objects are finite wedges of spheres $\bigvee_{i=1}^{k} \mathbf{S}^{n_i}$ for $n_i \geq 1$, (including the empty wedge, i.e. a point). Then Θ is an FP-theory, sorted by the set of spheres $\mathbf{S}^n$. The models of Θ in $\mathcal{S}et_*$ have been called Π-*algebras* (cf. [St, §4.2]). At times we restrict attention to the full subcategory $\Pi^{\mathrm{op}}_{\geq 2}$ of Π^{op}, whose objects are finite wedges of spheres of dimension ≥ 2, which corepresents simply-connected Π-algebras.

1.7. **Groups and abelian groups**. For $n \geq 1$, let Π_n denote the full subcategory of Π whose objects are wedges of copies of $\mathbf{S}^n$. Then $\mathfrak{G} \cong \Pi_1^{\mathrm{op}}$ is a single-sorted FP-theory which corepresents groups.

For $n \geq 2$, $\mathfrak{A} := \Pi_n^{\mathrm{op}}$ is a single-sorted FP-theory which corepresents abelian groups. Moreover, the suspension functor $\Sigma : \Pi_1 \to \Pi_2$ is a morphism of product theories. It corepresents the inclusion of the category of abelian groups into that of groups.

A geometric realization of the product completion of $\mathfrak{m}$ is given by the subcategory $\mathfrak{M}$ of $\mathfrak{G}$ whose objects are finite wedges of circles, and whose morphisms are generated by self maps of $\mathbf{S}^1$ with non-negative degree, together with the pinch map $\mathbf{S}^1 \to \mathbf{S}^1 \vee \mathbf{S}^1$. Similarly for commutative monoids, etc.

1.8**. Graded groups and abelian groups**. An FP-sketch corepresenting $\mathbb{N}$-graded groups is given by $\amalg_{\mathbb{N}}\mathfrak{G}$. Similarly, $\mathbb{N}$-graded abelian groups are corepresented by the FP-sketch $\amalg_{\mathbb{N}}\ \mathfrak{A} \cong \cup_{n\geq 2}\Pi_n^{\mathrm{op}}$.

If Θ is an FP-sketch singly-sorted by c, we denote the corresponding sorts in $\amalg_{\mathbb{N}}\Theta$ by c_n, $n \in \mathbb{N}$.

1.9**. Graded Lie rings**. Construct the product sketch $\mathfrak{L}(\mathbb{N})$ from $\amalg_{\mathbb{N}}\mathfrak{A}$ by adding universal bracket operation maps $b_{p,q} : c_p \times c_q \to c_{p+q}$ for $p, q, \geq 0$ which are additive in both variables, graded-commutative, and satisfy the graded Jacobi identity; then $\mathfrak{L}(\mathbb{N})$ corepresents graded Lie rings.

1.10**. Whitehead rings**. Let $\mathfrak{W}$ denote the opposite of the subcategory of Π which results from $\cup_{n\geq 2}\Pi_n$ by adding the universal Whitehead product maps $w_{p,q} : \mathbf{S}^{p+q-1} \to \mathbf{S}^p \vee \mathbf{S}^q$ for all $p, q \geq 2$. A $\mathfrak{W}^{\mathrm{op}}$-algebra is called a *Whitehead ring.*

1.11**. Relationship between graded Lie rings and Whitehead rings**. A functor from Whitehead rings to graded Lie rings is induced by the morphism of FP-sketches $L : \mathfrak{L}(\mathbb{N}) \to \mathfrak{W}$ defined by $c_p \mapsto \mathbf{S}^{p+1}$, and $b_{p,q} \mapsto w^{\mathrm{op}}_{p+1,q+1}$. If $X : \mathfrak{W} \to \mathcal{S}et_*$ is a Whitehead ring, commutative in degree 1, then $X \circ L$ is a graded Lie ring (with $X \circ L(0) = 0$).

In general, a graded Lie ring need not come from a Whitehead ring, since such rings satisfy additional relations. For example, for any element x of a Whitehead ring with $|x|$ even, we have $[[x, x], [x, x]] = 0$ (see [W, p. 536]). This relation comes from the hidden composition process:

$$[[\iota_{2n}, \iota_{2n}], [\iota_{2n}, \iota_{2n}]] = [[\iota_{2n}, \iota_{2n}] \circ \iota_{4n-1}, [\iota_{2n}, \iota_{2n}] \circ \iota_{4n-1}] = [\iota_{2n}, \iota_{2n}] \circ [\iota_{4n-1}, \iota_{4n-1}].$$

The Lie relation $2[\iota_{4n-1}, \iota_{4n-1}] = 0$ yields $2[[\iota_{2n}, \iota_{2n}], [\iota_{2n}, \iota_{2n}]] = 0$. From the Jacobi identity we have $3[[\iota_{2n}, \iota_{2n}], [\iota_{2n}, \iota_{2n}]] = 0$. On the other hand, $[[x, x], [x, x]]$ need not be 0 in a graded Lie ring.

However, a *rational* Whitehead ring *is* just a graded Lie algebra over $\mathbb{Q}$ (up to a shift in indexing), since Quillen showed in [Q2] that any such Lie algebra can be realized as $\pi_*X \otimes \mathbb{Q}$ for some space X.

1.12. *Remark.* As these examples show, there are FP-sketchable categories $\mathcal{C} = \Theta\text{-}\mathcal{M}dl(\mathcal{W})$ where it may be easier to think of a Θ-model as a contravariant functor $X : \Phi \to \mathcal{W}$, (with $\Theta \equiv \Phi^{\mathrm{op}}$), which takes designated coproducts into products. We then say that Φ *corepresents* $\mathcal{C}$ *contravariantly.*

1.13**. Algebras over the Steenrod algebra**. The category $\mathcal{K}$ of unstable algebras over the mod-p Steenrod algebra $\mathcal{A}_p$ is corepresentable by the FP-theory $\Theta := \mathbb{H}_p$, the full subcategory of $\mathrm{ho}\,\mathcal{T}_*$ (the homotopy category of pointed topological spaces) whose objects are finite products of $\mathbb{F}_p$-Eilenberg-Mac Lane spaces $\prod_{i=1}^k K(\mathbb{F}_p, n_i)$ for $n_i \geq 1$. See [Sc, §1.4].

$\mathbb{H}_p$ is sorted by the spaces $K(\mathbb{F}_p, n)$, $n \geq 1$. Therefore, for any space $X \in \mathcal{T}_*$, the values of the $\mathbb{H}_p$-algebra $\mathrm{Hom}_{\mathrm{ho}\,\mathcal{T}_*}(X, -)$ on objects are uniquely determined by the $\mathbb{N}$-graded $\mathbb{F}_p$-module $H^*(X; \mathbb{F}_p) \in \mathcal{K}$.

1.14. **Definition.** For an FP-theory Θ with null-object, an *ideal* in a Θ-model X is a sub-Θ-model $\iota : I \hookrightarrow X$ admitting of a map of Θ-models $\psi : X \to Y$ such that, for each $\vartheta \in \Theta$

$$I(\vartheta) \overset{\iota}{\hookrightarrow} X(\vartheta) \xrightarrow{\psi} Y(\vartheta)$$

is an exact sequence of pointed sets. (Notation: $I \triangleleft X$). We call I the *kernel* of ψ.

1.15. **Lemma.** *The intersection of a family* I_λ *of ideals in a* Θ*-model* X *is a again an ideal in* X*.*

Given a Θ-model X, the ideal $I(S)$ *generated by* a Θ-graded subset S of X is the intersection of all ideals containing S.

1.16**. Semi-categories**. For the description of certain structural phenomena there is no place for the identity morphisms. Therefore we need to weaken the concept of corepresenting category. In such situations it is appropriate and adequate to work with a *semi-category*; i.e., objects (as in a category), together with a collection of morphisms which is closed under compositions. (A semi-category bears a relationship to a category analogous to that of a semi-group to a monoid.) Alternatively, one can think of a semi-category as a 'directed graph with compositions'.

1.17**. (Co)limits in** Θ-$\mathcal{M}dl$.

For our purposes the question of existence of inverse and colimits in Θ-$\mathcal{M}dl(\mathcal{W})$ is most efficiently answered within a setting of locally presentable categories. Such categories are cocomplete by definition, and complete by [AR, 1.28].

1.18. **Theorem** ([AR, 1.53])**.** *Given an FP-sketch* Θ*, if* $\mathcal{W}$ *is locally presentable, so is* Θ-$\mathcal{M}dl(\mathcal{W})$*.* □

The limits and filtered colimits in Θ-$\mathcal{M}dl(\mathcal{W})$ and in $\mathcal{W}^\Theta$ are the same in both categories, hence can be computed object-wise. For the convenience of the reader, we outline two approaches to the construction of functorial arbitrary colimits in Θ-$\mathcal{M}dl$.

(1) Θ-$\mathcal{M}dl$ is a full subcategory of $\mathcal{S}et^\Theta$, which permits a reflection $R : \mathcal{S}et^\Theta \to \Theta$-$\mathcal{M}dl$. So, if $F : J \to \Theta$-$\mathcal{M}dl$, J small, then

$$\varinjlim{}^{\Theta\text{-}\mathcal{M}dl} F \;\cong\; R\left(\varinjlim{}^{\mathcal{S}et^\Theta}(J \xrightarrow{F} \Theta\text{-}\mathcal{M}dl \hookrightarrow \mathcal{S}et^\Theta)\right).$$

(2) The idea is to establish the existence of free Θ-models, and then obtain arbitrary colimits as quotients of free Θ-models. For each ϑ in Θ, Yoneda's lemma says that $\langle\vartheta\rangle := \mathrm{Hom}_\Theta(\vartheta, -) : \Theta \to \mathcal{S}et$ is free in the sense that, for each $X \in \Theta$-$\mathcal{M}dl$,

$$\mathrm{Hom}_{\Theta\text{-}\mathcal{M}dl}(\langle\vartheta\rangle, X) \longrightarrow X(\vartheta), \qquad f \mapsto f(\mathrm{Id}_\vartheta),$$

is a bijection. Moreover, $\langle\vartheta\rangle$ commutes with arbitrary inverse limits, hence belongs to Θ-$\mathcal{M}dl$. Now Θ has a product completion [TT]; i.e. an imbedding in $i : \Theta \to \hat{\Theta}$, where $\hat{\Theta}$ has arbitrary products. A product $\vartheta := \prod_I \vartheta_i$ of objects in Θ yields the free element in Θ-$\mathcal{M}dl$

$$i^*\left(\mathrm{Hom}_{\hat{\Theta}}(\vartheta, -)\right) \;\cong\; \coprod_I \langle\vartheta_i\rangle.$$

Given a functor $T : I \to \Theta$-$\mathcal{M}dl$, $\varinjlim\, T$ may now be constructed as follows: for each i in I, let K_i denote the kernel of the free cover $\varepsilon_i : \phi_i \to T(i)$, and let F denote the free Θ-model on the coproduct of the pointed sets $T(i)(\vartheta)$ for all $i \in I$. Then $\varinjlim\, T$ is the quotient of F by the smallest ideal containing the K_i for all $i \in I$, together with all those elements that correspond to relations imposed by the morphisms of I. This construction is evidently functorial.

1.19. **Definition.** For any category Θ, let Θ^δ denote the class of objects of Θ.

Thus Θ^δ is a semi-category (§1.16) whose inclusion into Θ corepresents 'underlying-functors', such as $U_* := U(\Theta)_* : \Theta\text{-}\mathcal{M}dl_* \to \Theta^\delta\text{-}\mathcal{M}dl_*$ or $U(\Theta) : \Theta\text{-}\mathcal{M}dl \to \Theta^\delta\text{-}\mathcal{M}dl$. As a consequence of approach (2) above to colimits, we have

1.20. **Corollary.** *There are 'free' functors* $F_* := F(\Theta)_* : \Theta^\delta\text{-}\mathcal{M}dl_* \to \Theta\text{-}\mathcal{M}dl_*$ *and* $F := F(\Theta) : \Theta^\delta\text{-}\mathcal{M}dl \to \Theta\text{-}\mathcal{M}dl$, *left adjoint to* $U(\Theta)_*$ *and* $U(\Theta)$, *respectively.*

Given FP-sketches $(\Theta, \mathcal{P})$ and $(\Psi, \mathcal{Q})$, we have the product FP-sketch $(\Theta \times \Psi, P \times Q)$. For an arbitrary category $\mathcal{C}$ we have the exponential isomorphisms of functor categories $(\mathcal{C}^\Psi)^\Theta \cong \mathcal{C}^{\Psi\times\Theta} \cong (\mathcal{C}^\Theta)^\Psi$.

1.21. **Proposition.** *Given FP-sketches* $(\Theta, \mathcal{P})$ *and* $(\Psi, \mathcal{Q})$ *and a category* $\mathcal{W}$ *with finite products, the exponential isomorphisms restrict to isomorphisms:*

$$\Theta\text{-}\mathcal{M}dl(\Psi\text{-}\mathcal{M}dl(\mathcal{W})) \;\cong\; (\Theta \times \Psi)\text{-}\mathcal{M}dl(\mathcal{W}) \;\cong\; \Psi\text{-}\mathcal{M}dl(\Theta\text{-}\mathcal{M}dl(\mathcal{W})).$$

2. Group objects in Θ-$\mathcal{M}dl$

In a category $\mathcal{C}$, a designation of a group object structure on an object x is given by a lifting G of $\mathrm{Hom}_{\mathcal{C}}(-, x)$ to the category of groups. The pair (x, G) is called a designated group object. A morphism $(x, G) \to (y, H)$ of group objects is given by a morphism $f : x \to y$ in $\mathcal{C}$ such that $f_* : \mathrm{Hom}(-, x) \to \mathrm{Hom}(-, y)$ determines a natural transformation $G \to H$. The definitions of (designated) 'monoid object', 'abelian group object', 'ring object' structure, and so on, are similar. When X is a Θ-model, we relate this idea to the presence of object-wise designations of group object structures at $X(\vartheta)$, ϑ in Θ. First, we introduce the following general concept:

2.1. **Definition.** Let $\mathfrak{X}$ be an FP-sketch with a distinguished object x. A designation of an *$\mathfrak{X}$-structure at an object* c in a category $\mathcal{C}$ is a morphism of FP-sketches $\psi : \mathfrak{X} \to \mathcal{C}$ of $\mathfrak{X}$ in $\mathcal{C}$ with $\psi(x) = c$.

If every object ϑ in an FP-sketch Θ admits an $\mathfrak{X}$-structure, we call Θ an *$\mathfrak{X}$-sketch.* A specific choice of an $\mathfrak{X}$-structure at each object of Θ will be called an *$\mathfrak{X}$-base* for Θ, and an *$\mathfrak{X}$-based sketch* is one with an $\mathfrak{X}$-base.

The reader need not be concerned about the apparent lack of coherence in an $\mathfrak{X}$-base of Θ. The main reason for designing such flexibility with $\mathfrak{X}$-structures will become apparent in 2.5.

2.2. **Example.** The inclusion of $\Pi_1^{\mathrm{op}} \cong \mathfrak{G}$ in Π^{op} (§1.6) designates a $\mathfrak{G}$-structure to the object $\mathbf{S}^1$. Similarly, one obtains a designation of an $\mathfrak{A}$-structure at $\mathbf{S}^n$, $n \geq 2$. These designations are unique up to symmetry of coproduct summands in Π_n. Since Π is the finite coproduct co-completion of the spheres $\mathbf{S}^n$, every Π-algebra X has, on each wedge of spheres W, a unique group structure which is the product of the appropriate groups $X(\mathbf{S}^n)$ (abelian if $n \geq 2$).

Two observations are in order here: (1) the $\mathfrak{G}$-structures at each W do not suffice to designate a group object structure on an arbitrary Π-algebra; (2) the $\mathfrak{G}$-base we constructed for Π^{op} is not unique.

To see the relationship between group objects and $\mathfrak{G}$-structures, we have the following:

2.3. **Lemma.** *In a pointed category* $\mathcal{C}$, *a group object structure on* X *determines and is determined by a* $\mathfrak{g}$*-structure at* X. *Moreover, if* $\mathcal{C} = \mathcal{W}^\Theta$, *a* $\mathfrak{g}$*-structure at* X *is given by a* $\mathfrak{g}$*-structure at each* $X(\vartheta)$ *in* $\mathcal{W}$ *such that, for each morphism* $f : \vartheta \to \vartheta'$, *the map* $f_* : X(\vartheta) \to X(\vartheta')$ *is a morphism of* $\mathfrak{g}$*-structures.* $\square$

A similar result holds for abelian group objects; in particular:

2.4. **Corollary.** *In* Θ-$\mathcal{M}dl_*$ *an* $\mathfrak{A}$*-structure at* X *is given by an abelian group structure on each* $X(\vartheta)$ *such that* $f_* : X(\vartheta) \to X(\vartheta')$ *is a homomorphism of groups for every* $f : \vartheta \to \vartheta'$ *in* Θ. □

The following generalizes the well-known fact that the fundamental group of a topological group is commutative (cf. [W, III, Thm. (5.21)]):

2.5. **Lemma.** *Let* $\mathcal{W}$ *be a category with null object and finite products. If* Θ *is an FP* $\mathfrak{m}$*-sketch with null object, then each object* X *in* Θ-$\mathcal{M}dl(\mathcal{W})$ *has at most one* $\mathfrak{G}$*-structure. In this case* X *is automatically abelian and the* $\mathfrak{A}$*-structure map at* $X(\vartheta)$ *agrees with the composite* $\mathfrak{m} \to \Theta \xrightarrow{X} \mathcal{W}$, *for* every choice *of an* $\mathfrak{m}$*-structure at* ϑ.

If the FP-sketch Θ is an $\mathfrak{m}$-sketch, Lemma 2.5 entitles us to speak of *the* abelian objects in Θ-$\mathcal{M}dl(\mathcal{W})$. These form a full subcategory of Θ-$\mathcal{M}dl(\mathcal{W})$, and each of its morphisms is a morphism of abelian group objects. In contrast, there are categories like $\mathcal{S}et$ where most objects have many choices of a group object structure. Selecting one of them for a set S amounts to designating a group object structure at S. There is nothing natural about such a designation, and a function between two sets with a designated group object structure usually fails to be a morphism of group objects.

2.6. **Lemma.** *If* Θ *is a FP* $\mathfrak{g}$*-sketch,* X *is a* Θ*-model, and* $p : Y \to X$ *has a designated group object structure in* Θ-$\mathcal{M}dl/X$, *then it is a designated abelian group object structure.*

2.7. **Corepresenting abelian Θ-models**. Given an FP $\mathfrak{g}$-sketch Θ, we construct a morphism of sketches $\Theta \to \Theta^{\mathrm{ab}}$ which corepresents the inclusion of abelian Θ-models in Θ-$\mathcal{M}dl_*$. We call Θ^{ab} the *abelianized* category of Θ.

2.8. **Definition.** For an FP $\mathfrak{g}$-sketch Θ define Θ^{ab} to be the category with the same objects as Θ, but with morphisms obtained as follows: first form Θ', the largest quotient of Θ so that all functors from $\mathfrak{g}$ to Θ factor uniquely through $\mathfrak{a}$. So Θ' has a unique $\mathfrak{a}$-structure at each $\vartheta \in \Theta$. Now let Θ^{ab} be the largest quotient of Θ' for which every $u : \vartheta \to \vartheta'$ is a morphism of abelian group objects; i.e., Θ^{ab} is constructed by taking Θ' modulo the equivalence relation generated by $u \circ \mu_\vartheta \sim \mu_{\vartheta'} \circ (u \times u)$, for each pair of $\mathfrak{a}$-structure maps μ_ϑ and $\mu_{\vartheta'}$ on ϑ and ϑ', respectively.

2.9. **Example.** The functor $\mathfrak{G} \to \mathfrak{G}^{\mathrm{ab}}$ is the opposite of the suspension functor $\Sigma : \Pi_1 \to \Pi_2$.

2.10. **Lemma.** *For an FP* $\mathfrak{g}$*-sketch* Θ, Θ^{ab} *corepresents the subcategory of abelian objects in* Θ-$\mathcal{M}dl(\mathcal{W})$, *and the abelianization functor* $\Theta \to \Theta^{\mathrm{ab}}$ *corepresents the inclusion of the category of abelian* Θ*-models into* Θ-$\mathcal{M}dl(\mathcal{W})$.

2.11. **Corollary.** *If* Θ *is an abelian category, then* $\Theta^{\mathrm{ab}} = \Theta$.

If the FP-sketch Θ is also a $\mathfrak{g}$-sketch, we establish here the existence of an abelianization functor on Θ-$\mathcal{M}dl$.

2.12. **Definition.** Let $\mathcal{C}$ be a category whose abelian group objects form a full subcategory $\mathrm{Ab}\mathcal{C}$ of $\mathcal{C}$. Abelianization on $\mathcal{C}$ is an augmented functor $\mathcal{A}b : \mathcal{C} \to \mathrm{Ab}\mathcal{C}$ which is idempotent in the sense that, for each object X the commutative augmentation diagram below

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "\rho_X"'] & \mathcal{A}bX \arrow[d, "\rho_{\mathcal{A}bX}"] \\ \mathcal{A}bX \arrow[r, "\mathcal{A}b\rho_X"'] & \mathcal{A}b\mathcal{A}bX \end{tikzcd}$$

has isomorphisms arriving at $\mathcal{A}b\mathcal{A}bX$.

2.13. **Proposition.** *If Θ is an FP $\mathfrak{g}$-sketch, then the category of Θ-$\mathcal{M}dl$ has a localization functor $X \mapsto X_{\rm ab}$.*

Proof. The abelianization of a Θ-model X can be constructed as follows: choose a $\mathfrak{g}$-base for Θ, and let ΓX denote the ideal of X generated by all elements of the form $f_*(u)f_*(v)(f_*\mu_\vartheta(u,v))^{-1}$, where $f:\vartheta\to\vartheta'$ ranges over all morphisms in Θ, $u,v\in X(\vartheta)$, and multiplication in $X(\vartheta')$ is with $X(\mu_{\vartheta'})$.

Then $X_{\rm ab}:=X/\Gamma X$ is the abelianization of X: it is a quotient of X such that, for every ϑ in Θ, $X(\vartheta)$ has the structure of an abelian group object and every morphism in Θ induces a homomorphism of abelian groups. So it is an abelian object in the category of Θ-$\mathcal{M}dl$ (cf. 2.4). Finally, if A is an abelian Θ-model, then any morphism $X\to A$ sends ΓX to 0. So the universal property of abelianization follows. □

3. Quillen Algebras and Modules over a Θ-model

In [Q3, §2], Quillen gave a general definition of modules over a given object in an arbitrary category $\mathcal{D}$. Here we spell out features of this concept in the case where $\mathcal{D}=\Theta$-$\mathcal{M}dl(\mathcal{W})$ for an FP $\mathfrak{g}$-sketch Θ.

3.1. **Definition.** Given a fixed object X in a category $\mathcal{D}$, let $\mathcal{D}/X$ denote the category of objects over X (cf. [Mc, II,6]). Any $p:Y\to X$ in $\mathcal{D}/X$ equipped with a section $s:X\to Y$ (so $p\circ s=\mathrm{Id}_X$) will be called an *algebra over X in the sense of Quillen* (cf. [Q3, §2]). The category of such algebras over X and section preserving morphisms will be denoted by $\mathcal{D}$-$\mathcal{A}lg/X$. When $Y=X\times Z$ and p is the projection, we say Y is a *trivial* algebra over X.

Now consider the case where $\mathcal{D}=\Theta$-$\mathcal{M}dl_*$ for an FP $\mathfrak{g}$-sketch Θ. Suppose a group object structure has been designated on the X-algebra $p:Y\to X$ in Θ-$\mathcal{A}lg/X:=\mathcal{D}$-$\mathcal{A}lg/X$. A choice of a $\mathfrak{g}$-structure at ϑ in Θ yields a split short exact sequence of groups $K(\vartheta)\to Y(\vartheta)\to X(\vartheta)$. The argument which proves 2.5 can be adapted to show that the group object structure restricted to $K(\vartheta)$ agrees with the $\mathfrak{g}$-multiplication and is, therefore, commutative. Consequently Θ-$\mathcal{A}lg/X$ has an intrinsically defined full subcategory of abelian group objects. This justifies the following terminology.

3.2. **Definition.** Given an FP $\mathfrak{g}$-sketch Θ, and an X in Θ-$\mathcal{M}dl_*$, an abelian group object in Θ-$\mathcal{A}lg/X$ will be called an *X-module (in the sense of Quillen)*, and the category of such will be denoted by Θ-$\mathcal{M}od/X$.

3.3. **X-action algebras and modules**. A section of an epimorphism of groups $q:Y\to X$ determines an action of X on $\mathrm{Ker}\,(q)$ and a corresponding description of Y as a semidirect product of groups. An analogous construction is available for X-algebras in Θ-$\mathcal{M}dl/X$ whenever Θ is an FP $\mathfrak{g}$-sketch. The key to the notion of a semidirect product of Θ-models is the notion of an X-action algebra. Its definition is based on the following observation:

Choose a $\mathfrak{g}$-base for Θ, and consider an X-algebra $q : Y \to X$ with section σ and $K := \mathrm{Ker}\,(q)$. For each ϑ in Θ we obtain a semidirect product decomposition of groups $Y(\vartheta) \cong K(\vartheta) \rtimes X(\vartheta)$.

3.4. **Lemma.** *If $f : \vartheta \to \vartheta'$ is a morphism in Θ, then $Y(f)$ determines and is determined by a function*

$$\gamma_f : K(\vartheta) \times X(\vartheta) \to K(\vartheta'), \qquad \text{such that} \qquad \gamma_f(1_{K(\vartheta)}, x) = 1_{K(\vartheta')}.$$

Proof. If $k \in K(\vartheta)$ and $x \in X(\vartheta)$, we have

$$\begin{aligned} Y_f(k\sigma_\vartheta(x)) &= K_f(k)\sigma_{\vartheta'}(f_X(x)) \left[(K_f(k)\sigma_{\vartheta'})^{-1} Y_f(k\sigma_\vartheta(x))\right] \\ &= K_f(k)\sigma_{\vartheta'}(f_X(x))\; \gamma(k, x) \\ &= K_f(k)\,(\sigma_{\vartheta'}(X_f(x)).\gamma_f(k, x))\; \sigma_{\vartheta'}(X_f(x)) \end{aligned}$$

The property $\gamma_f(1_{K(\vartheta)}, x) = 1_{K(\vartheta')}$ follows as $\sigma : X \to Y$ is a morphism of Θ-models. □

3.5. **Definition.** Given an FP $\mathfrak{g}$-sketch Θ, and a Θ-model X, an X*-action algebra* is a Θ-model K together with

(i) for each ϑ in Θ, an action c_ϑ of $X(\vartheta)$ on $K(\vartheta)$ through group automorphisms;
(ii) for each $f : \vartheta \to \vartheta'$, a function $\gamma_f : K(\vartheta) \times X(\vartheta) \to K(\vartheta)$ satisfying $\gamma_f(1_{K(\vartheta)}, x) = 1_{K(\vartheta')}$.

3.6. **Definition.** Given an FP $\mathfrak{g}$-sketch Θ, the *semidirect product* of a Θ-model X by an X-action algebra K is the Θ-model $K \rtimes X$ over X given by

(i) $(K \rtimes X)(\vartheta) := K(\vartheta) \rtimes_{c_\vartheta} X(\vartheta)$, the semidirect product of groups;
(ii) $(K \rtimes X)(f)(k, x) := (K_f(k)(X_f(x).\gamma_f(k, x)), X_f(x))$;
(iii) $q : K \rtimes X \to X$ given by projection onto the second coordinate;
(iv) $\sigma = (1_K, \mathrm{Id}) : X \to K \rtimes X$ given by inclusion as the second coordinate.

By Lemma 3.4, $q : K \rtimes X \to X$ is indeed an X-algebra, with kernel K.

3.7. **Definition.** An X*-action module* is an abelian group object in the category of X-action algebras.

3.8. **Proposition.** *The category of X-action modules is equivalent to the category of X-modules in the sense of Quillen, under the functors taking an X-action module K to the semidirect product $K \rtimes X$, and an X-module $q : M \to X$ to* $\mathrm{Ker}\,(q)$*, respectively.*

Let us continue to assume that Θ is an FP $\mathfrak{g}$-sketch. We have a section-forgetting functor $\Phi : \Theta\text{-}\mathcal{A}lg/X \to \Theta\text{-}\mathcal{M}dl/X$. An abelian object $Y \to X$ with section s provides a designation of an abelian group object structure on $\Phi(Y \to X, s)$ in $\Theta\text{-}\mathcal{M}dl/X$. Every such designation arises in this fashion. A morphism of abelian group objects in $\Theta\text{-}\mathcal{A}lg/X$ provides a morphism of the corresponding designations of abelian group object structures in $\Theta\text{-}\mathcal{M}dl/X$.

We alert the reader to the somewhat subtle fact that, in $\Theta\text{-}\mathcal{M}dl/X$, the collection of objects which possess an intrinsic and unique abelian group object structure is, in general, far smaller than the collection of objects which possess a designation of an abelian group object structure. This remains true even if Θ is an $\mathfrak{a}$-sketch whose models are object-wise vector spaces over a field.

4. Model categories of Θ-models

To define cohomology in FP-sketchable categories, we need a framework for "doing homotopy theory", in the form of a *model category*: that is, a bicomplete category $\mathcal{M}$ equipped with three distinguished classes of maps: weak equivalences, fibrations, and cofibrations, satisfying certain axioms (cf. [Q1, I, §§1,5] or [H, 7.1])).

However, an algebraic category $\mathcal{Z}$ itself rarely has a useful model category structure, so we embed it in a larger model category, such as the category $\mathcal{Z}^{\Delta^{\mathrm{op}}}$ of simplicial objects over $\mathcal{Z}$ (also denoted by $s\mathcal{Z}$). Many categories $\mathcal{Z}$ – including $\mathcal{S}et$, $\mathcal{S}et_*$, $\mathcal{G}p$, and $\mathcal{T}_*$ – have a standard model category structure on $s\mathcal{Z}$; see [Q1, II, 3]. This is cofibrantly generated (cf. [H, 11.1.2]) when $\mathcal{Z} = \mathcal{S}et$ or $\mathcal{S}et_*$.

Moreover, for $\mathcal{Z} = \Theta\text{-}\mathcal{M}dl$, we can use the adjoint functors of Lemma 1.20 to transport the model category structure on $s\mathcal{S}et_*$ to $s\Theta\text{-}\mathcal{M}dl$ (as Quillen did implicitly in [Q1, II,3]). Formally, by applying [Bl, Thm. 4.15], we obtain the following:

4.1. **Proposition.** *For any FP-theory Θ, there is a cofibrantly generated model category structure on $s\Theta\text{-}\mathcal{M}dl$, in which a map f is a weak equivalence (respectively, a fibration) in $s\Theta\text{-}\mathcal{M}dl$ if and only if $U(\Theta)f$ is such. Thus a map $f : W_\bullet \to Y_\bullet$ of simplicial Θ-models is:*

- *(i)* a weak equivalence *if for each $\vartheta \in \Theta^\delta$, the map $\mathrm{pr}_\vartheta\, U(\Theta)f$ is a weak equivalence of simplicial sets – i.e., induces an isomorphism in homotopy groups between fibrant replacements (these are not needed if Θ is a $\mathfrak{G}$-theory, by Lemma 4.19 below).*
- *(ii)* a fibration *if for each $\theta \in \Theta^\delta$, the map $\mathrm{pr}_\vartheta\, U(\Theta)f$ is a Kan fibration.*

The cofibrations, which are determined by the left lifting property, can also be described explicitly with the aid of the following:

4.2. **Definition.** Given a simplicial object $X_\bullet$ in a cocomplete category $\mathcal{E}$, its n-th *latching object* is defined

$$L_n X_\bullet := \coprod_{0 \le i \le n-1} X_{n-1}/\sim,$$

where for any $x \in X_{n-2}$ and $0 \le i \le j \le n-1$ we set $s_j x$ in the i-th copy of X_{n-1} equivalent to $s_i x$ in the $(j+1)$-st copy of X_{n-1}.

4.3. **Definition.** A map $f : W_\bullet \to Y_\bullet$ of simplicial Θ-models is a *free map* if for each $n \ge 0$ there is a free Θ-model V_n such that Y_n is isomorphic to the pushout $(V_n \amalg_{L_n W_\bullet} L_n Y_\bullet) \amalg W_n$.

The construction of a free map should be thought of as inductively attaching 'free cells' V_n to $W_\bullet$. In particular, a simplicial Θ-model $Y_\bullet$ is *free* (that is, the map $0 \to Y_\bullet$ is free), if for each $n \ge 0$ there is a Θ^δ-graded set T^n such that $Y_n = F_\Theta(T^n)$, and each degeneracy map $s_j : Y_n \to Y_{n+1}$ takes T^n to T^{n+1}.

4.4. **Fact.** A map of simplicial Θ-models is a cofibration if and only if it is a retract of a free map.

(In the cases of interest to us, any retract of a free map is itself free.)

4.5**. Simplicial categories**.

The model category $\mathcal{C} = s\Theta\text{-}\mathcal{M}dl$ supports additional structure, which is needed for some of our constructions and results. First of all (like any category of simplicial objects over a bicomplete category), it is *simplicial* – that is, for any $X, Y, Z \in \mathcal{C}$

and simplicial sets $K, L \in s\mathcal{S}et$, we have functorial constructions $\underline{\mathrm{Hom}}(X,Y) \in s\mathcal{S}et$, $X \otimes K \in \mathcal{C}$, and $X^K \in \mathcal{C}$, with appropriate properties (cf. [Q1, Ch. II,§1]).

In fact, the coproduct in Θ-$\mathcal{M}dl$ induces a functor $\hat{\otimes} : \Theta\text{-}\mathcal{M}dl \times s\mathcal{S}et \to \mathcal{C}$, defined $A \times K \mapsto A\hat{\otimes}K$ (where $(A\hat{\otimes}K)_n = \coprod_{K_n} A$), and $A_\bullet \otimes K = \mathrm{diag}(A_\bullet \hat{\otimes} K)$ for any simplicial object $A_\bullet \in \mathcal{C}$. Here $\mathrm{diag}\, B_{\bullet\bullet}$ is the diagonal of a bisimplicial object $B_{\bullet\bullet} \in s\mathcal{C}$. Moreover:

4.6. **Fact.** The functor $(-)^K$ is preserved by left adjoints, while $(-) \otimes K$ (as well as $(-)\hat{\otimes}K$) are preserved by right adjoints.

Furthermore, $s\Theta$-$\mathcal{M}dl$ is actually a *simplicial model category* – that is, for any cofibration $i : A \hookrightarrow B$ and fibration $p : X \twoheadrightarrow Y$ the induced map:

$$\underline{\mathrm{Hom}}(B,X) \xrightarrow{(i^*,p_*)} \underline{\mathrm{Hom}}(A,X) \times_{\underline{\mathrm{Hom}}(A,Y)} \underline{\mathrm{Hom}}(B,Y) \tag{4.7}$$

is a fibration in $s\mathcal{S}et$, which is a weak equivalence if either i or p is (see [Q1, Ch. II, §2]). Moreover, in order for this to hold, it is enough to check when $A \hookrightarrow B$ is one of the canonical cofibrations $\partial\mathbf{\Delta}[n] \hookrightarrow \mathbf{\Delta}[n]$ $(n \geq 1)$.

4.8. **Definition.** In a simplicial model category $\mathcal{C}$ we define two maps $f, g : X \to Y$ to be *strictly simplicially homotopic* if they are homotopic as 0-simplices in $\underline{\mathrm{Hom}}(X,Y)$ – i.e., if there is a simplex $\sigma \in \underline{\mathrm{Hom}}(X,Y)_1$ with $d_0\sigma = f$ and $d_1\sigma = g$. Note that $\underline{\mathrm{Hom}}(X,Y)_1 = \mathrm{Hom}_{\mathcal{C}}(X \otimes \mathbf{\Delta}[1], Y)$, so we can think of σ as a *strict simplicial homotopy* from f to g.

More generally, if J is a generalized interval (i.e., a union of 1-simplices laid end to end), then any map in $\mathrm{Hom}_{\mathcal{C}}(X \otimes J, Y)$ is called a (ordinary) *simplicial homotopy* between f and g if the obvious boundary condition holds.

4.9. **Fact** ([Q1, Ch. II, §2, Prop. 5])**.** In a simplicial model category, if X is cofibrant and Y fibrant then all versions of homotopy between two maps $f, g : X \to Y$ coincide, and $[X,Y] = \pi_0\underline{\mathrm{Hom}}(X,Y)$.

4.10. **Proposition.** *If* $B_\bullet = F(\hat{B}_\bullet)$ *is the free simplicial* Θ*-model on a* Θ^δ*-graded simplicial set* $\hat{B}_\bullet$*, and* $X_\bullet \in s\Theta\text{-}\mathcal{M}dl$ *is fibrant, then*

$$[B_\bullet, X_\bullet]_{\mathcal{C}} \cong [\hat{B}_\bullet, U(\Theta)X_\bullet]_{s\,\mathrm{gr}_{\Theta^\delta}\,\mathcal{S}et_*}.$$

Here F is the free functor of Lemma 1.20, and $U(\Theta) : \Theta\text{-}\mathcal{M}dl \to \mathcal{S}et_*$ is its right adjoint. We use the same names for their extensions to simplicial objects.

Proof. By Proposition 4.1, $f : X_\bullet \to Y_\bullet$ is a fibration (resp., weak equivalence) in $s\Theta$-$\mathcal{M}dl$ if and only if $U(\Theta)f$ is such in $s\,\mathrm{gr}_K\,\mathcal{S}et$. In particular, mapping into a fibrant simplicial Θ-model $X_\bullet$ we see that:

$$\begin{aligned} \underline{\mathrm{Hom}}(B_\bullet, X_\bullet) &= \mathrm{Hom}_{\mathcal{C}}(B_\bullet \otimes \mathbf{\Delta}[\bullet], X_\bullet) = \mathrm{Hom}_{\mathcal{C}}(F(\hat{B}_\bullet) \otimes \mathbf{\Delta}[\bullet], X_\bullet) \\ &= \mathrm{Hom}_{\mathcal{C}}(F(\hat{B}_\bullet \otimes \mathbf{\Delta}[\bullet]), X_\bullet) = \mathrm{Hom}_{s\,\mathrm{gr}_{\Theta^\delta}\,\mathcal{S}et_*}(\hat{B}_\bullet \otimes \mathbf{\Delta}[\bullet], U(\Theta)X_\bullet) \end{aligned} \tag{4.11}$$

Since $s\mathcal{S}et_*$, and thus also $s(\mathcal{S}et_*^{\Theta^\delta}) \cong (s\mathcal{S}et_*)^{\Theta^\delta}$, are simplicial model categories, applying $U(\Theta)$ to $\partial\mathbf{\Delta}[n] \hookrightarrow \mathbf{\Delta}[n]$ in (4.7) (and taking Fact 4.6 into account), we see that $\mathcal{C} = s\Theta$-$\mathcal{M}dl$ is a simplicial model category. Therefore, Fact 4.9 implies that we may use simplicial homotopies to compute homotopy classes of maps in $[B_\bullet, X_\bullet]$, and thus the adjunction of (4.11) passes to homotopy, as required. □

4.12. **Resolution model categories**.

Dwyer, Kan, and Stover provide another way to describe the model category structure on $s\Theta\text{-}\mathcal{M}dl$ as a *resolution* (or: E^2-) model category (cf. [DKS1]; see also [Bou]). In this approach, one chooses certain cogroup objects in a given category $\mathcal{C}$ (in our case: the free Θ-models in $\Theta\text{-}\mathcal{M}dl$), and uses these to define weak equivalences and cofibrations in $s\mathcal{C}$.

In particular, given an FP-sketch Θ, for $\vartheta \in \Theta$ and $n \geq 1$, the *n-simplicial ϑ-sphere* is the simplicial Θ-model $\Sigma^n_\vartheta := F_\vartheta \hat{\otimes} S^n$, where $S^n := \mathbf{\Delta}[n]/\partial\mathbf{\Delta}[n]$. F_ϑ is the free Θ-model generated by ϑ, and $A\hat{\otimes}X$ was defined in §4.5. In fact, each n-simplicial ϑ-sphere Σ^n_ϑ is free (and thus cofibrant). Set $\Sigma^0_\vartheta := F_\vartheta \hat{\otimes}\mathbf{\Delta}[0]$.

4.13. **Definition.** For $\vartheta \in \Theta$ and $n \geq 0$, the (n,ϑ) *homotopy group* of a simplicial Θ-model $Y_\bullet$ is $\pi_{(n,\vartheta)}Y_\bullet := [\Sigma^n_\vartheta, Y_\bullet]_{\mathrm{ho}\, s\Theta\text{-}\mathcal{M}dl}$.

4.14. *Remark.* Because S^n has two non-degenerate simplices, in dimensions 0 and n respectively, the homotopy groups defined here have more information than the usual ones: they also record the component in $\pi_{(0,\vartheta)}Y_\bullet$ where a given map $f : \Sigma^n_\vartheta \to Y_\bullet$ lands.

More precisely, if we set $\hat{\Sigma}^n_\vartheta := F_\vartheta\hat{\otimes}\mathbf{\Delta}[n]/F_\vartheta\hat{\otimes}\partial\mathbf{\Delta}[n]$ (for $n \geq 1$), then the map of simplicial sets $S^n \to \mathbf{\Delta}[0]$ has a section, which induces:

$$\begin{tikzcd} \hat{\Sigma}^n_\vartheta \arrow[r, "i"] & \Sigma^n_\vartheta \arrow[r, "p"'] & \Sigma^0_\vartheta \arrow[l, bend right, "s"'] \end{tikzcd},$$

and thus a natural splitting:

$$\begin{tikzcd} \pi_{(n,\vartheta)}Y_\bullet \arrow[r, "p_\#"'] & \pi_{(0,\vartheta)}Y_\bullet \arrow[l, bend right, "s_\#"'] \end{tikzcd} \tag{4.15}$$

for each simplicial Θ-model $Y_\bullet$ and $\vartheta \in \Theta$, where $\mathrm{Ker}\,(p_\#) \cong [\hat{\Sigma}^n_\vartheta, Y_\bullet]$ is actually the more traditional n-th homotopy group of $Y_\bullet$ (over the base-point component).

If we think of $Y_\bullet$ as a Θ-object in $s\mathcal{S}et_*$, and use Proposition 4.10 to see that this adjunction carries over to the homotopy groups, we conclude:

4.16. **Proposition.** *For any simplicial Θ-model $Y_\bullet$, $\hat{\pi}_nY_\bullet := (\pi_{(n,\vartheta)}Y_\bullet)_{\vartheta\in\Theta}$ has the structure of a Θ-model.*

Note also that any map of FP $\mathfrak{G}$-sketches $T : \Theta' \to \Theta$ induces a natural transformation $\hat{\pi}_nY_\bullet \to \hat{\pi}_n(T^*Y_\bullet)$ for any simplicial Θ-model $Y_\bullet$ and $n \geq 1$.

4.17. **Proposition.** *If Θ is an $\mathfrak{M}$-theory (§1.12), then for each $n \geq 1$ the Θ-model $\hat{\pi}_nY_\bullet$ has a natural designated abelian group object structure in the category $\Theta\text{-}\mathcal{M}dl/\hat{\pi}_0Y_\bullet$ for any simplicial Θ-model $Y_\bullet$.*

Proof. By Proposition 4.16, $\hat{\pi}_nY_\bullet$ has a natural Θ-model structure. If $\overline{S}^n$ is a fibrant replacement for S^n in $s\mathcal{S}et$, then the standard homotopy cogroup structure on the n-sphere is represented by a pinch map $\nabla : \overline{S}^n \to \overline{S}^n \vee \overline{S}^n$ (and so on). This induces maps $F_\vartheta\hat{\otimes}\overline{S}^n \to F_\vartheta\hat{\otimes}(\overline{S}^n \vee \overline{S}^n)$ (and so on) over $F_\vartheta\hat{\otimes}\mathbf{\Delta}[0]$. Since Σ^n_ϑ is fibrant and cofibrant in $s\Theta\text{-}\mathcal{M}dl$, it is homotopy equivalent to $F_\vartheta\hat{\otimes}\overline{S}^n$ (naturally in ϑ), so it also has a natural homotopy cogroup object structure over Σ^0_ϑ, making $\hat{\pi}_nY_\bullet$ a group object in $\Theta\text{-}\mathcal{M}dl/\hat{\pi}_0Y_\bullet$. The claim then follows by Lemma 2.6. $\square$

4.18. **Corollary.** *If Θ is an $\mathfrak{M}$-theory (§1.12), then the Θ-model $[\hat{\Sigma}^n_\vartheta, Y_\bullet]_{\vartheta\in\Theta}$ is abelian for any simplicial Θ-model $Y_\bullet$ and $n \geq 1$.*

We also note the following:

4.19. **Lemma.** *If Θ is a $\mathfrak{G}$-theory, a map of simplicial Θ-models $f : W_\bullet \to Y_\bullet$ is a fibration if and only if the underlying map of Θ^δ-graded groups is a surjection onto the base point component.*

Proof. This follows from Proposition 4.1 (ii) and [Q1, II, §3, Prop. 1], since each Θ-model X has the underlying structure of a (Θ^δ-graded) group. □

4.20. **Corollary.** *A map of simplicial Θ-models $f : W_\bullet \to Y_\bullet$ is a weak equivalence if and only if it induces isomorphisms in $\hat{\pi}_n$ for all $n \geq 0$.*

Proof. This follows from Propositions 4.10 and 4.1 (i), since all simplicial Θ-models are fibrant, while Σ^n_ϑ is free, and so cofibrant. □

4.21. *Remark.* If Θ is merely an FP-sketch, rather than a theory, the above definitions are still valid, but they may be less useful. The reason is that Θ-$\mathcal{M}dl$ itself may have a non-trivial model category structure – e.g., when $\Theta = \Delta^{\mathrm{op}}$ (Example 1.4). In that case the construction of the resolution model category should take this into account, and will then differ from the model category structure of Proposition 4.1. This was the main point of [DKS1].

5. Cohomology of Θ-models

In [Q1, II, §5], Quillen proposed a general method for defining cohomology in any model category $\mathcal{C}$. In the case of interest to us here, where $\mathcal{C} = s\Theta$-$\mathcal{M}dl$ for Θ an FP $\mathfrak{g}$-theory, the cohomology groups have the additional property of being representable (in $\mathrm{ho}\,\mathcal{C}$) by suitable Eilenberg-Mac Lane objects. These can be used to describe a bijective correspondence between $H^1_\Theta(X; M)$ and the equivalence classes of central extensions of X by M.

First, we need the following:

5.1. **Definition.** Let $\mathcal{C}$ be any category, and X an object in $\mathcal{C}$. A *Q-module over X* is an object in the over category $\mathcal{C}/X$ equipped with a designated abelian group object structure. If M has a designated abelian group object structure in $\mathcal{C}$, the projection $M \times X \to X$ will be called a *trivial Q-module*.

5.2. **Definition.** Let $\mathcal{C}$ be any category, X an object in $\mathcal{C}$, and $p : M \to X$ a Q-module over X. Assume that $s\mathcal{C}$ has a simplicial model category structure satisfying Fact 4.9 (e.g., if $\mathcal{C}$ is FP-sketchable). Then the (the André-Quillen) *cohomology with coefficients in M* is the total left derived functor of $\mathrm{Hom}_{\mathcal{C}/X}(-, M)$. If $c(X)_\bullet$ is the constant simplicial object defined by X, then this is defined for any $Y_\bullet \in s(\mathcal{C}/X) = (s\mathcal{C})/c(X)_\bullet$ by applying $\mathrm{Hom}_{\mathcal{C}/X}(-, M)$ dimensionwise to a cofibrant replacement $B_\bullet$ for $Y_\bullet$ (that is, a cofibrant object equipped with a weak equivalence $Y_\bullet \to B_\bullet$ – any two such are homotopy equivalent.) The total left derived functor takes values in the homotopy category of cosimplicial abelian groups (or equivalently, of cochain complexes). See [Q3, §2].

The n-th (André-Quillen) *cohomology group* of $Y_\bullet \in s(\mathcal{C}/X)$ with coefficients in M is defined to be:

$$H^n_X(Y_\bullet; M) := \pi^n \,\mathrm{Hom}_{\mathcal{C}/X}(B_\bullet, M) \tag{5.3}$$

(where the n-th cohomotopy group of a cosimplicial abelian group is simply the n-th cohomology group of the corresponding cochain complex – cf. [BK, X,§7.1]).

5.4. **Definition.** Note that if $\tilde{M} = M \times X \to X$ is a trivial Q-module, then $H^n_X(Y_\bullet; \tilde{M}) \cong \pi^n \operatorname{Hom}_{\mathcal{C}}(B_\bullet, M)$. We will denote this group simply by $H^n(Y_\bullet; M)$.

5.5**. Representing cohomology**. Definition 5.2 makes sense for any category $\mathcal{C}$, as long as $s\mathcal{C}$ is equipped with an appropriate simplicial model category structure. However, when Θ is a $\mathfrak{G}$-theory, we have an alternative description for the cohomology groups, using the model category structure of §4.12.

5.6. **Definition.** Let Θ be a $\mathfrak{G}$-theory. Given a Θ-model X, we write BX for any simplicial Θ-model with $\hat{\pi}_0 BX = X$ and $\hat{\pi}_k BX = 0$ for $k > 0$.

Given a Q-module M over X and an integer $n \geq 1$, an *n-dimensional extended M-Eilenberg-Mac Lane object* $K^X(M,n)$ is a simplicial Θ-model $K_\bullet$ such that:

- $K_\bullet$ is equipped with a designated abelian group object structure in $\operatorname{ho} s(\Theta\text{-}\mathcal{M}dl/X)$.
- $\hat{\pi}_0 K^X(M,n) \cong X$.
- $\hat{\pi}_n K^X(M,n) \cong M$ as an X-module (see Proposition 4.17); and
- $\hat{\pi}_k K^X(M,n) = 0$ for $k \neq 0, n$.

The homotopy fiber of $K^X(M,n) \to BX$ will be called an *n-dimensional M-Eilenberg-Mac Lane object*, and denoted by $K(M,n)$.

5.7. **Proposition.** *For any $X \in \Theta\text{-}\mathcal{M}dl$, Q-module M over X, and $n \geq 1$, there exist a BX, as well as an n-dimensional extended M-Eilenberg-Mac Lane object $K^X(M,n)$ – and thus also $K(M,n)$ – all unique up to homotopy.*

Proof. There is a fibrant (though not cofibrant) model for $K^X(M,n)$ of the form $\bar{W}^n M$, where $\bar{W}$ is the Eilenberg-Mac Lane classifying space functor applied in the category $\Theta\text{-}\mathcal{M}dl/X$ (cf. [Ma, §21]). We may take $BX = c(X)_\bullet$. Evidently $K(M,n) \simeq \bar{W}^n_{\Theta\text{-}\mathcal{M}dl} M$, where $\bar{W}_{\Theta\text{-}\mathcal{M}dl}$ is now taken in $\Theta\text{-}\mathcal{M}dl$. □

5.8. **Theorem.** *If Θ is an $\mathfrak{A}$-theory, for any Θ-model X, Q-module M over X, and $B_\bullet \in s\Theta\text{-}\mathcal{M}dl/X$, there is a bijective correspondence (natural in X, M, and $B_\bullet$):*

$$H^n_\Theta(B_\bullet/X; M) \;\cong\; [B_\bullet, K^X(M,n)]_{\operatorname{ho} s\Theta\text{-}\mathcal{M}dl/X} \quad \textit{for } n \geq 1.$$

Proof. Any map of X-algebras $\phi : B_\bullet \to M$ which is an n-cocyle yields a map $\varphi : B_\bullet \to K^X(M,n)$ over BX (using the model for $K^X(M,n)$ given in Proposition 5.7). Therefore, we have canonical natural transformations $H^n_\Theta(-/X; M) \to [-, K^X(M,n)]_{\operatorname{ho} s\Theta\text{-}\mathcal{M}dl/X}$, which are isomorphisms when applied to a coproduct of spheres, by Definitions 4.13-5.6.

Every cofibrant $B_\bullet$ can be constructed (up to homotopy) by successively attaching cells along maps from spheres (this is cofibrant – or rather, free – approximation in $s\Theta\text{-}\mathcal{M}dl/X$). We may verify that $H^*_\Theta(-/X; M)$ and $[-, K^X(M,*)]_{\operatorname{ho} s\Theta\text{-}\mathcal{M}dl/X}$ both satisfy the Eilenberg-Steenrod axioms for a cohomology theory; the Mayer-Vietoris long exact sequence in homotopy follows from the pushout condition as in the proof of Proposition 4.10. □

A morphism $T : \Theta \to \Psi$ of FP-sketches corepresents a 'structure changing functor' $T^* : \Psi\text{-}\mathcal{M}dl \to \Theta\text{-}\mathcal{M}dl$ (which extends to simplicial objects). Thus a given Ψ-model X turns into a Θ-model T^*X. The following establishes the relationship between the cohomologies of X and T^*X in their respective categories:

5.9. **Definition.** Given a map of theories $T : \Theta \to \Psi$, a simplicial Ψ-model $Y_\bullet$ and $\pi_0 Y_\bullet$-module M, for each $n \geq 1$ the *n-th relative cohomology group* of $Y_\bullet$ with coefficients in M, denoted by $H^n_T(Y_\bullet, M)$, is defined as follows.

Let $\varepsilon : P_\bullet \to Y_\bullet$ be a cofibrant replacement in $s\Psi\text{-}\mathcal{M}dl$; then $T^*\varepsilon : T^*P_\bullet \to T^*Y_\bullet$ is still a fibration and weak equivalence in $s\Theta\text{-}\mathcal{M}dl$ (though $T^*P_\bullet$ is not usually cofibrant). So if $\eta : Q_\bullet \to T^*Y_\bullet$ is a cofibrant replacement in $s\Theta\text{-}\mathcal{M}dl$, there is a lifting:

$$\begin{tikzcd} * \arrow[r] \arrow[d, "\mathrm{cof}"'] & T^*P_\bullet \arrow[d, "\simeq"', "T^*\varepsilon"] \\ Q_\bullet \arrow[ur, dotted, "f"] \arrow[r, "\simeq", "\eta"'] & T^*Y_\bullet \end{tikzcd}$$

in $s\Theta\text{-}\mathcal{M}dl$, where f is a weak equivalence, too. Now if $K \simeq K^A(M,n)$ in $s\Psi\text{-}\mathcal{M}dl$ (for $A = \pi_0 Y_\bullet$), then, since $T^* : \Psi\text{-}\mathcal{M}dl \to \Theta\text{-}\mathcal{M}dl$ preserves fibrations, weak equivalences, and module relations, we have $T^*K \simeq K^{T^*A}(T^*M, n)$ in $s\Theta\text{-}\mathcal{M}dl$. Moreover, T^* induces a map of simplicial sets from $\mathrm{map}_{B\pi_0 Y_\bullet}(Y_\bullet, K) := \mathrm{map}_{B\pi_0 Y_\bullet}(P_\bullet, K)_{s\Theta\text{-}\mathcal{M}dl}$ to $\mathrm{map}_{B\pi_0 T^*Y_\bullet}(T^*Y_\bullet, T^*K) := \mathrm{map}_{B\pi_0 T^*Y_\bullet}(Q_\bullet, T^*K)_{s\Psi\text{-}\mathcal{M}dl}$.

Denote the homotopy fiber of this map (cf. [Q1, I, 3]) by $Z_\bullet$, and set $H^n_T(Y_\bullet; M) := \pi_{n+1} Z_\bullet$.

5.10. **Proposition.** *Given a map of FP $\mathfrak{g}$-theories $T : \Theta \to \Psi$, the relative cohomology groups are homotopy invariant, and fit into a natural long exact sequence:*

$$(5.11)\quad \ldots \to H^n_\Theta(Y_\bullet, M) \to H^n_{\Theta'}(T^*Y_\bullet, T^*M) \to H^n_T(Y_\bullet, M) \to H^{n+1}_\Theta(Y_\bullet, M) \to \ldots.$$

Proof. The homotopy invariance is evident from the construction; (5.11) is just the long exact sequence of the fibration sequence:

$$Z_\bullet \to \mathrm{map}_{B\pi_0 Y_\bullet}(Y_\bullet, K) \to \mathrm{map}_{B\pi_0 T^*Y_\bullet}(T^*Y_\bullet, T^*K),$$

combined with the natural isomorphisms

$$\pi_i\, \mathrm{map}_{B\pi_0 Y_\bullet}(Y_\bullet, K) \cong \pi_0 \Omega^i\, \mathrm{map}_{B\pi_0 Y_\bullet}(Y_\bullet, K) \cong \pi_0\, \mathrm{map}_{B\pi_0 Y_\bullet}(Y_\bullet, K^A(M, n-i)),$$

and similarly for $\mathrm{map}_{B\pi_0 T^*Y_\bullet}(T^*Y_\bullet, T^*K)$. $\square$

6. Full fibers of a functor

We now turn to the subject of this paper: the various fibers, or preimages, of one or more functors between FP-sketchable categories. We start with a general result on fibers of such a functor.

6.1. **Definition.** Given a functor $T : \mathcal{C} \to \mathcal{D}$ and an object D in $\mathcal{D}$,

(i) the *strict fiber* of T is the subcategory of $\mathcal{C}$ associated to those morphisms which T sends to the identity on D.
(ii) the *full fiber* of T at D is the *full* subcategory $T^{-1}D$ of $\mathcal{C}$ consisting of those objects C satisfying $T(C) = D$.

Our primary interest is directed towards the isomorphism classes of objects of $T^{-1}(D)$, i.e., the components of the groupoid of its isomorphisms. However, the category $T^{-1}(D)$ is structurally much richer than the connected component set of the groupoid of its isomorphisms. Here we provide information about the classifying space $BT^{-1}D$ (see [Se] or [Q4, §2]) in the following setting:

Let Ψ and Θ be FP $\mathfrak{m}$-sketches with the same object set, and let $\varphi : \Psi \to \Theta$ be a morphism of FP-sketches which is the identity function on object sets. Set

$\mathcal{C} := \Theta\text{-}\mathcal{M}dl$ and $\mathcal{D} := \Psi\text{-}\mathcal{M}dl$, respectively. The fiber of $T := \varphi^* : \mathcal{C} \to \mathcal{D}$ over any object of $\mathcal{D}$ is automatically small. Moreover, for fixed $D : \Psi \to \mathcal{S}et$, $G := \operatorname{Aut}(D)$ acts on $T^{-1}(D)$ via functors as follows: given an automorphism $f : D \to D$ and a Θ-model $X : \Theta \to \mathcal{S}et_*$ with $X \circ \varphi = D$, let $(f.X)(\vartheta) := X(\vartheta)$ on objects. If $u : \vartheta \to \vartheta'$ is a morphism of Θ, define $(f.X)(u)$ by the diagram of functions of sets

$$\begin{tikzcd}
X(\vartheta) \arrow[rrr, "(f.X)(u)"] \arrow[d, equal] & & & X(\vartheta') \arrow[d, equal] \\
D(\vartheta) \arrow[r, "f(\vartheta)"'] & D(\vartheta) = X(\vartheta) \arrow[r, "X(u)"'] & X(\vartheta') = D(\vartheta') \arrow[r, "f(\vartheta')^{-1}"'] & D(\vartheta')
\end{tikzcd}$$

6.2. **Proposition.** *In the situation above, if the full fiber F of T over D in $\mathcal{D}$ is not empty, then BF is the universal* $\operatorname{Aut}(D)$*-space with respect to the closure under finite intersection of the family of isotropy groups of the objects in F.*

Proof. Set $G := \operatorname{Aut}(D)$. The canonical CW-structure on BF can be given the structure of a G-CW-space (cf. [tD, II]) by using G-orbits in the nerve of F to index equivariant cells. Further, given a subgroup H of G, the category F^H of objects and morphisms of F which are fixed under H is filtering via 0-maps (we use the hypothesis that Ψ and Θ be FP $\mathfrak{m}$-sketches to guarantee the existence of a 0-morphism between any two models). So it is contractible by [Q4, p. 92]. Thus the claim follows upon observing that the inclusion $F^H \to F$ determines a homeomorphism $B(F^H) \to (BF)^H$. □

6.3. **Example.** Consider the forgetful functor $U : \mathcal{G}p \to \mathcal{S}et_*$. If D is a pointed set, $U^{-1}(D)$ is the full subcategory of $\mathcal{G}p$ whose objects provide a group structure on D. We may view $\operatorname{Aut}(D)$ as the symmetric group on $D - \{*\}$. So if $|D| = n+1$, we have an action of Σ_n on BF, whose stabilizer at a group H in $U^{-1}(D)$ is the subgroup of automorphisms of H under this action.

Thus BF is the universal Σ_n-space with respect to the closure of the family of homogeneous sets $\Sigma_n/\operatorname{Aut}(H)$ under intersection of conjugacy classes of such stabilizers.

In particular, if $|D| = p$ is a prime, there is exactly one isomorphism class of groups in the fiber of U, namely the cyclic group C_p. Automorphisms of C_p correspond to the group of units of the field $\mathbb{F}_p$, hence form the cyclic group of order $(p-1)$. It follows that BF is the universal Σ_{p-1}-space with respect to the family of those cyclic subgroups of Σ_{p-1} whose order divides $p-1$.

7. Complementary subcategories for an $\mathfrak{A}$-theory Θ

Unfortunately, further analysis of the fiber of a functor $T : \mathcal{C} \to \mathcal{D}$ requires additional assumptions on both the categories and the functors; so we specialize to the following situation (hopefully still of general interest in the context of FP-sketchable categories):

From now on we assume that Θ is an $\mathfrak{A}$-based theory (§2.1), and consider two functors, each of which retains at least some of the information lost by the other. We term such functors 'complementary'. Specifically, we are interested in

- the abelianization functor $\mathcal{A}b : \Theta\text{-}\mathcal{M}dl \to (\Theta\text{-}\mathcal{M}dl)_{\text{ab}}$, and
- the forgetful functor $U : \Theta\text{-}\mathcal{M}dl \to \Xi\text{-}\mathcal{M}dl$, associated to the inclusion of a subcategory Ξ of Θ which is 'complementary' to Θ^{ab} in the sense that it corepresents some of the information lost under abelianization.

By Corollary 2.4, if Θ-models are not all abelian, there must be maps in Θ which fail to be homomorphisms. For any $u : \vartheta \to \vartheta'$ in Θ, the obstruction to u being a homomorphism is the *cross-effects* map $u(x+y)-u(x)-u(y)$. Thus we concentrate on the situation described in the following:

7.1. **Definition.** A *complementary subcategory* to Θ^{ab} in an $\mathfrak{A}$-based theory Θ is a subcategory Ξ such that

(a) Ξ has the same objects as Θ;
(b) Ξ^{ab} (§2.7) is an $\mathfrak{A}$-subcategory of Ξ – that is, there is an inclusion $j' : \Xi^{\mathrm{ab}} \hookrightarrow \Xi$ for which $\Xi \to \Xi^{\mathrm{ab}}$ is a retraction.
(c) Ξ includes all cross-effect maps $c_u(x,y) := u(x+y) - u(x) - u(y)$ in Θ.

Ξ^{ab} will be called the *underlying* $\mathfrak{A}$*-category* of Θ (with respect to the complementary subcategory Ξ).

7.2. *Remark.* Ξ^{ab} is a subcategory of Θ, with the same object set, which includes the given $\mathfrak{A}$-structure at each object ϑ, and, in addition, some or all of those maps $\eta : \vartheta \to \vartheta'$ of Θ which are homomorphisms with respect to the $\mathfrak{A}$-structure. It is thus a subcategory of Θ^{ab} which embeds in Θ (which is not generally true of Θ^{ab} as a whole).

Ξ, which is in fact determined by Ξ^{ab}, is 'complementary' to Θ^{ab} in the sense that they 'intersect' only in the underlying category Ξ^{ab}, which should be thought of as the 'ground category'.

7.3. **Examples.** In many examples – such as Π-algebras, Lie algebras, associative algebras, and so on – we have a simple algebraic description of the complementary subcategories Ξ, since all cross-effect maps are generated by binary products:

(i) If the theory Θ corepresents associative algebras over a ground ring R, say, then the minimal complementary subcategory Ξ would represent rings ($\mathbb{Z}$-algebras), with $\Xi^{\mathrm{ab}} = \mathfrak{A}$ corepresenting abelian groups, while Θ^{ab} corepresents R-modules. However, in this case we could also take the maximal complementary subcategory $\Xi = \Theta$; intermediate choices could have Ξ corepresenting k-algebras for some subring $\mathbb{Z} \subseteq k \subseteq R$, for example.
(ii) If $\Theta = \Pi_{\geq 2}$, corepresenting Π-algebras (§1.6), the minimal complementary subcategory Ξ corepresents Whitehead rings (§1.9), and then $\Xi^{\mathrm{ab}} = \mathfrak{A}(\mathbb{N}_+)$ corepresents graded abelian groups. On the other hand, we could take Ξ^{ab} to be the image of the suspension functor $\Sigma : \Pi \to \Pi_{\geq 2}$, which would yield a larger complementary subcategory of Θ (though still not all of it).
(iii) If $\Theta = \mathbb{H}_p$, corepresenting unstable algebras over the the mod p Steenrod algebra $\mathcal{A}_p$ (§1.13), then the minimal complementary subcategory Ξ corepresents graded rings, with $\Xi^{\mathrm{ab}} = \mathfrak{A}(\mathbb{N}_+)$. However, it would be more natural to let Ξ corepresent graded-commutative $\mathbb{F}_p$-algebras, so Ξ^{ab} corepresents graded $\mathbb{F}_p$-modules, and Θ^{ab} corepresents unstable modules over $\mathcal{A}_p$. In fact, we could let Ξ corepresent unstable algebras over various subalgebras of $\mathcal{A}_p$.

These examples show that a given Θ may have more than one complementary subcategory. On the other hand, not every theory has a complementary subcategory (as in the case $\Theta = \mathfrak{G}$, corepresenting groups).

7.4. **Lemma.** *The forgetful functor induces an equivalence of categories* $(\Xi\text{-}\mathcal{M}dl)_{\mathrm{ab}} \to \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$.

7.5. Corepresenting abelianized Θ-models.

There are two equivalent ways of corepresenting the category $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ of abelian Θ-models (analogous to the two ways of defining the homology of a pair of pointed spaces) – in addition to that described in Lemma 2.10:

7.6. **Definition.** Let Θ be an FP-theory, Ξ' a sub-semi-category of Θ (§1.16), and $\mathcal{C}$ some pointed category, such as $\mathcal{S}et_*$. A Θ-model $X : \Theta \to \mathcal{C}$ in $\mathcal{C}$ is called a *relative* (Θ, Ξ')*-object* in $\mathcal{C}$ if $X|_{\Xi'} = 0$. The category of all such functors will be denoted by $(\Theta, \Xi')\text{-}\mathcal{C}$.

7.7. *Remark.* Of course, this definition makes sense only if we place some restrictions on Θ and Ξ' – in particular, we want to make sure that the morphisms in Ξ' do not include any identities, and we usually want Ξ' and Θ to have the same objects.

We shall be interested in the case when Ξ' is obtained from a complementary subcategory Ξ for some $\mathfrak{A}$-theory Θ by omitting all morphisms which are in Ξ^{ab} – so that the morphisms of Ξ' are exactly the cross-effects of Θ. We call this a *complementary* semi-category of Θ.

7.8. **Proposition.** *If* Θ *is an* $\mathfrak{A}$*-theory and* Ξ' *a complementary semi-category, the inclusion* $(\Theta, \Xi')\text{-}\mathcal{M}dl \hookrightarrow \Theta\text{-}\mathcal{M}dl$ *is naturally isomorphic to the inclusion* $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}} \hookrightarrow \Theta\text{-}\mathcal{M}dl$.

Proof. This follows from Corollary 2.4, if we note that $\Theta\text{-}\mathcal{M}dl_* \cong \Theta\text{-}\mathcal{M}dl$, because any $\mathfrak{A}$-theory is automatically pointed. □

7.9. **Definition.** Let Θ be any pointed small category and $\mathcal{A} \subseteq \Xi \subseteq \Theta$ subcategories of Θ, all three with the same objects, such that all isomorphisms and retractions of Ξ are already in $\mathcal{A}$. Then we can define the *relative quotient category* $(\Theta/\Xi)_{\mathcal{A}}$, again with the same objects, by setting all morphisms of Θ which come from Ξ, but are not in $\mathcal{A}$, equal to 0.

7.10. *Remark.* When Θ is a $\mathfrak{G}$-theory, Ξ is a complementary subcategory, and $\mathcal{A} = \Xi^{\mathrm{ab}}$, then the set of morphisms in Ξ which are not in $\mathcal{A}$ is generated by the cross-effects, so $(\Theta/\Xi)_{\Xi^{\mathrm{ab}}}$ is (equivalent to) the largest quotient of Θ in which the cross-effect maps vanish. This construction could have an unexpected effect if there are morphisms in Ξ^{ab} which factor through cross-effect maps; but this just means that abelianization may be more destructive than expected. With appropriate assumptions on Θ (see, e.g., §8.2 below), this will not happen; in any case, $(\Theta/\Xi)_{\Xi^{\mathrm{ab}}}$ corepresents the abelianization, if it exists, so it must be equivalent to Θ^{ab} of §2.8 when both are defined.

Applying this construction to an $\mathfrak{A}$-theory Θ and complementary subcategory Ξ, we obtain an 'exact sequence of categories under Ξ^{ab}':

(7.11)
$$\begin{tikzcd} & \Xi^{\mathrm{ab}} \arrow[dl, "j'"'] \arrow[d, "j"] \arrow[dr, "j''"] & \\ \Xi \arrow[r, "i"'] & \Theta \arrow[r, "q"'] & (\Theta/\Xi)_{\Xi^{\mathrm{ab}}} \end{tikzcd}$$

in which the functors j, j', j'', and i are all inclusions, and q is the quotient functor.

7.12. **Proposition.** *If* Θ *is an* $\mathfrak{A}$*-theory, the quotient functor* $q : \Theta \twoheadrightarrow (\Theta/\Xi)_{\Xi^{\mathrm{ab}}}$ *induces the inclusion* $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}} \hookrightarrow \Theta\text{-}\mathcal{M}dl$, *so in particular* $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ *is corepresented by* $(\Theta/\Xi)_{\Xi^{\mathrm{ab}}}$.

Proof. This follows again from Corollary 2.4. □

Clearly, i induces the forgetful functor $U : \Theta\text{-}\mathcal{M}dl \to \Xi\text{-}\mathcal{M}dl$. The inclusions of $\Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$ into $\Xi\text{-}\mathcal{M}dl$ (and thus into $\Theta\text{-}\mathcal{M}dl$, via i, as well as into $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$, via q) also induce appropriate forgetful functors $V' : \Xi\text{-}\mathcal{M}dl \to \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$, $V = V'U$, and $V'' = V|_{(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}}$ into the abelian category $\Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$ (Lemma 2.11).

8. Θ-models and the fiber of abelianization

We now consider the special case of the abelianization functor $\mathcal{A}b : \Theta\text{-}\mathcal{M}dl \to (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$: for this purpose, let Θ be an $\mathfrak{A}$-theory, equipped with a complementary subcategory Ξ, an underlying abelian category Ξ^{ab}, and a relative quotient category $(\Theta/\Xi)_{\Xi^{\mathrm{ab}}}$ corepresenting $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$, as in Section 7.

These yield a diagram of FP-sketchable categories:

$$\begin{tikzcd}
\Theta\text{-}\mathcal{M}dl \arrow[r, "U"'] \arrow[d, "\mathcal{A}b"'] & \Xi\text{-}\mathcal{M}dl \arrow[r, "V"'] \arrow[d, "\mathcal{A}b"'] & \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl \arrow[d, "="] \\
(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}} \arrow[r, "U'"'] & (\Xi\text{-}\mathcal{M}dl)_{\mathrm{ab}} \arrow[r, "V'"'] & \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl
\end{tikzcd} \tag{8.1}$$

in which the horizontal arrows are forgetful functors, V' is an equivalence of categories (by Lemma 7.4), the bottom row consists of abelian categories with the vertical arrows abelianization functors. Note that U' is just the restriction of U to the subcategory $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$, and similarly for V'.

8.2. **Assumption.** For any Θ-model X, any set of Ξ^{ab}-generators (see Lemma 1.20) for $VU(X/I(X))$ can serve as a set of Θ-generators for X.

8.3. *Remark.* This technical assumption is needed in order for there to be any chance of recovering X from X_{ab} – for example, to rule out the possibility of nontrivial *perfect* Θ-models (those with trivial abelianization). In practice, this is guaranteed by Proposition 8.12, and holds in the motivating examples (see Introduction).

We want to investigate *two* data – the abelianization X_{ab}, and the complementary structure induced by the inclusion $\Xi \hookrightarrow \Theta$. Note that the abelianization functor $\mathcal{A}b : \Theta\text{-}\mathcal{M}dl \to (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ is not induced by a map of theories.

8.4**. The abelianization functor**. If Θ is an FP $\mathfrak{A}$-theory with complementary subcategory Ξ (§7.1), and X is a Θ-model, then a Θ-ideal in X (§1.14) is a sub-Θ-model $I \subseteq X$ such that for any n-fold cross-effect map $\varphi : \prod_{i=1}^{n} \kappa_i \to \vartheta$ in Ξ and any $1 \le j \le n$, the composite

$$\prod_{i \neq j}^{n} X(\kappa_i) \times I(\kappa_j) \xrightarrow{\prod_{i\neq j} \mathrm{Id}_X \times \iota} \prod_{i=1}^{n} X(\kappa_i) \xrightarrow{X(\varphi)} X(\vartheta) \tag{8.5}$$

factors through $\iota(\vartheta) : I(\vartheta) \hookrightarrow X(\vartheta)$.

8.6. **Definition.** If the composite in (8.5) is zero for any φ, we say that the ideal I is *central.*

8.7. **Definition.** If Θ is an FP $\mathfrak{A}$-theory as above, a *central extension* of Θ-models is a sequence

$$M \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} X$$

such that the ideal $\mathrm{Im}\,(i) = \mathrm{Ker}\,(p)$ is central. Two such extensions

$$\begin{array}{ccccc} M & \rightarrowtail & E & \twoheadrightarrow & X \\ \| & & \downarrow & & \| \\ M & \rightarrowtail & E' & \twoheadrightarrow & X \end{array}$$

are *equivalent* if the dotted arrow can be filled in to make the diagram commute.

8.8. **Theorem.** *Let Θ be an FP $\mathfrak{A}$-theory with complementary subcategory Ξ as above. For any Θ-model X and abelian Θ-model M, the equivalence classes of central extensions of M by X are in natural bijective correspondence with $H^1(X;M)$ (§5.4).*

Proof. I. Assume given an extension $0 \to M \xrightarrow{i} E \xrightarrow{p} X \to 0$ as above. First, construct a free simplicial resolution $Q_\bullet \to X$ (§4.3) by setting $Q_0 := FU(X)$ and $Q_1 := F\bar{T} \amalg L_1 Q_\bullet$, with $\bar{T} \in \Theta^\delta$-$\mathcal{M}dl$ in degree $\vartheta \in \Theta^\delta$ given by:

$$\bar{T}(\vartheta) \ := \ \{\langle f_*\langle x\rangle \cdot \langle f_* x\rangle^{-1}\rangle \mid f : \vartheta' \to \vartheta \text{ in } \Theta,\ x \in X(\vartheta')\} \ ,$$

where $\langle w\rangle \in UX(\vartheta)$ corresponds to $w \in X(\vartheta)$. Set $d_0|_{F\bar{T}} := \bar{d}_0$, with $\bar{d}_0 : F\bar{T} \to Q_0$ is defined:

$$\langle f_*\langle x\rangle \cdot \langle f_* x\rangle^{-1}\rangle \mapsto f_*\langle x\rangle \cdot \langle f_* x\rangle^{-1} \in FU(X) \ .$$

$d_1|_{F\bar{T}} := 0$; and the rest determined by the simplicial identities.

Since p is surjective, we can choose a section of graded sets $\sigma : UX \to UE$ which respects products, such that $\sigma(0) = 0$. Now define $\varphi : Q_1 \to M$ by $\varphi(\langle f_*\langle x\rangle \cdot \langle f_* x\rangle^{-1}\rangle) := f_*\sigma(x)\cdot\sigma(f_* x)^{-1}$. Note that $p(f_*\sigma(x)\cdot\sigma(f_* x)^{-1}) = f_* p(\sigma(x))\cdot p(\sigma(f_* x))^{-1} = 0$, so this is well-defined, and since φ factors through $\mathrm{Ker}\,\{\varepsilon : Q_0 \to X\}$, this is in fact a cocycle in $\mathrm{Hom}(Q_\bullet, M)$, so defines a class in $H^1(X;M)$. Moreover, φ is a coboundary if and only if it extends to $Q_\bullet$ – that is, if and only if σ is a map of Θ-models (so $0 \to M \xrightarrow{i} E \xrightarrow{p} X \to 0$ splits as a semi-direct product).

Note that what we have shown is part of what is needed to prove that, as for simplicial sets (cf. [Ma, Thm. 21.13]), fibrations with base BX and fibre BM are classified by $[BX, K(M,1)]$.

II. Conversely, given a cohomology class $\mu \in H^1(BX;M) \cong [BX, K(M,1)]$, let $Y_\bullet$ denote the homotopy fiber of μ (that is, the pullback of $\hat{X} \xrightarrow{m} K(M,1) \leftarrow *$, where m is a fibration representing the homotopy class μ). As in [Q1, I,§3], there is a homotopy fibration sequence

$$0 = \Omega BX \to BM = \Omega K(M,1) \to Y \to BX \xrightarrow{m} K(M,1)$$

and thus a long exact sequence in $\hat{\pi}_*$:

$$0 \to M = \hat{\pi}_0 BM \to E := \hat{\pi}_0 Y \to X = \hat{\pi}_0 BX \to 0 = \hat{\pi}_0 K(M,1)$$

(cf. [Q1, I,§3, Prop. 4]), which yields the required central extension. □

8.9. **Definition.** For any Θ-model X, the *abelianizing ideal* $J(X) \lhd X$ is generated by the image under X of all cross-effects in Ξ (compare [BS, §5.1]).

8.10. **Proposition.** *If Θ is an $\mathfrak{A}$-theory, the abelianization functor on the category of Θ-models (§2.7) is naturally isomorphic to $X \mapsto X/J(X)$.*

8.11. **Fibers of abelianization**.

Given $W \in (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$, we would like to describe all Θ-models X equipped with a map $\rho : X \to W$ which is (up to isomorphism) the augmentation of the abelianization functor $\mathcal{A}b : \Theta\text{-}\mathcal{M}dl \to (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$. As noted, $X_{\mathrm{ab}} \cong X/J(X)$, where $J(X)$ is generated by the image of the cross-effect maps of Θ. On the other hand, if $I(X) \supset J(X)$ is the ideal in X generated by all morphisms in Θ except for those of Ξ^{ab}, then $VU(X/I(X)) \in \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$ is defined to be the (K-graded) *module of indecomposables* of X.

8.12. **Proposition.** *Assumption 8.2 holds, in particular, if the semi-category FP-sketch* $\Theta \setminus \Xi^{\mathrm{ab}}$ *(§§1.16,7.7) is a* directed preorder *(in the sense of* [Mc, IX, §1]*).*

8.13. **Lemma.** *For any* $X \in \Theta\text{-}\mathcal{M}dl$, *the augmentation* $\varepsilon : F_\Theta(X_{\mathrm{ab}}) \twoheadrightarrow X_{\mathrm{ab}}$ *factors (non-canonically) as* $F_\Theta(X_{\mathrm{ab}}) \xrightarrow{f} X \xrightarrow{\rho} X_{\mathrm{ab}}$.

(The map f is prescribed by choosing Θ-generators for X corresponding to the elements of VUX_{ab}.)

This implies that the fiber of $\mathcal{A}b : \Theta\text{-}\mathcal{M}dl \to (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ over W can be described in terms of appropriate quotients of $F_\Theta W$. In order to analyze the possible quotients, let Q denote $\mathrm{Ker}\,(\varepsilon_{\mathrm{ab}})$, which fits into a commuting diagram in $\Theta\text{-}\mathcal{M}dl$ with exact rows and columns (the bottom row is actually in $(\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$):

(8.14)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & J(F_\Theta W) \arrow[r, "="] \arrow[d] & J(F_\Theta W) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K := \mathrm{Ker}\,(\varepsilon) \arrow[r] \arrow[d] & F_\Theta W \arrow[r, "\varepsilon"] \arrow[d, "\rho_W"'] & W \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & (F_\Theta W)_{\mathrm{ab}} \arrow[r, "\varepsilon_{\mathrm{ab}}"] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and more generally for any Θ-model X with abelianization W we have

(8.15)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J' \arrow[r, "="'] \arrow[d] & J(F_\Theta W) \arrow[r] \arrow[d] & J(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L := \mathrm{Ker}\,(f) \arrow[r] \arrow[d] & F_\Theta W \arrow[r, "\varepsilon"] \arrow[d, "\rho_W"'] & X \arrow[r] \arrow[d, "\rho_X"'] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & (F_\Theta W)_{\mathrm{ab}} \arrow[r, "\varepsilon_{\mathrm{ab}}"] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

8.16. **Summary.** Let $\langle \mathfrak{X}, \preceq \rangle$ denote the partially ordered set of Θ-models X with abelianization W, where $\rho : X \to W$ precedes $\rho' : X' \to W$ if ρ factors through ρ'.

Then $X = W$ is terminal in $\mathfrak{X}$, so that $K := \mathrm{Ker}\,(\varepsilon)$ is initial among the possible values for L (see (8.14)), and the set of maximal objects in $\mathfrak{X}$ corresponds to minimal sub-Θ-ideals $L' \lhd K$ surjecting onto Q.

All other $X \in \mathfrak{X}$ are obtained by adding subideals I' of $J(F_\Theta W)$ to such minimal L', so that $X \cong F_\Theta W/(I' + L')$.

This correspondence is not one-to-one, since any automorphism of $F_\Theta W$ which takes K to itself induces an automorphism between the corresponding quotients X. But we shall not pursue this point any further.

9. Graded varieties

If the theory Θ is graded, the procedure described in the previous sections can be carried out by induction on an algebraic version of the Postnikov tower. The successive stages of the tower yield central extensions, which can be classified cohomologically.

9.1. **Notation.** If Θ is a positively graded theory, we denote by Θ_n the full subcategory of objects in degree n, and by $\Theta_{\leq n}$ the subcategory of objects in degree $\leq n$, with $\mathrm{tr}_n : \Theta\text{-}\mathcal{M}dl \to \Theta\text{-}\mathcal{M}dl_{\leq n}$ the truncation functor (induced by the inclusion $\Theta_{\leq n} \hookrightarrow \Theta$). Its right adjoint $\iota_n : \Theta\text{-}\mathcal{M}dl_{\leq n} \to \Theta\text{-}\mathcal{M}dl$ is itself an embedding of categories.

9.2. **Lemma.** *The truncation functor* $\mathrm{tr}_n : \Theta\text{-}\mathcal{M}dl \to \Theta\text{-}\mathcal{M}dl_{\leq n}$ *has a left adjoint* $P_n : \Theta\text{-}\mathcal{M}dl_{\leq n} \hookrightarrow \Theta\text{-}\mathcal{M}dl$.

Proof. Given X in $\Theta\text{-}\mathcal{M}dl_{\leq n}$, set $Q_k := (\phi_\Theta\,\mathrm{tr}_n)^{k+1}X$ and $P_nX \cong \pi_0 Q_\bullet$, where ϕ_Θ is as in Lemma 1.20. □

9.3. **Assumption.** We now assume that the $\mathfrak{A}$-theory Θ (and thus its complementary subcategory Ξ and Ξ^{ab}, too) have a positive grading on the set of objects, and:

(a) there are no degree-decreasing morphisms in Θ;
(b) all degree-preserving maps in Θ (including the abelian group structure maps) are included in the subcategory Ξ^{ab}.

9.4. *Remark.* These assumptions imply, in particular, that all cross-effect maps are strictly degree-increasing, and that $\Theta \setminus \Xi^{\mathrm{ab}}$ is indeed a directed preorder, so Assumption 8.2 is satisfied (by Proposition 8.12).

An example to keep in mind is the category Θ corepresenting graded algebras over a ground ring k. In this case the complementary subcategory Ξ corepresents graded rings, $\Xi^{\mathrm{ab}} \cong \mathfrak{A}(\mathbb{N})$ corepresents graded abelian groups, and Θ^{ab} corepresents graded k-modules.

Write $X\langle n\rangle$ for the $(n-1)$-connected cover of a Θ-model X, so that we have a short exact sequence in Ξ^{ab}:

$$0 \to X\langle n+1\rangle \to X \to \iota_n\,\mathrm{tr}_n\,X \to 0 \tag{9.5}$$

for any $X \in \Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$.

9.6. **Lemma.** *Given* $Y \in \Theta\text{-}\mathcal{M}dl_{\leq n}$ *and* $W \in (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ *such that* $\mathrm{tr}_n W = Y_{\mathrm{ab}}$, *there is a unique* Θ*-model* $\sigma_{n+1}Y$ *equipped with a map* $s : \sigma_{n+1}Y \to \iota_nY$ *such that* $\mathrm{tr}_n\,s$ *is the identity, and* $(\sigma_{n+1}Y)_i \cong W_i$ *for* $i > n$.

Proof. The abelianization map $r : Y \to \mathrm{tr}_n\,W$ determines a unique map $\hat{r} : P_nY \to W$, and $K := \mathrm{Ker}\,(\hat{r})$ is a $\Theta\text{-}\mathcal{M}dl$-ideal in P_nY. so $K\langle n+1\rangle$ is, too, by Assumption 9.3(b). Set $\sigma_{n+1}Y := P_nY/K\langle n+1\rangle$. □

9.7. **The inductive procedure**.

In the context of Diagram (8.1), assume given $W \in (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ and $\rho' : X' \to W' = UW$ in $\Xi\text{-}\mathcal{M}dl$; we want $X \in \Theta\text{-}\mathcal{M}dl$ such that $W \cong X_{\mathrm{ab}}$ and $\rho' = U(\rho : X \to X_{\mathrm{ab}})$. Consider the short exact sequence

$$0 \to M' \xrightarrow{i'} X' \xrightarrow{\rho'} W' \to 0 \tag{9.8}$$

by Lemma 9.6, we may assume that at the n-th stage we have already determined $\sigma_{n+1}X \in \Theta\text{-}\mathcal{M}dl$, so we only need to attach M'_{n+1} to it (in dimension $n+1$) in order to obtain $\mathrm{tr}_{n+1} X$ (and thus $\sigma_{n+2}X$) as required.

9.9. **Proposition.** *If* $\bar{\sigma}_{n+1}X' := \mathrm{tr}_{n+1}\, \sigma_{n+1}X'$, *then*

$$H^1_{\Xi\text{-}\mathcal{M}dl}(\bar{\sigma}_{n+1}X', M'_{n+1}) \cong H^1_{\Xi\text{-}\mathcal{M}dl}(\mathrm{tr}_n X', M'_{n+1}) \oplus H^1_{\mathcal{A}_{\leq n+1}\text{-}\mathcal{A}lg}(W''_{n+1}, M''_{n+1}).$$

Proof. This follows from the fact that in $\Xi\text{-}\mathcal{M}dl$

$$\bar{\sigma}_{n+1}X' \cong \mathrm{tr}_n X' \times W'_{n+1}, \tag{9.10}$$

since $K = \mathrm{Ker}\,(\hat{r})$ contains $I(X')$ (for $Y = X'$ in the proof of Lemma 9.6). By Assumption 9.3(b) the forgetful functor from $\Xi\text{-}\mathcal{M}dl$ to $\Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$ is an equivalence of categories when restricted to any one degree, so there are splittings of H^1 as indicated.

Note that there are natural maps

$$H^1_{\Xi\text{-}\mathcal{M}dl}(\mathrm{tr}_n X', M'_{n+1}) \xrightarrow{s^*} H^1_{\Xi\text{-}\mathcal{M}dl}(\bar{\sigma}_{n+1}X', M'_{n+1}) \xrightarrow{V_*} H^1_{A_n\text{-}\mathcal{A}lg}(W''_{n+1}, M''_{n+1}),$$

where V_* is induced by the forgetful functor; we know s^* is one-to-one by (9.10), $V_* \circ s^* = 0$ by construction, and $\mathrm{Ker}\,(V_*) \supseteq \mathrm{Im}\,(s^*)$ for the same reason. $\square$

By Theorem 8.8 there is a class

$$\lambda'' \in H^1_{\mathcal{A}_n}(W''_{n+1}, M''_{n+1}) \cong H^1_{\mathcal{A}}(\bar{\sigma}_{n+1}X'', M''_{n+1}),$$

classifying the extension

$$0 \to M''_{n+1} \to \mathrm{tr}_{n+1} X'' \to \bar{\sigma}_{n+1}X'' \to 0 \tag{9.11}$$

in $\Xi^{\mathrm{ab}}\text{-}\mathcal{M}dl$ (where we have denoted the $\mathcal{A}$-algebra $\Sigma^{n+1}M''_{n+1}$, which has M''_{n+1} in degree $n+1$, and 0 elsewhere, simply by M''_{n+1}). Similarly, we have $\lambda' \in H^1_{\Xi\text{-}\mathcal{M}dl}(\bar{\sigma}_{n+1}X', M'_{n+1})$ classifying

$$0 \to M'_{n+1} \to \mathrm{tr}_{n+1} X' \to \bar{\sigma}_{n+1}X' \to 0 \tag{9.12}$$

in $\Xi\text{-}\mathcal{M}dl$.

We may summarize our results so far in

9.13. **Theorem.** *The obstruction to extending* $\bar{\sigma}_{n+1}X$ *to* $\mathrm{tr}_{n+1} X$ *(and thus to* $\bar{\sigma}_{n+1}X$*) lies in* $H^1_{\Theta/\Xi}(\sigma_{n+1}X, M'_{n+1})$ *(cf. §5.9); the difference obstructions for the various extensions lie in* $H^0_{\Theta/\Xi}(\sigma_{n+1}X, M'_{n+1})$.

9.14. **The fiber of a single functor**.

Note that in the graded case we can also consider the fiber of the forgetful functor $U : \Theta\text{-}\mathcal{M}dl \to \Xi\text{-}\mathcal{M}dl$ alone, without assuming that $W \in (\Theta\text{-}\mathcal{M}dl)_{\mathrm{ab}}$ is given. For this purpose we need the following fact:

9.15. **Proposition.** *The class* $\kappa \in H^1_{\Xi\text{-}\mathcal{M}dl}(\mathrm{tr}_n X', X'_{n+1})$ *comes from a* unique *class* $\tilde{\kappa}$ *in* $H^1_{\Xi\text{-}\mathcal{M}dl}(\mathrm{tr}_n X', M'_{n+1})$.

Proof. The short exact sequence of $\operatorname{tr}_n X'$-modules

$$0 \to M'_{n+1} \xrightarrow{i'} X'_{n+1} \xrightarrow{\rho'} W'_{n+1} \to 0$$

induces a fibration sequence

$$B_{\Xi\text{-}\mathcal{M}dl} W'_{n+1} \to K_{\Xi\text{-}\mathcal{M}dl}(M'_{n+1}, 1) \to K_{\Xi\text{-}\mathcal{M}dl}(X'_{n+1}, 1) \to K_{\Xi\text{-}\mathcal{M}dl}(W'_{n+1}, 1)$$

which in turn yields a long exact sequence

$$0 \to H^1_{\Xi\text{-}\mathcal{M}dl}(\operatorname{tr}_n X', M'_{n+1}) \to H^1_{\Xi\text{-}\mathcal{M}dl}(\operatorname{tr}_n X', X'_{n+1}) \to H^1_{\Xi\text{-}\mathcal{M}dl}(\operatorname{tr}_n X', W'_{n+1}) \to \dots$$

in cohomology; but

$$H^0_{\Xi}(\operatorname{tr}_n X', \mathcal{W}'_{n+1}) = [B_{\Xi\text{-}\mathcal{M}dl} \operatorname{tr}_n X, B_{\Xi\text{-}\mathcal{M}dl} W'_{n+1}] = \operatorname{Hom}_{\Xi}(\operatorname{tr}_n X, W'_{n+1}) = 0.$$

Now note that the extension $0 \to W'_{n+1} \to \bar{\sigma}_{n+1} X' \to \operatorname{tr}_n X \to 0$ is trivial. □

Thus for the inductive stage of the fiber of a single functor, in addition to κ (which reduces to $\tilde{\kappa}$), we need only the class λ'', classifying the extension (9.11). In the previous approach (Theorem 9.13), these were replaced by the single class λ', for the extension (9.12).

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