EMIL POST

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1 BIOGRAPHY

Emil Leon Post was born on February 11 1897 into an Orthodox Jewish family in Augustów, a town at that time within the Russian empire, but after 1918 in the province of Białystok in Eastern Poland. In 1897 his father Arnold emigrated to join his brother in America. Seven years later, in May 1904, with the success of the family clothing and fur business in New York, his wife Pearl, together with Emil and his sisters Anna and Ethel, joined him. The family lived in a comfortable home in Harlem.

Figure 1. Emil Post, June 1924

As a child, Post was particularly interested in astronomy, but an accident at the age of twelve foreclosed that choice of career. As he reached for a lost ball under a parked car, a second car crashed into it, and he lost his left arm below the shoulder. As a high school senior, Post wrote to several observatories inquiring whether his handicap would prevent him pursuing a career as an astronomer. The responses he received, though not uniformly negative, were sufficient to discourage him from following his childhood ambition; instead, he turned towards mathematics.

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Post attended Townsend Harris High School, a free secondary school for gifted students on the campus of City College, and received a B.S. in mathematics from City College in 1917 (where he benefited from a college education free of tuition fees). While still an undergraduate, he had already done a good deal of the work for a paper on generalized differentiation, presented to the October 1923 meeting of the American Mathematical Society, but only published in full seven years later [Post, 1930]. Post was a graduate student at Columbia University from 1917 to 1920, and it was at Columbia that his interest in modern mathematical logic was aroused by a seminar on the recently published *Principia Mathematica* [Whitehead and Russell, 1910-13] conducted by Cassius J. Keyser. His doctoral thesis, written under the supervision of Keyser, is a systematic study of the propositional calculus of *Principia*, including a proof of completeness.

Post spent the academic year 1920-21 as a Procter Fellow at Princeton University, and it was during this postdoctoral year that he discovered results anticipating the later incompleteness theorems of Kurt Gödel and the undecidability results of Church and Turing. The excitement caused by these startling discoveries precipitated his first attack of manic-depressive illness, a condition that plagued him for the rest of his life. He recovered sufficiently from this first attack to take up an instructorship at Cornell University, but a second attack led to a withdrawal from university teaching, and during the 1920s, he supported himself by teaching at George Washington High School in New York.

Under the care of a general practitioner, Dr. Levy, Post developed a routine that was designed to avoid undue excitement leading to manic attacks. His regimen involved strictly restricting the time spent on his research to three hours a day, from 4 to 5 p.m. and then from 7 to 9 p.m.

In 1932, Post was appointed to the faculty of City College of New York. He left after only one month, but returned in 1935 for the rest of his career. In spite of restrictions on his research time due to his treatment regimen and a teaching load of sixteen hours a week, Post was able to publish some of his most remarkable and influential papers during this period. His marriage to Gertrude Singer in 1929 undoubtedly introduced a measure of stability into his life, and she assisted him both by typing his papers and letters, and taking care of day to day financial affairs. Their daughter Phyllis Goodman remarks that: "My mother ... was the buffer in daily life that permitted my father to devote his attention to mathematics (as well as to his varied interests in contemporary world affairs). Would he have accomplished so much without her? I, for one, don't think so" [Davis, 1994, p. xii].

Post was a remarkably successful and popular teacher at City College; his unusual pedagogical methods are described below in §8. He continued to struggle with manic-depressive illness, and in 1954, he succumbed once more. He died of a heart attack on 21 April 1954 after electro-shock treatment at an upstate New York mental hospital.

2 INFLUENCES ON POST

To a remarkable extent, Post as a logician was a homegrown American original. Most of the other outstanding logicians of the 1920s and 1930s, such as Gödel, Herbrand, Bernays, Turing and Church, were influenced more or less directly by Hilbert's formalist program, and the problems that grew out of it. Post, on the other hand, after the initial stimulus provided by *Principia Mathematica*, created his own research project and problems more or less single handed, at a time when logic and the foundations of mathematics were not considered respectable topics for mathematical work in the United States.

An important early influence on his ideas about logic was the philosopher-logician C.I. Lewis. Lewis's monograph *A Survey of Symbolic Logic* [Lewis, 1918] is an expository work largely devoted to bridging the gap between the tradition of algebraic logic of Boole and Schröder and *Principia Mathematica*, but it has a remarkable sixth chapter that had a strong effect on Post's early thinking. Lewis contrasts the logistic method of Whitehead and Russell with what he calls a "heterodox" view of the nature of mathematics and logistic.

Lewis points out that the foundational portion of *Principia Mathematica* falls short of being a completely formal system. Rather, the primitive propositions of that work are presented in terms of the intuitive meanings associated with the notation. Thus, for example, the formation rules of the language are presented by phrases such as "If p is an undetermined elementary proposition, "not-p" is an elementary propositional function," while the rule of detachment (*modus ponens*) is stated as follows: "Anything implied by a true elementary proposition is true" [Whitehead and Russell, 1910-13, Vol. 1, pp. 92, 94]. Whitehead and Russell fail to make the basic distinction between axioms and rules of inference, since they are lumped together under the heading of "Primitive Propositions."

Lewis presents his heterodox view of the logistic method in the following striking definition:

> *A mathematical system is any set of strings of recognizable marks in which some of the strings are taken initially and the remainder derived from these by operations performed according to rules which are independent of any meaning assigned to the marks* [Lewis, 1918, p. 355].

Lewis illustrates his definition by presenting the propositional calculus of *Principia Mathematica* in a strictly formalist style. He uses the purposely meaningless words "quid" and "quod" to refer to formulas and propositional connectives. With this terminology, he restates the formation rule above as follows: "The combination of any quid preceded immediately by the mark $\sim$ may be treated as a quid." The rules of substitution and *modus ponens* he states in this way:

> (7) In any string in the initial set, or in any string added to the list according to rule, any quid whatever may be substituted for p or q or r, or for any quid consisting of only one mark. When a quid is substituted

> for any mark in a string, the same quid must also be substituted for that same mark wherever it appears in the string.
>
> (8) The string resulting from the substitution of a quid consisting of more than one mark for a quid of one mark, according to (7), may be added to the list of strings.
>
> (9) In any string added to the list, according to (8), if that portion of the string which precedes any mark ⊃ is identical with some other string in the set, preceded by (, then the portion of that string which follows the mark ⊃ referred to may be separately repeated, with the omission of the final mark), and added to the set [Lewis, 1918, p. 357].

Lewis's strictly formal statement of the rule of detachment makes a striking contrast with the informal and inexact version of Whitehead and Russell, which invokes the extraneous notion of truth in setting down the rule.

Post refers several times to the influence of Lewis's formalistic view of logistic systems on his early work [Post, 1921a, p. 165], [Post, 1994, pp. 23,377], and his research at Princeton during his postdoctoral year can be seen as an attempt to work out Lewis's formalistic ideas in the context of the full system of Whitehead and Russell, and to use them in obtaining general results about formal systems.

Post also acknowledged the influence of his thesis advisor, Cassius J. Keyser (1862-1947), on his logical work. This may seem somewhat surprising, in view of the fact that Keyser, although he took part in a debate with Bertrand Russell on the axiom of infinity [Keyser, 1904][Russell, 1904], was best known for his numerous books of popular essays on mathematics and mathematical philosophy. Nevertheless, Post thanks Keyser warmly in the introduction to [Post, 1921a], and his monograph [Post, 1941] is dedicated to Keyser, "in one of whose pedagogical devices the author belatedly recognizes the true source of his truth-table method."

3 THE DOCTORAL DISSERTATION

Post's first published paper in logic [Post, 1921a] is a shortened version of his doctoral dissertation; according to a letter from Post to Hermann Weyl of 1941, the paper was accepted only on condition that its original length be cut by one-third [Davis, 1994, p. xviii].

This article is outstanding for several reasons. Its main mathematical result is the first published proof of completeness and decidability of the propositional fragment of *Principia Mathematica*. (An earlier proof was given by Paul Bernays in his Habilitationsschrift of 1918 at Göttingen, published in abbreviated form in [Bernays, 1926].) The method of the paper is a conscious departure from the methods of Whitehead and Russell, and Post gives a clear statement of the modern metalogical attitude to formal systems. Furthermore, a great deal of Post's most significant later work in logic is present in embryo in this paper.

In the introduction to his paper, Post points out the restricted viewpoint adopted by the authors of *Principia Mathematica*. The aim of Whitehead and Russell is to develop the logical foundations of mathematics in a fixed axiomatic system (a form of the theory of types). The ideal of the work is to develop mathematics within a logical formalism that is viewed as a universal language. Hence, Whitehead and Russell eschew as far as possible any considerations about their system that go beyond this restricted purpose. Post points out that in this way, they "gave up the generality of outlook which characterized symbolic logic," and goes on to say:

> It is with the recovery of this generality that the first portion of our paper deals. We here wish to emphasize that the theorems of this paper are *about* the logic of propositions but are *not included* therein. More particularly, whereas the propositions of 'Principia' are *particular* assertions introduced for their interest and usefulness in later portions of the work, those of the present paper are about the set of *all* such assertions [Post, 1921a, pp. 163-4].

Post concludes his introduction with a clear explicit statement of his metalogical viewpoint, saying: "We have consistently regarded the system of 'Principia' and the generalizations thereof as purely *formal developments*, and so have used whatever instruments of logic or mathematics we found useful for a study of these developments," referring in a footnote to the chapter of Lewis's book discussed above. Post had thus arrived at a logical method similar to that of the Hilbert school, though in his case, there is no explicit restriction to finitary methods (however, the methods he used were in fact mostly constructive).

The main part of the paper begins with a precise formulation of the propositional calculus of *Principia Mathematica*, inspired by the formulation of Lewis described above. Post then immediately introduces a truth-table development. Denoting the truth-value of a proposition by $+$ if it is true and by $-$ if it is false, he gives the now familiar truth-tables for the primitive connectives

p	$\sim p$
$+$	$-$
$-$	$+$

$p,\ q$	$p \vee q$
$++$	$+$
$+-$	$+$
$-+$	$+$
$--$	$-$

and goes on to provide a clear explanation of the truth-table method. He uses the adjective "positive" for tautologies, "negative" for contradictions, and "mixed" for contingent formulas (the terminology of "tautology" and "contradiction" is derived from the independent development of the truth-table method in [Wittgenstein, 1921],[Wittgenstein, 1922]). In a footnote, he remarks that the truth-tables for the primitive connectives are described by Whitehead and Russell [Whitehead and Russell, 1910-13, Vol. 1, pp. 8,115], but not the general notion of truth-table, and points out that the idea is implicit in the expansion technique used by the algebraic logicians Boole, Jevons and Venn [Lewis, 1918, p. 74, p. 175f]. As we have seen above, he later cited his thesis advisor Cassius J. Keyser as a more immediate source of the method.

Post now states his "Fundamental Theorem," the result that a propositional function is assertible in the system if and only if it is positive. This provides both a completeness theorem as well as a decision procedure for the system (Post emphasizes the decision procedure aspect rather than completeness). The proof proceeds by showing that any formula in the system is provably equivalent to one in full disjunctive normal form, that is to say, any function $f(p_1, \ldots, p_n)$ can be proved equivalent to a disjunction of conjunctions of the form $Q_1 \wedge \cdots \wedge Q_n$, where each Q_i is either p_i or $\sim p_i$. Post observes that if $f(p_1, \ldots, p_n)$ is a positive function, then the equivalent full disjunctive normal form is easily seen to be provable (since it must contain all possible conjunctions of the form described above), and hence f itself is provable.

Post remarks that while the conversion to full disjunctive normal form and the corresponding decision procedure were both implicit in the earlier literature of algebraic logic, the significance of his theorem is quite different. Although Schröder had given a version of the truth-table decision procedure in his treatise on the algebra of logic, formal and informal logic are bound together in his development in such a way as to prevent the system as a whole from being completely determined. The paper continues by considering three different generalizations of his results, two of which were the basis of a good deal of his later research.

The first generalization consists of generalizing the truth-table method to an arbitrary finite set of truth-functions as primitive connectives. The basis of $\sim$ and $\vee$ chosen by Whitehead and Russell is functionally complete, since all truth-functions can be expressed in this primitive vocabulary (a fact proved explicitly by Post). However, not all choices of primitive connectives result in a functionally complete system; the basis of $\sim$ and $\equiv$ is an example. Post announces but does not prove his complete classification of all possible classes of truth-functions generated by a finite set of connectives. We discuss this result below in §5.

Post next considers "Generalization by Postulation," a much farther-reaching idea. He envisages sets of postulates for generalized propositional logics, with a finite set of primitive connectives, and a finite set of axioms and rules, including the rule of unrestricted substitution, and goes on to consider generalized versions of completeness and consistency for such systems. The usual definition of consistency does not work here, since such systems may not have a recognizable negation connective; similar remarks apply to the notion of completeness. Post replaces these with the following broader notions. He defines a postulate system to be *consistent* if not all formulas are assertible, and *closed* if the addition of any unprovable formula as an axiom results in an inconsistent system. These notions are now usually referred to as "Post-consistency" and "Post-completeness." Earlier in the paper, Post had shown the propositional calculus of *Principia Mathematica* to be consistent and complete in these senses, as corollaries to his Fundamental Theorem. Using his abstract definitions of consistency and completeness, he proves a few general results about closed (Post-complete) systems, remarking of one of them that it "begins to approximate to the truth-table method." This remark looks forward to the work of his postdoctoral year, in which he tried to obtain general decidability results using the postulational generalization.

The last section of the paper is given over to the final generalization, many-valued logic. Post replaces his earlier truth-values + and − by m distinct "truth-values" $t_1, t_2, \ldots, t_m$, where m is any positive integer. He generalizes classical negation by replacing with a cyclic permutation of the m truth-values; the generalization of classical disjunction has the higher of its two inputs (assuming the ordering $t_1 \geq t_2 \geq \cdots \geq t_m$). Thus the truth-tables for the generalized connectives are as follows:

p	$\sim_m p$
t_1	t_2
t_2	t_3
$\ldots$	$\ldots$
t_m	t_1

$p,\ q$	$p \vee_m q$
$t_1\ t_1$	t_1
$\ldots$	$\ldots$
$t_{i_1}\ t_{j_1}$	t_{i_1}
$\ldots$	$\ldots$
$t_{i_2}\ t_{j_2}$	t_{j_2}
$t_m\ t_m$	t_m

where $i_1 \leq j_1$ and $i_2 \geq j_2$.

Generalizing his earlier results for two-valued logic, Post shows that his m-valued connectives are functionally complete. He then goes on to consider the m-valued logical systems that result by defining the asserted functions to be those that always take values $t_1, t_2, \ldots, t_\mu$, where $0 < \mu < m$. He provides a brief sketch of the method used to give a complete set of postulates for the resulting systems. —) Jan Łukasiewicz had proposed the idea of a three-valued logic slightly earlier than Post [Łukasiewicz, 1920]. Łukasiewicz was inspired by the idea of a non-Aristotelian logic, in which the third truth-value would represent the notion of indeterminacy, associated with propositions about future events. Post seems mostly interested in his generalization as pure mathematics, remarking that "whereas the highest dimensioned intuitional point space is three, the highest dimensioned intuitional propositional space is two." He goes on to say, though, that just as higher dimensional spaces in geometry can be interpreted by using some other element than point, so the higher dimensioned logical spaces can be interpreted by using elements other than propositions. This idea is carried out in the last section of the paper by interpreting the propositions of m-valued logic as vectors $(p_1, p_2, \ldots, p_{m-1})$ of two-valued propositions, with the proviso that if one proposition is true, then all those that follow are true.

Post himself did not continue his work in many-valued logic, but later mathematicians such as D.L. Webb, Alan Rose and A.L. Foster continued the development of these systems. The algebraic version of Post's systems, the theory of "Post algebras," is investigated in papers such as [Rosenbloom, 1942], [Epstein, 1960], [Traczyk, 1963], [Traczyk, 1964].

4 THE POSTDOCTORAL YEAR

During the academic year 1920-21, Post held a Procter Fellowship at Princeton University (Alan Turing also held a Procter Fellowship 1937-38). It was during this year that he made his most dramatic discoveries in logic, anticipating by a decade or more the work of Gödel, Church, Turing and others on incompleteness and undecidability. Unfortunately, for reasons we discuss below, Post failed to publish his ideas until after other authors had caught up with his groundbreaking work.

He started from the postulational generalization of [Post, 1921a], hoping to obtain general results about the decidability of postulate systems that are abstract versions of the formalism of classical propositional logic. The simplicity of such systems encouraged him in the belief that such results were obtainable, and in addition, he had several positive results to his credit. He had solved the decision problem in the important special case of two-valued propositional logic, and in addition had provided a complete classification of all possible subsystems of this logic based on two-valued connectives. Finally, he had proved the decidability of all abstract propositional logics where all of the connectives were unary (one-place operators), a result published in the abstract [Post, 1921b].

Post's hope was that progress on the decision problem for such generalized propositional calculi (which he called the "general finiteness problem") was possible because of their simple and transparent mathematical structure. His ideas appear nowadays wildly optimistic, but we have the benefit of hindsight; Jacques Herbrand, in a paper surveying the work of the Hilbert school [Herbrand, 1930], expressed a similarly optimistic view towards the decision problem for first-order logic (known as the *Entscheidungsproblem*).

Post published nothing in logic between [Post, 1921b] and [Post, 1936]. Nevertheless, we know of his work of 1920-21 because of the retrospective survey he wrote in 1941 [Post, 1965], on which this section is based. This article falls into two parts. In the first part, Post demonstrates a series of reductions through which the decision problem, or "general finiteness problem" is reduced to a system of an extremely simple type, which he calls a system in "normal form." In the second part, the application of a diagonal argument leads him to the conclusion that the decision problem for systems in normal form is unsolvable, together with the incompleteness results following from this conclusion.

Post defines the notion of reduction as follows: "A system S_1 is reduced to a system S_2 if a 1-1 correspondence is (effectively) set up between the enunciations of S_1, and certain of the enunciations of S_2, so that an enunciation of S_1 is asserted when and only when its correspondent in S_2 is asserted" [Post, 1965, p. 351], [Post, 1994, p. 382]. (This definition is in fact Post's later interpretation of his tacit assumptions; at the time, he did not have a formal definition.) Post uses the word "enunciation" to denote a formula of a system. The most important property of reduction is that if we can solve the decision problem for a system S_2, and S_1 can be reduced to S_2, then the decision problem for S_1 is also solvable.

Post begins from the formulation of a generalized propositional logic described in the previous section, calling it "Canonical Form A." He shows that any system in canonical form A can be reduced to a second form, "Canonical Form B." In Canonical Form B, the unrestricted rule of substitution present in Form A is replaced by a much weaker version. Post shows, however, that the effect of the substitution rule can be obtained by using the idea of axiom schemes. For example, in formulating the propositional calculus, we can take as axioms all formulas of the form $(\sim A \vee (\sim B \vee A))$. This method of dispensing with the rule of substitution is usually attributed to John von Neumann, who employed it in [von Neumann, 1927]. In the conventional formulation, this involves an infinite set of axioms. Post, on the other hand, requires his rules to be finite, so he adds an axiomatization of the notion of formula, or "enunciation," in his terminology. To achieve this, he adds a primitive function $e(\cdot)$ to his system, whose intended interpretation is "P is an enunciation," together with rules such as: $e(P), e(Q) \Longrightarrow e((P \vee Q))$.

Post's next reduction shows that the entire first-order fragment of *Principia Mathematica* can be reduced to canonical form B. Considerable complications are induced by the fact that different types of variable are involved here, while canonical form B only allows one type of variable. A corollary of this work is that if the decision problem for all systems in canonical form B were solvable, then the Hilbertian *Entscheidungsproblem* would be solvable. More important from Post's point of view was the fact that it appeared that all of *Principia Mathematica* could in a similar fashion be reduced to a system in canonical form B. Hence, in spite of its apparently modest appearance, canonical form B already encapsulates the full complexity of the whole system of Whitehead and Russell, at least as far as the decision problem is concerned.

At this point in the narrative of his postdoctoral research, Post introduces a digression into the combinatorial problem of 'tag,' a seemingly innocuous problem that arose early in his research. Let us suppose that we are given a finite set of primitive symbols, say the set of numerals $0, 1, \ldots, \mu$ for simplicity. Then a *tag system* is given by a finite set of rules of the form $0 \to \sigma_0, \ldots, \mu \to \sigma_\mu$, where $\sigma_0, \ldots, \sigma_\mu$ is a sequence of finite strings of primitive symbols, possibly empty, together with a positive integer ν. These rules are applied as follows. Given a non-empty sequence B of symbols, append to the end of B the string associated with the first symbol in B, then delete the first ν symbols (all, if there are less than ν symbols). For example, consider the tag system with the primitive symbols $0, 1$, the rules $0 \to 00, 1 \to 1101$ and $\nu = 3$ (a system that Post himself found intractable). Then if we start with the sequence 10, the rules successively produce $101, 1101, 11101, 011101, \ldots$. The problem of tag is to decide, given an initial string, whether or not the process terminates in the empty string or not. One can think of this as a process in which one end of the string, advancing at a constant rate, is trying to catch up with the other end of the string – hence the colourful name of 'tag,' suggested by Post's fellow-student and colleague B.P. Gill.

Post came upon the problem in the following way. Early in his work, he had discovered a method that, given two formulas in a system in canonical form A (that is to say, two terms of first-order logic), determines whether or not there is a set of substitutions that makes the two expressions identical. He called this method the "the L.C.M. process" – this is Post's name for the unification algorithm that plays a very important part in automated theorem proving in first-order logic, and was described in print in the groundbreaking paper on the resolution method [Robinson, 1965]. (Post's "least common multiple" appears to be an alternative name for Robinson's "most general unifier.") In attempting to extend the algorithm to higher-order logic, Post encountered the problem of tag. An effort to extend the result of [Post, 1921b] to more general systems led again to the same problem. Thus this combinatorial problem in symbol manipulation seemed like an essential stepping stone towards his hoped-for general results on decidability, and Post made it the major project of his postdoctoral year.

It should come as no surprise that the two problems mentioned above are unsolvable. The problem of tag was proved unsolvable in [Minsky, 1961] – an attractive exposition of this result is in the book [Minsky, 1967]. As for the higher-order unification problem, the second-order case was proved unsolvable in [Goldfarb, 1981]. At the initial stage of his research, though, Post did not have an inkling of unsolvable problems, but instead hoped that the simple and primitive character of tag systems would lead to equally simple decision procedures. This hope led rapidly to disillusionment, as he found that problems of ordinary number theory cropped up in his attempts at solving the problem of tag, disappointing his hope that "the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics" [Post, 1965, p. 373], [Post, 1994, p. 398]. Although the solution of the general problem of "tag" appeared hopeless (and hence the entire program of the solution of decision problems), Post had assumed that it was a very minor, if essential, stepping stone towards his broader program. However, in the late summer of 1921, Post found a further series of reductions that led to a final canonical form very close to the apparently special form of "tag." We now describe this final series of reductions.

Canonical form B assumes that there are infinitely many variables, and that the formation rules for formulas are given in terms of functional operations $f(p_1, \ldots, p_n)$. Post begins his series of reductions by reducing systems in form B to systems in canonical form C, in which all formulas are built from a finite alphabet Σ, and the formulas consist simply of the set Σ^* of all finite strings built from symbols in Σ. A system in canonical form C has a finite set of strings as primitive assertions, and a finite set of rules, or "productions" having the form $\sigma_1, \ldots, \sigma_k \Longrightarrow \tau$, where $\sigma_1, \ldots, \sigma_k, \tau$ are strings composed of the symbols in Σ, together with added symbols from a set of auxiliary symbols $\Delta = \{P_1, P_2, \ldots, P_m, \ldots\}$ disjoint from Σ, and every auxiliary symbol in τ appears in one of $\sigma_1, \ldots, \sigma_k$. In an application of the rule, the auxiliary symbols in Δ are considered as syntactic variables ranging over arbitrary strings (possibly empty).

As an example of such a system, consider the alphabet $\{a, b\}$, and let the primitive assertions consist of the strings a, b, aa, bb. The two rules of the system are:

$$P_1 \Longrightarrow aP_1a, \;\; P_1 \Longrightarrow bP_1b.$$

It is easy to see that the assertions of this system consist of all palindromes in the alphabet $\{a, b\}$, that is to say, all non-empty strings that read the same forward as backwards.

As a second example, consider the alphabet $\{p, \sim, \vee, (,)\}$, with the single primitive assertion p, and the rules:

$$P_1 \Longrightarrow \sim P_1, \;\; P_1, P_2 \Longrightarrow (P_1 \vee P_2).$$

Then the assertions of this system are exactly the formulas of the propositional calculus of *Principia Mathematica* containing the single variable p.

Post's next move is to give a series of four successive reductions, in the course of which the decision problem is reduced to that of systems of the very simple Post normal form. A system of canonical form C is said to be in *normal form* if it has a single primitive assertion, and all productions are of the form $\sigma P \Longrightarrow P\tau$. Thus the general decision problem, even for a system as apparently complicated as *Principia Mathematica*, can be reduced to that for a system of rules that can only operate on strings by first deleting an initial substring, then tacking on a new string at the end. This amazing result was described by Marvin Minsky as "one of the most beautiful theorems in mathematics" [Minsky, 1967, p. 240]. Post's proof of the theorem is not inherently difficult, but it is rather lengthy and not very easy to follow; a clear exposition of the theorem, following the lines of Post's original construction, with numerous helpful examples, is provided in Chapter 13 of [Minsky, 1967]. An alternative approach to the proof, based on the theory of Turing machines and semi-Thue systems, is given in Chapter 6 of [Davis, 1958].

The normal form is very close to that of tag systems, since in both cases, the rules involve removing a string of symbols at the beginning, and tacking on a string at the end. The simplicity of the final normal form, together with his earlier success in solving the decision problem for a simple case of canonical form A, aroused Post's hopes anew. However, as he said later, "just when hope was thus renewed for a solution of the general finiteness problem, a fuller realization of the significance of the previous reductions led to a reversal of our entire program" [Post, 1965, p. 402][Post, 1994, p. 418].

4.1 The Anticipation

The reduction of the whole of first-order logic to canonical form B, and the subsequent reduction of systems in canonical form B to systems in normal form made it seem very likely that the whole of *Principia Mathematica* could be reduced to a system in normal form (although Post never in fact worked out all the details of this reduction).

In spite of its apparently great deductive power, a system as complicated as *Principia Mathematica* can be presented as a set of elementary rules for rewriting strings of symbols. This led Post to a far-reaching generalization [Post, 1994, p. 405], [Post, 1994, p. 420]:

> In view of the generality of the system of *Principia Mathematica*, and its seeming inability to lead to any other generated sets of sequences on a given set of letters than those given by our normal systems, we are led to the following generalization.
>
> *Every generated set of sequences on a given set of letters* $a_1, a_2, \ldots, a_\mu$ *is a subset of the set of assertions of a system in normal form with primitive letters* $a_1, a_2, \ldots, a_\mu, a'_1, a'_2, \ldots, a'_\mu$, *i.e., the subset consisting of those assertions of the normal system involving only the letters* $a_1, a_2, \ldots, a_\mu$.

The italicized statement in the quotation equates an intuitive notion, the notion of sets of sequences that are generated by finite means, with a precisely defined mathematical notion. It is analogous to Church's thesis [Church, 1936] identifying functions that are computable in the intuitive sense with general recursive (equivalently, Turing-computable) functions. Post's hypothesis, by analogy, has been called "Post's thesis" [Davis, 1982, p. 21]; under some plausible assumptions, it is in fact equivalent to Church's thesis.

Post now observed that a simple application of Cantor's diagonal argument leads immediately to an apparent counter-example to his thesis. Let us represent sets of positive integers as sets of strings on a single letter 'a', where a string of length n represents the integer n. It is easy to see that we can effectively enumerate all possible normal systems generating such sets of strings. But now the diagonal method apparently leads to a generated set of strings that is not one of those in the enumeration. That is to say, let the diagonal set D be defined by the prescription: a string $a \ldots a$ of length n is in D if and only if it is not generated by the nth normal system in the enumeration. This appears to contradict Post's thesis.

However, a contradiction would only arise if D were in fact a generated set. D would be generated by a finite process, if we make the assumption that the decision problem is solvable for the class of normal systems. What we have shown, in fact, is that this hypothesis together with Post's thesis, leads to a contradiction. Post already had a suspicion that there were inherent difficulties in his quest for general decision methods, based on his difficulties with the problem of tag. However, no difficulties appeared to stand in the way of his thesis. Hence, he held fast to his thesis, concluding that the decision problem for normal systems was in fact unsolvable. He recorded the philosophical influences that led him to the resolution of the contradiction as follows [Post, 1965, p. 407],[Post, 1994, p. 421]:

> In thus resolving this dilemma, the writer was greatly influenced by having heard, not long before, of Brouwer's rejecting the law of the excluded middle. This revolution in the writer's thought was largely energized by the immediately prior reading of Poincaré's *Foundations of Science*.

The informal sketch of the unsolvability of the decision problem for normal systems given above depends on the assumption of Post's thesis, which, since it equates an informal notion with a mathematical notion, cannot be considered a mathematical hypothesis. However, we can restate the result in such a way as to make into a purely mathematical theorem.

Post concludes the mathematical part of his 1941 paper with a sketch of this restatement. He begins by observing that there is a universal normal system in the following sense. There is a normal system K that has the following property: For every normal system S and formula P in the vocabulary of S, there is a formula (S, P) of K so that (S, P) is assertible in K if and only if P is assertible in S. Post's system K, which he calls *the complete normal system*, is thus a complete inference system for assertions of the form "The string P is assertible in the normal system S," and is the analogue in the realm of normal systems of Turing's universal computing machine [Turing, 1936].

Although the normal systems are abstract versions of formal axiomatic systems, Post observed that we can also use them to represent decision procedures. Suppose that we are given a normal system M, and that in addition, there is a normal system M' that contains all of the primitive symbols of M, and in addition a new letter b (possibly with some other additional symbols). Assume that if P is a formula of M, that P is an assertion of M' if and only if it is an assertion of M, and that bP is an assertion of M' if and only if it is not an assertion of M. Then there is a decision procedure for M, since to decide whether or not a formula P is an assertion of M or not, we can systematically enumerate all of the assertions of M', and observe whether P or bP is an assertion of M'. (This idea is the source of Post's later theorem [Post, 1944, p. 290] that a set of positive integers is recursive if and only if both it and its complement are recursively enumerable.) If such a system M' exists, then Post says that there exists a *finite-normal-test* for the system M [Post, 1965, pp. 412-413], [Post, 1994, p. 425].

In the informal sketch given above of the result that there is no general decision procedure for normal systems, we can replace the appeal to Post's thesis by restricting our attention to normal systems from the outset, and by replacing the informal notion of decision procedure by the concept of finite-normal test. We can thus prove the fundamental result: *There is no finite-normal-test for the complete system K.* A rigorous proof of this result depends on a formal version of the diagonal method sketched above.

Throughout his research of 1920-21, Post concentrated on the problem of decision procedures. However, his idea of representing decision procedures as complete formal theories, as in his notion of "finite-normal-test," leads immediately to incompleteness results for formal systems.

The complete system K is able to give correct positive answers to all questions of the form: "Is the formula P an assertion of normal system S?" Could there be a system that in addition gives correct negative answers to all such questions? The fundamental theorem above indicates that such a system cannot exist. Let L be a normal system that includes all the primitive letters of the complete normal system K among its primitive letters, together with another primitive letter b. Furthermore, let us assume that for any normal system S and formula P of S,

(S, P) is an assertion of L if and only P is an assertion of S, while if $b(S, P)$ is an assertion of L, then P is not an assertion of S. Any such L, Post calls a *normal deductive-system* adjoined to K. In other words, a normal deductive-system adjoined to K gives a correct and complete set of positive answers to questions of the above type, while all of the negative answers it gives to such questions are correct.

The same argument used to prove the fundamental theorem then demonstrates a basic incompleteness theorem: *No normal deductive-system is complete, so that if L is any such system, there is always a normal system S and formula P of S so that P is not an assertion of S, but $b(S, P)$ is not an assertion of L.* This is an abstract form of Gödel's incompleteness theorem, though stated with respect to normal systems rather than arithmetical propositions. Post concludes with the striking words, *A complete symbolic logic is impossible*, and remarks [Post, 1965, pp. 416-417],[Post, 1994, pp. 428-429] :

> This is an iconoclastic result from the formal logician's point of view since it means that logic must not only in some parts of its description (as in the operations), but in its very operation be informal. Better still, we may write
>
> *The Logical Process is Essentially Creative.*

4.2 Post's failure to publish

In view of the truly remarkable nature of the results that Post achieved in the early 1920s during his postdoctoral year, one may well wonder why he did not publish any of this material until fifteen years had passed. There were certainly a number of external reasons for this. The first outbreak of his manic-depressive illness followed immediately on his great discoveries, and the loss of his academic position at Cornell University, together with his being forced to make a living as a high school teacher, were hardly conducive to research and publication. In the summer of 1924, during a year in which he had some association with Cornell University, Post made some progress on his project, but he was never able to bring it to completion.

Another reason given by Post in later years was the general lack of interest in the United States in logical matters. In the covering letter that he sent in 1941 with [Post, 1965] to Hermann Weyl, the editor of the *American Journal of Mathematics*, he points out that, for example, the original paper giving the results of [Post, 1941] was returned to him by the editors of the *Annals of Mathematics* at the height of his postdoctoral work "without any editorial commitment, and with a very mixed report from the referee." He continues:

> It therefore seemed to me to be hopeless to seek publication of Part One of the present paper. And without it, the then revolutionary Part

> Two would have seemed but idle chatter. An attempt to obtain a full proof development was interrupted by ill health and led to a constantly receding date of ultimate publication [Davis, 1994, p. xviii].

This last quotation hints at an internal reason for Post's failure to stake his claim to the incompleteness and undecidability results in time. Kurt Gödel, in the famous incompleteness paper [Gödel, 1931] that led to most of the later research on the limitations of formal systems, set himself a precise but restricted goal. He showed that a powerful, but quite specific, axiomatic system for mathematics, the simple theory of types with the natural numbers as a ground type and the Peano axioms for number theory, was incomplete, and furthermore, that any of a very broad class of ω-consistent extensions of this system shared this incompleteness. At the same time, Gödel left somewhat indefinite the class of systems to which his incompleteness results applied, and it was only later, with the publication of [Church, 1936] and [Turing, 1936] that it became clear that the incompleteness results were completely general, applying to any formal system including a minimum of elementary number theory.

Post, on the other hand, set himself the goal of proving his thesis, that the mathematical concept of normal system exactly characterizes the informal notion of a generated set of strings. In the introduction to a strange but fascinating Appendix to [Post, 1965], he explains his plan:

> While the formal reductions of Part I should make it a relatively simple matter to supply the details of the development outlined in §9 and the beginning of §10, that development owes its significance entirely to the universal character of our characterization of an arbitrary generated set of sequences as given in §7. Establishing this universality is not a matter for mathematical proof, but of psychological analysis of the mental processes involved in combinatory mathematical processes. Because these seemed to be sufficiently simple to be exhaustively described, the writer gave up a direct use of *Principia Mathematica* as a partial verification of the characterization in question, planning rather that the incompleteness of the logic of *Principia Mathematica* would be a corollary of the more general result [Post, 1965, p. 418],[Post, 1994, p. 429].

The remainder of the Appendix consists of quotations from a diary that Post kept under the title "Time Accounts" from the spring of 1916 to the spring of 1922 (this diary is not among the Post papers in Philadelphia, and may have been destroyed or lost).

It is of course possible to provide quite convincing plausibility arguments for Church's thesis, as also for Post's thesis, and [Church, 1936] and [Turing, 1936] both contain such heuristic considerations. Turing's arguments are more convincing than those of Church; the article [Sieg, 1997] contains an illuminating detailed analysis of Church's arguments and a comparison with those of Turing. Nevertheless, it seems hopeless to produce an argument that would establish the result with complete certainty, and Post's search for such an analysis was a quixotic venture doomed to failure. The result was that his results of 1920-21 were superseded by the work of others, a development that undoubtedly caused Post a great deal of anguish.

Post's unpublished paper of 1941 was an attempt to recoup some of his losses. However, the paper was rejected by Hermann Weyl, who explained his decision by saying:

> I have little doubt that twenty years ago your work, partly because of its then revolutionary character, did not find its due recognition. However, we cannot turn the clock back; in the meantime Gödel, Church and others have done what they have done, and the American Journal is no place for historical accounts . . . [Davis, 1994, p. xix]

The paper [Post, 1943] is a greatly shortened version of [Post, 1965], containing only the reduction from canonical form C to normal form, although a concluding lengthy footnote summarizes the content of the rejected paper.

On October 29 1938, Post made the acquaintance of Kurt Gödel at a regional meeting of the American Mathematical Society in New York, and spoke with him about his work on absolutely unsolvable problems. On the same day, he wrote Gödel a touching letter in which he said:

> I am afraid that I took advantage of you on this, I hope but our first meeting. But for fifteen years I had carried around the thought of astounding the mathematical world with my unorthodox ideas, and meeting the man chiefly responsible for the vanishing of that dream rather carried me away [Gödel, 2003, p. 169].

On the following day, Post, still excited by his meeting with Gödel, wrote a second letter in which he summarized his work of the 1920s, following the outline of [Post, 1965] while emphasizing the incompleteness of any formal system with respect to the fixed subject matter of provability in normal systems. Explaining his failure to prove Gödel's results at the time, he said:

> May I finally say that nothing that I had done could have replaced the splendid actuality of your proof. For while corollary your theorem may be of the existence of an absolutely unsolvable problem, the absolute unsolvability of that problem has but a basis in the nature of physical induction at least in my work and I still think in any work. Of course with sufficient labor that induction could have gone far enough to include your particular system theorematically. That that could be done for Principia Mathematica I saw then. My only excuse for not doing so – well there are many and having written so much I might add them. Chiefly I thought I saw a way of so analyzing "all finite processes of the human mind" [something of the sort of thing Turing does in his computable number paper] that I could establish the above

> conclusions in general and not just for Principia Mathematica. Secondly that the absolute unsolvability of my problem would not achieve much recognition for others merely on the basis of an incompleteness proof for Principia Mathematica. And lastly while the above general analysis enticed me it seemed foolish to do all the labor involved in the more special Principia Theorem.

Post concluded his letter by apologizing to Gödel for his excitement when they met:

> It was a real pleasure to meet you and I hope my egotistical outbursts have spent themselves with that first meeting and this letter. Needless to say I have the greatest admiration for your work, and after all it is not ideas but the execution of ideas that constitute a mark of greatness [Gödel, 2003, pp. 171-72].

In a brief reply, Gödel reassured Post that he had noticed "nothing of what you call egotistical outbursts in your letters or in the talk I had with you in New York; on the contrary it was a pleasure to speak with you" [Gödel, 2003, p. 173].

In spite of his disappointment in being overtaken by other logicians, Post continued to work in the area of recursion theory, and made some of his most important and influential contributions in the period from 1936 to 1954. Before we take up this thread again, however, it is necessary to make a digression in order to describe Post's work on Boolean clones. Although his fundamental results in this area were already announced in [Post, 1921a], they were finally published twenty years later after major changes in exposition.

5 CLASSIFYING SETS OF BOOLEAN FUNCTIONS

In this section, we discuss Post's basic results on classes of Boolean functions closed under composition. Although his work in this area belongs with the most fundamental results of modern mathematical logic, it has remained somewhat difficult of access.

When Post rewrote his research for publication, following an unfortunate suggestion of a referee [Post, 1941, p. 4], [Post, 1994, p. 256] he adopted the notation of Jevons for logical formulas (his earlier version was in terms of the more familiar truth-tables). Jevons uses capital letters $A, B, C, \ldots$ as Boolean variables (interpreted as classes), while the corresponding lower case letters $a, b, c, \ldots$ represent the complements of these variables [Jevons, 1864]. The addition symbol $+$ represents Boolean union (inclusive disjunction), while the universal and empty classes are symbolized by 1 and 0. As an example of Jevons's notation, the disjunctive normal form expression usually symbolized as $(A \wedge B) \vee (\sim A \wedge B \wedge \sim C)$ is written in Jevons's notation as $AB + aBc$. Post proceeds in his classification by means of what he calls an "expansion" of a logical function, a pair $\alpha : \beta$ of expressions in disjunctive normal form, expressed in Jevons's notation, the first being a disjunctive normal form of the function itself, the second a disjunctive normal form of its negation. The expansion is not unique; for example, one expansion of the implication function $A \supset B$ is $a + B : Ab$, while another is $AB + aB + ab : Ab$.

Although Jevons's notation is concise, and not difficult to learn, Post's use of it leads to quite unnecessary obscurities. For example, the familiar class of monotone functions in Post's terminology becomes the class of functions "satisfying the $[A : a]$ condition" [Post, 1941, p. 35], [Post, 1994, p. 287]. This idiosyncratic phrase derives from a characterization of monotone functions in terms of their expansions. A Boolean function f is monotone if and only if it satisfies the $[A : a]$ condition: for any two terms in an expansion of f, there is a letter which is capital in the term that is in the first expression and small in the term that is in the second expression.

The result, unfortunately, is a monograph that modern logicians seem to find impossible to read. Post provides a complete and painstaking proof of his major results; reviewing the work in the *Journal of Symbolic Logic*, H.E. Vaughan [Vaughan, 1941] remarked that the book is "self-contained and very clearly written," and noted only four serious misprints. Nevertheless, the contemporary logician Roman Murawski wrote [Murawski, 1998] of Post's characterization of functionally complete sets of connectives (proved in §26 of his monograph):

> Did Post prove this theorem? In [Post, 1941] one finds no proof satisfying the standards accepted today. The reason for that was Post's baroque notation (it was in fact an unprecise adaptation of the imprecise notation of Jevons from his *Pure Logic*, cf. [Jevons, 1864]), other reason was the fact that Post seemed to be simultaneously pursuing several different topics.

Although these accusations of imprecision and lack of rigour are quite unjustified, Murawski's remarks (which echo similar but somewhat more guarded observations in [Pelletier and Martin, 1990, p. 463]) form an eloquent testimony to the unreadability of Post's ill-fated monograph.

In view of the problematic nature of Post's exposition, in most of this section we shall expound his results in unabashedly modern terminology. The exposition below in fact has some substantive differences from Post's original statement of his results; towards the end of this section, we shall return to Post's original version to explain where these differences lie.

5.1 A tour of Post's lattice

Our object of study in this section is the family of all finitary functions on the two-element set $S = \{0, 1\}$; we can identify the elements with truth-values, with 1 as "true," and 0 as "false." An n-place operation on S, for $n > 0$, maps n-tuples of Boolean values into Boolean values. There are 2^{2^n} such functions for each n; familiar logical operations such as $\wedge$ and $\vee$ are among the sixteen 2-place Boolean functions. (We might also consider the constants 0 and 1 as zero-place functions, but here we prefer to think of them as 1-place constant functions.)

The most basic n-place functions (apart from the constant functions) are the *projections* π_i^n, defined by the equation: $\pi_i^n(x_1, \ldots, x_n) = x_i$. If we think of functions as given by expressions involving variables $x_1, \ldots, x_i, \ldots, x_n, \ldots$, then the projections correspond to the variables themselves. We can construct new functions from old by the operation of *composition.* If f is a k-place operation, and $g_1, \ldots, g_k$ a list of n-place functions, then we define an n-place function h by composition:

$$h(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)).$$

We define a family of Boolean functions to be a *clone* if it contains all the projection functions and is closed under composition (Post uses a more general notion of "iteratively closed class," described below in §5.2).

Figure 2 shows the inclusion diagram of all Boolean clones. Post's results are perhaps most easily understood in terms of a commentary on this diagram. The first and most obvious feature of the diagram is its left-right symmetry. This symmetry arises from the phenomenon of duality. If $f(x_1, \ldots, x_n)$ is a Boolean function, then the function g defined by the equation

$$g(x_1, \ldots, x_n) = \sim f(\sim x_1, \ldots, \sim x_n)$$

is the *dual* of f; the dual of the dual of f is just f itself. Equivalently, the truth-table of the dual of f is obtained from the truth-table of f by replacing 0 by 1 and 1 by 0. For example, $\wedge$ is the dual of $\vee$, and vice versa; the dual of the implication operator $(x \supset y)$ is the subtraction operator $y \setminus x$, or equivalently, $(y \wedge \sim x)$. A function is *self-dual* if it is equal to its dual. If C is a clone, then the class of functions dual to those in C is also a clone, so that to every clone in our diagram there is a corresponding dual clone, explaining the left-right symmetry. The fourteen systems in the centre of the diagram are self-dual classes (that is to say, they are their own duals).

There are only countably many Boolean clones, and Post's classification gives a complete, explicit description of all of them. Figure 2 shows a central section, consisting of 30 "sporadic" clones, together with eight infinite descending chains, four on the left and four on the right. The supplementary Figure 3 shows some of the rather intricate pattern of containments in the central part of the lattice. Every one of the clones is finitely generated, that is to say, there is an explicitly describable finite set G of functions so that every function in the clone can be defined from functions in G. In the remaining part of this section, we shall take a tour of Post's lattice, giving a description of all of the clones and their generators.

At the bottom of the lattice, we find six classes generated by negation and constants: $R_1 = \{\emptyset\}$ (the set of all projections), $R_4 = \{\sim\}$, $R_6 = \{1\}$, $R_8 = \{0\}$, $R_{11} = \{1, 0\}$, $R_{13} = \{\sim, 1\}$. Here, the notation $\{\sim, 1\}$ (for example) denotes the smallest clone containing the functions $\sim$ and 1.

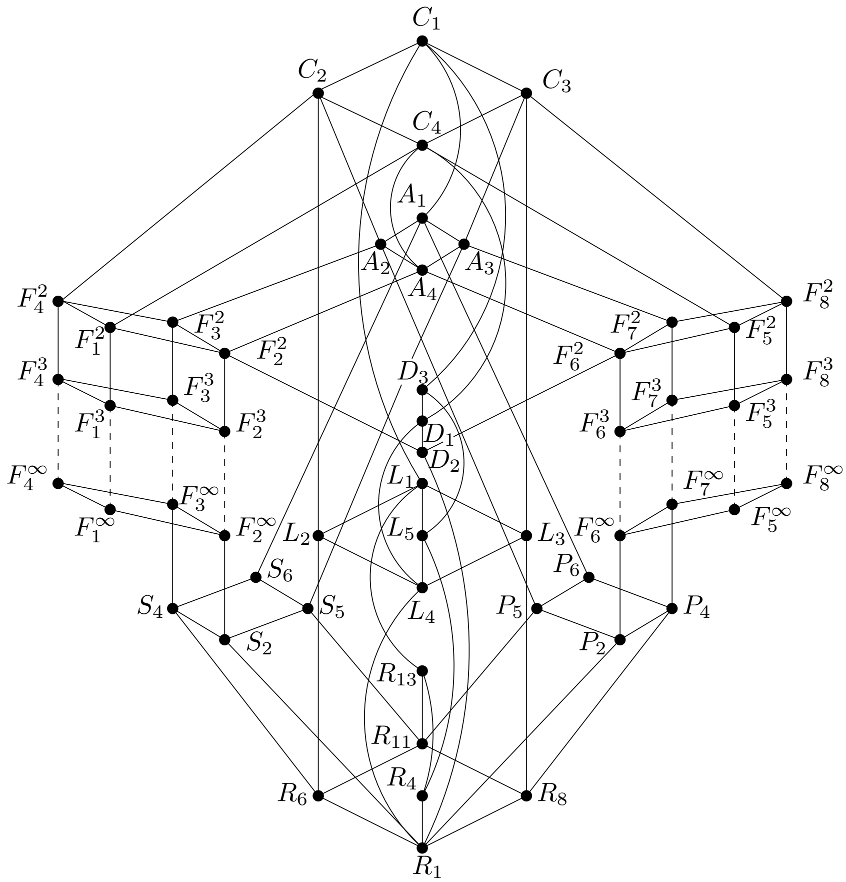

Figure 2. Post's lattice

Proceeding up the lattice, we encounter two groups of four on left and right, and one group of five in the middle. On the left, we have four classes generated by disjunction and the constants: $S_2 = \{\vee\}$, $S_4 = \{\vee, 1\}$, $S_5 = \{\vee, 0\}$ and $S_6 = \{\vee, 0, 1\}$. On the right, the dual classes are: $P_2 = \{\wedge\}$, $P_4 = \{\wedge, 0\}$, $P_5 = \{\wedge, 1\}$, $P_6 = \{\wedge, 0, 1\}$. In the middle, there are five classes describable in terms of the biconditional $\equiv$ and its dual $\oplus$ (that is to say, addition modulo 2). If we write $\oplus_3$ for the three place operation $x \oplus y \oplus z$, then we have: $L_1 = \{\equiv, \sim\}$, $L_2 = \{\equiv\}$, $L_3 = \{\oplus\}$, $L_4 = \{\oplus_3\}$, $L_5 = \{\oplus_3, \sim\}$. The class L_1 can also be described abstractly as the set of all *linear* Boolean functions, that is to say, those expressible in the form $x_1 \oplus \cdots \oplus x_n \oplus c$, where c is a constant (0 or 1).

At this point, it is convenient to make a digression to introduce some definitions. We define two infinite sequences, $c_2, c_3, \ldots, c_n, \ldots$ and $d_2, d_3, \ldots, d_n, \ldots$ of

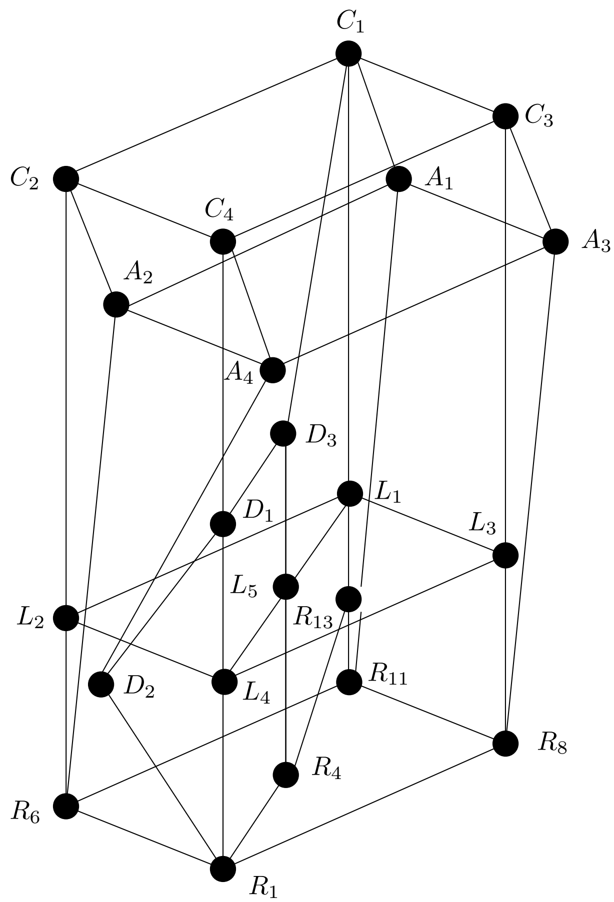

Figure 3. Post's lattice: the central part

functions, where c_n and d_n each have $n+1$ input variables. For $n \geq 2$, we define:

$$c_n(x_1, x_2, \ldots, x_{n+1}) = \bigwedge_{1 \leq i \leq n+1} (x_1 \vee \cdots \vee \widehat{x_i} \vee \cdots \vee x_{n+1}),$$

$$d_n(x_1, x_2, \ldots, x_{n+1}) = \bigvee_{1 \leq i \leq n+1} (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}),$$

where the notation "$\widehat{x_i}$" means that the variable x_i is omitted from the term in question. For example, function d_2 is given by the definition

$$d_2(x_1, x_2, x_3) = (x_2 \wedge x_3) \vee (x_1 \wedge x_3) \vee (x_1 \wedge x_2).$$

The function d_n belongs to the family of *threshold functions*, a family that plays an important role in theoretical computer science [Vollmer, 1999]. The threshold function T^n_m, for $m \leq n$, is the n-place function defined by the condition: $T^n_m(x_1, \ldots, x_n) = 1$ if and only if $x_1 + \cdots + x_n \geq m$ (that is to say, at least m of the inputs $x_1, \ldots, x_n$ are equal to 1). For example, $(x \vee y) = T^2_1$, and $(x \wedge y) = T^2_2$. With this definition, we have $d_n = T^{n+1}_n$. The function c_n satisfies the dual condition: $c_n(x_1, \ldots, x_n) = 1$ if and only if $x_1 + \cdots + x_n > 1$. It is easy to see that $c_2 = d_2$. The functions c_n and d_n are duals of each other, so that the function c_2 is self-dual.

In the centre of the diagram is a group of three classes, $D_2 = \{d_2\}$, $D_1 = \{d_2, \oplus_3\}$ and $D_3 = \{d_2, \oplus_3, \sim\}$. D_3 is the class of all self-dual functions; hence there are exactly seven classes consisting entirely of self-dual functions.

Immediately above D_3 lies a group of four classes, A_1, A_2, A_3 and A_4. The top element in this group, A_1, is the important class of all monotone functions. If we order the truth-values by setting $0 < 1$, then n-tuples of truth-values can be ordered by defining $\langle x_1, \ldots, x_n \rangle \leq \langle y_1, \ldots, y_n \rangle$ if and only if $x_i \leq y_i$, for all i, $1 \leq i \leq n$. An n-place function f is *monotone* if for all $x_1, \ldots, x_n, y_1, \ldots, y_n$,

$$\langle x_1, \ldots, x_n \rangle \leq \langle y_1, \ldots, y_n \rangle \Rightarrow f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n).$$

The generators of these classes are given by $A_1 = \{\wedge, \vee, 0, 1\}$, $A_2 = \{\wedge, \vee, 1\}$, $A_3 = \{\wedge, \vee, 0\}$ and A_4 $\{\wedge, \vee\}$.

Proceeding to the top of the lattice, we find C_1, the class of all Boolean functions, C_2, the class of all functions that preserves 1, in the sense that if f is an n-place function in C_2, then $f(1, 1, \ldots, 1) = 1$, C_3, the class of all functions preserving 0, and C_4, the class of all functions preserving both 1 and 0. For generators, we have $C_1 = \{\vee, \sim\}$, $C_2 = \{\wedge, \supset\}$, $C_3 = \{\vee, \setminus\}$, $C_4 = \{x \wedge (y \vee \sim z), \vee\}$.

Finally, we have to describe the eight infinite descending chains arranged symmetrically on each side of the lattice. Starting with the four chains on the left, we can describe them using the following sets of generators, where $2 \leq m < \infty$:

$$F^m_1 = \{x \vee (y \setminus z), c_m\}, F^\infty_1 = \{x \vee (y \setminus z)\};$$

$$F^m_2 = \{x \vee (y \wedge z), c_m\}, F^\infty_2 = \{x \vee (y \wedge z)\};$$

$$F^m_3 = \{c_m, 1\}, F^\infty_3 = \{x \vee (y \wedge z), 1\};$$

$$F^m_4 = \{\supset, c_m\}, F^\infty_4 = \{\supset\}.$$

The four chains on the righthand side of the lattice are the duals of those on the left. Thus, a set of generators for each class is given by the duals of the generators of the corresponding dual class:

$$F^m_5 = \{x \wedge (y \vee \sim z), d_m\}, F^\infty_5 = \{x \wedge (y \vee \sim z)\};$$

$$F^m_6 = \{x \wedge (y \vee z), d_m\}, F^\infty_6 = \{x \wedge (y \vee z)\};$$

$$F^m_7 = \{d_m, 0\}, F^\infty_7 = \{x \wedge (y \vee z), 0\};$$

$$F^m_8 = \{\setminus, d_m\}, F^\infty_8 = \{\setminus\}.$$

It is possible to give a more structural description of the eight infinite chains by giving properties that characterize the functions in each clone. We generalize the property of being 0-preserving that picks out the functions in the class C_3. Define a function f to be *doubly 0-preserving* if, given two n-tuples of truth-values, $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$, there is a j, where $1 \leq j \leq n$, so that $x_j = y_j = 1$. More generally, define f to be *m-tuply 0-preserving* if whenever $x_1^1, \ldots, x_n^1$, $x_1^2, \ldots, x_n^2$, $\ldots$, $x_1^m, \ldots, x_n^m$ is a sequence of length m, consisting of n-tuples of truth-values such that $f(x_1^1, \ldots, x_n^1) = f(x_1^2, \ldots, x_n^2) = \cdots = f(x_1^m, \ldots, x_n^m) = 1$, there is a j, where $1 \leq j \leq n$, so that $x_j^1 = x_j^2 = \cdots = x_j^m = 1$. A function is *infinitely 0-preserving* if it is m-tuply 0-preserving for all m. This terminology, taken from [Pippenger, 1997, p. 22], is justified by the fact that a function is 0-preserving just in case it is singly 0-preserving.

The class F_8^m consists of all m-tuply 0-preserving functions, F_7^m results by restricting such functions to be monotone. F_5^m consists of all m-tuply 0-preserving functions that are also 1-preserving, while F_6^m is the subclass of all monotone functions in F_5^m. Finally, the classes F_5^∞, F_6^∞, F_7^∞, F_8^∞ are characterized in exactly the same way, except that the crucial property is that of being infinitely 0-preserving.

The classes F_1^m, F_2^m, F_3^m, F_4^m and F_1^∞, F_2^∞, F_3^∞, F_4^∞ are characterized in the same way as their dual counterparts, except that the dual property, of being m-tuply 1-preserving is employed.

The descriptions of the classes in terms of their generators is adapted from the paper [Lyndon, 1951] (see also [Böhler *et al.*, 2003]). In that paper, Lyndon provides a complete axiomatization for the equational logics corresponding to each of Post's classes.

5.2 Post's iterative classes

In the exposition of Post's results, following the contemporary trend, we have included all projection functions in our functionally closed classes. This is a natural approach, since (for example) we can consider the formula $\sim x$ as expressing a function on any number of variables, simply by regarding the variables other than x as irrelevant. In other words, when we include an n-place function in a class, then we also include all those functions that arise from it by adding irrelevant variables.

Post, however, did not make this assumption, and as a result, his classification includes twenty iteratively closed classes that are not clones. We first need to define Post's own notion of iteratively closed class. Let π be a map from $\{1, \ldots, n\}$ onto $\{1, \ldots, k\}$, where $k \leq n$. Define $\{0, 1\}^n/\pi$ to be the class of all $x \in \{0, 1\}^n$ satisfying the condition that for $i, j \leq n$, if $\pi(i) = \pi(j)$ then $x_i = x_j$. If $x \in \{0, 1\}^k$, then there is a unique vector $y = \pi^{-1}(x)$ in $x \in \{0, 1\}^n/\pi$ so that for all i, $1 \leq i \leq n$, $y_i = x_{\pi(i)}$. If f is a Boolean function defined on $\{0, 1\}^n$, then we define the function f^π on $\{0, 1\}^k$ by setting $f^\pi(x) = f(\pi^{-1}(x))$. We now define a family

F of Boolean function to be *iteratively closed* if it satisfies the conditions:

1. If f is an n-place function in F and π a map from $\{1, \ldots, n\}$ onto $\{1, \ldots, k\}$, where $k \leq n$, then f^π is in F;

2. If $f(x_1, \ldots, x_i, \ldots, x_k)$ is a k-place function in F, and $g(x_1, \ldots, x_n)$ an n-place function in F, then the $k + n - 1$-place function h defined by

$$h(x_1, \ldots, x_{k+n-1}) = f(x_1, \ldots, g(x_i, \ldots, x_{i+k-1}), x_{i+k}, \ldots, x_{k+n-1})$$

   is also in F.

That is to say, Post defines a non-empty class to be iteratively closed if it is closed under permutation and identification of variables, and also under composition. We now give a short description of the twenty iteratively closed classes that are not clones.

There are nine classes consisting entirely of 1-place functions. These are $O_1 = \{\pi^1_1\}$, $O_2 = \{1\}$, $O_3 = \{0\}$, $O_4 = \{\pi^1_1, \sim\}$, $O_5 = \{\pi^1_1, 1\}$, $O_6 = \{\pi^1_1, 0\}$, $O_7 = \{1, 0\}$, $O_8 = \{\pi^1_1, 1, 0\}$ and $O_9 = \{\pi^1_1, 1, 0, \sim\}$. Since Post does not consider the empty class to be iteratively closed, his containment diagram has no least element, and hence is not a lattice, since O_1, O_2 and O_3 are incomparable minimal elements.

Seven classes consist of sets of functions reducible to 1-place functions, that is to say, all functions in these classes are obtained by adding irrelevant variables to 1-place functions. Let 1^* be the class of all n-place constant functions taking the value 1, and 0^* the class of all n-place constant functions taking the value 0. Then we have the additional iterative classes $R_2 = \{1^*\}$, $R_3 = \{0^*\}$, $R_5 = \{\pi^1_1, 1^*\}$, $R_7 = \{\pi^1_1, 0^*\}$, $R_9 = \{1^*, 0^*\}$, $R_{10} = \{\pi^1_1, 1^*, 0^*\}$ and $R_{12} = \{\pi^1_1, 1^*, 0^*, \sim\}$.

Finally, there are four classes defined in terms of conjunction and disjunction. $S_1 = \{\pi^1_1, \vee\}$, $S_3 = \{\pi^1_1, 1^*, \vee\}$, $P_1 = \{\pi^1_1, \wedge\}$ and $P_3 = \{\pi^1_1, 0^*, \wedge\}$.

5.3 Applications of Post's classification

Post's classification, together with the accompanying containment diagram, encodes an enormous quantity of information. Many results, some of them far from obvious, can be read off immediately from the diagram.

As an example of such a result, consider the following assertion: If S is an infinite set of Boolean functions, then there is a finite set S_0 contained in S so that every function in S can be defined from functions in S_0, together with the projection functions. This does not seem at all obvious. However, it follows immediately by inspecting Post's lattice. The lattice, although it contains infinite descending chains, contains no infinite ascending chains, and the result follows easily from this fact.

Post's lattice also makes it easy to see which class is generated by a given finite set of Boolean functions. In particular, we can readily give a criterion for a set of functions F to be functionally complete, so that all functions are definable from F. To see how this works, let us examine the top element C_1 of Post's lattice. The class C_1 covers exactly five other classes; that is to say, exactly five classes, namely C_2, C_3, L_1, A_1, D_3, lie immediately below C_1 in the diagram. It follows from this that if F is a set of Boolean functions, then it is functionally complete if and only if for every one of these five classes, there is a function in F that is not in the class.

The last sentence of the previous paragraph can be rephrased to give Post's criterion for functional completeness. A set of functions is functionally complete if and only if it contains (a) a function that does not preserve 0, (b) a function that does not preserve 1, (c) a non-monotone function, (d) a non-linear function, (e) a function that is not self-dual. Since each one of these conditions is easy to check, given the truth-table for a function, the criterion leads to a simple algorithm to determine whether or not a given finite set of functions is functionally complete. A self-contained proof of Post's theorem on functional completeness can be found in [Pelletier and Martin, 1990]. Similar criteria can be given for all the other classes in Post's diagram.

For several decades after the publication of [Post, 1941], Post's classification was consigned to a backwater of logic, and was almost forgotten. However, with the increasing interest in Boolean functions and propositional logic, this situation has changed, and his fundamental results are now frequently employed by theoretical computer scientists. A survey of some of these applications of Post's classification can be found in the excellent expository article [Böhler *et al.*, 2003].

6 RECURSION THEORY AND DECISION PROBLEMS

Although he had been overtaken by the work of other logicians, Post was well placed to contribute to the newly developing fields stemming from the discovery of the incompleteness of formal systems by Gödel, and of an exact mathematical definition of computable functions by Church and Turing. His point of view was substantially different from the approach followed by other logicians, so that his papers of the 1930s and 1940s are highly original, breaking new ground in several directions.

Most of the work in recursion theory in the mid-1930s emphasized computable functions on the natural numbers. Post's earlier work differed in its overall thrust, since he was mainly concerned with formal axiomatic systems, expressed as rules for manipulating strings of symbols. Consequently, Post was able to develop the theory in new directions, emphasizing both generated sets of numbers and problems involving rules for symbolic manipulation.

6.1 Post machines

Post's first publication in logic after a fifteen year gap was [Post, 1936], a short paper in which he describes a computational model very similar to that of the Turing machine. Turing's article was received for publication on May 28 1936, while Post's was received October 7 1936; a footnote to Post's article by the editor, Alonzo Church, informs the reader that "the present article, however, although bearing a later date, was written entirely independently of Turing's." Post, on the other hand, unlike Turing, had the advantage of having read [Church, 1936], while Turing's logical influences seem to have been confined to the incompleteness results [Gödel, 1931], the Cambridge lectures of Max Newman and the textbook [Hilbert and Ackermann, 1928].

Unlike Turing, Post does not use the word "machine," but instead speaks of a "problem solver or worker" (however, the term "Post machine" [Uspensky, 1983] is now standard terminology for the model). The analogue of a Turing machine's "tape" is provided by a "symbol space" consisting of a two way infinite sequence of spaces or boxes, each of which is either empty or unmarked. The worker is assumed to be performing the following primitive acts:

(a) Marking the box he is in (assumed empty),

(b) Erasing the mark in the box he is in (assumed marked),

(c) Moving to the box on his right,

(d) Moving to the box on his left,

(e) Determining whether the box he is in, is or is not marked.

The *set of directions* for the worker consists of a sequence of numbered instructions, each instruction having one of the following forms:

(A) Perform operation (a), (b), (c) or (d) and then follow direction j,

(B) Perform operation (e) and according as the answer is yes or no correspondingly follow direction number i or direction number j,

(C) Stop.

One box is to be singled out and called the starting point. A problem is given in symbolic form by a finite number of boxes being marked with a stroke, and the output is given in the same form. With these conventions, a set of instructions serves to define a computable function, where the positive integer n is symbolized by a sequence of n consecutive marked boxes.

Post conjectures (correctly) but does not prove that his formulation is equivalent to Church's definition of computable functions. His formulation is strikingly similar to Turing's earlier definition, though Turing's notion of machine configurations or "m-configurations" does not appear in Post's development. This role is played by the numbers assigned to the instructions in Post's sets of directions. Although Turing's notion of computing machine, rather than Post's, has become the standard model in theoretical computer science, Post's formulation is a little more like standard machine code (since programs typically do consist of numbered instructions). To put the matter in computer science terms: Post shows that a computer that can only perform the primitive acts of reading, writing, moving and following jump instructions conditional on a zero test is a universal computational model.

Post evidently had plans to continue in a series of articles on the topic (as appears in the title of his paper) showing how "Formulation 1" could be extended to broader computational models. He must have been discouraged from this by the appearance of the more elaborate development in [Turing, 1936], and one can sense his disappointment in a remark recorded in a notebook in February 1938:

> *Turing:* In large measure removed what was left in my point of view. His *Finite number of mental states* hypothesis I did not have, and if verified should be a cornerstone in the finite process development [Grattan-Guinness, 1990, p. 82].

Post's notebooks in Philadelphia (see §8) contain some of his work on "Formulation 2." In it, he considered rules operating in a two-dimensional symbol space, somewhat reminiscent of later work on cellular automata, such as Conway's Game of Life [Berlekamp *et al.*, 1982, Chapter 25] [Wolfram, 2002].

Post ends his paper with some remarks on the significance of Church's thesis. He says of his conjecture that his formulation 1 is equivalent to the Gödel-Church development: "We offer this conclusion at the present moment as a *working hypothesis*. And to our mind such is Church's identification of effective calculability with recursiveness." To this last sentence he adds a footnote, in which he says: "Actually the work done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification" [Post, 1936, p. 105].

6.2 Recursively Enumerable Sets

In February 1944, Post gave an invited address to the New York meeting of the American Mathematical Society, and the published version of his talk [Post, 1944] is his most influential paper. In it, he sets forth a program to determine the relative complexity of undecidable problems, based on abstract notions of reducibility. As part of this program, he poses a problem, later known as "Post's problem" that was one of the driving forces behind the rapid development of recursion theory in the decades following Post's address.

Since he was addressing a general mathematical audience, Post adopted an informal mode of presentation that contrasts with the rather formal exposition that was standard in the field of recursive functions at that time. He remarks:

> We must emphasize that, with a few exceptions explicitly so noted, we have obtained formal proofs of all the consequently mathematical theorems here developed informally. Yet the real mathematics involved must lie in the informal development. For in every instance the informal "proof" was first obtained; and once gotten, transforming it into

> the formal proof turned out to be a routine chore [Post, 1944, p. 284] [Davis, 1965, p. 305][Post, 1994, p. 462].

Although Post seems to have felt that publication of final results required a more formal mode, in fact his informal expository style has now become the norm in presentations of research in recursion theory.

Post begins his article with a beautifully written survey of the area, stating Church's thesis as the identification of the intuitive concept of effectively calculable function with the technical concept of recursive function. He goes on to explain the intuitive concept of a *generated set* of positive integers, and asserts his own thesis as the identity of this concept with that of recursively enumerable set. The technical definition of recursively enumerable set is given in terms of his own concept of normal system, a set of positive integers being recursively enumerable if it is generated by a normal system with the alphabet $\{1, b\}$, the positive integer n being represented by a string of n 1's; if a set of integers is generated by a set of rules B, then we say that B is a *basis* for the set generated by these rules. He goes on to point out the generality of this concept, since the assertions of an arbitrary formal system of formal logic can (by numerical encoding) be regarded as a recursively enumerable set.

With this identification, the decision problem for a formal system can be seen as the decision problem for a recursively enumerable set. A corollary of Church's thesis is that the intuitive concept of a set with a solvable decision problem is coextensive with the notion of recursive set. Hence, questions of algorithmic decidability can be placed in the abstract framework of recursively enumerable sets of positive integers and their decision problems. The basic result on the existence of unsolvable problems (for which Post references [Church, 1936], [Rosser, 1936] and [Kleene, 1936]) is stated as: *There exists a recursively enumerable set of positive integers which is not recursive.*

Post continues with a version of Gödel's incompleteness theorem, a reworked version of his earlier incompleteness results from [Post, 1965]. The set of all bases $B_1, B_2, \ldots$ can be effectively enumerated, and hence so can the set T of all pairs (B, n), where n is in the set generated by the basis B. This set T is thus the same set as is generated by the complete normal system described in §4.1. If we define $\overline{T}$ as the set of all pairs (B, n) not in T, then $\overline{T}$ is not recursively enumerable, by a diagonal argument. Thus there is no formal logical system that proves a proposition of the form "The integer n is not in the set generated by the basis B" if and only if it is true. Post concludes with words echoing his earlier remarks from his unpublished manuscript of 1941 [Post, 1944, p. 295] [Davis, 1965, p. 316]:

> The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.* To the writer's mind, this conclusion must inevitably result in at least a partial reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth as being of the essence of mathematics.

With these words, Post sums up his earlier work on undecidability and incompleteness. The remainder of his paper sets out a program of research that was to determine a great deal of the direction of the new field of recursion theory. The emphasis is now on the classification of recursively enumerable sets, with the measure of complexity provided by notions of reducibility.

In his postdoctoral work at Princeton, Post had used an informal notion of reducibility in relating systems in his various canonical forms. If we translate Post's earlier notion of reducibility, cited above in §4, into the language of set theory, then it amounts to that of *one-one reducibility.* If S and T are sets of positive integers, then S is *many-one reducible* to T if there is a computable function f defined on the positive integers so that n is in S if and only if $f(n)$ is in T. S is *one-one reducible* to T if the function f is injective (one-one).

The intuitive idea of reducibility is that it measures the degree of difficulty of a problem. If a problem P is reducible to a problem Q, then a solution to Q would also lead to a solution of P. Different notions of reducibility arise, depending on the complexity of the reduction process. The reducibility relation is always assumed to be transitive and reflexive, so that we can classify sets in a partial ordering of degrees of unsolvability, where two sets are equivalent under the ordering if and only they are reducible to each other (in this case, they are said to be of the same *degree*). The most restrictive notion of reducibility defined by Post is that of one-one reducibility; later he introduces more and more general notions of reducibility, the most general notion being that of *Turing reducibility.*

Because the complete set K encodes positive answers to all questions of the form "Is the integer n in the set generated by the basis B?", it is not hard to see that every recursively enumerable set is one-one reducible to K. Conversely, it is easy to prove that if a set is many-one reducible to K, then it is recursively enumerable. Hence, the complete set K is of the highest degree of unsolvability relative to one-one reducibility (*1-complete*), and consequently also relative to many-one reducibility (*m-complete*).

Are there degrees of unsolvability of recursively enumerable sets strictly between the degree of recursive sets and that of K? This is the general problem addressed by Post in the later parts of his paper; he answers the question for some of the stronger (that is, more restrictive) concepts of reducibility, but not for the most general notion, Turing reducibility. As an initial step in his quest for intermediate degrees, Post begins by singling out a key property of the complete set K.

Fix a numerical encoding of all of the pairs (B, n). Then the image K of the set T under this encoding is a recursively enumerable set that is not recursive. The diagonal method that proves K not to be recursive is constructive. In Post's terminology, the set K is a *creative* set, that is to say, a recursively enumerable set C for which there exists a recursive function f giving a unique positive integer $n = f(i)$ for each basis B_i for a recursively enumerable set W_i of positive integers such that whenever W_i is a subset of $\overline{C}$ (the complement of C), then n is also in $\overline{C}$, but not in W_i. This key property of the complete set K is an abstract version of the Gödel incompleteness theorem, in the sense that Gödel's construction provides a constructive method that given a formal system for arithmetic in which only true statements are provable, produces another statement (the Gödel sentence for the system) that is also true, but unprovable in the system.

If S is a creative set, then its complement $\overline{S}$ contains an infinite recursively enumerable subset. This follows from the fact that starting from the empty set, we can construct successively larger subsets of $\overline{S}$ using the function $f(i)$ to obtain an infinite recursively enumerable subset of $\overline{S}$. If A and B are recursively enumerable, A is creative, and A is many-one reducible to B, then B is also creative. It follows that if B is an m-complete recursively enumerable set, that $\overline{B}$ contains an infinite recursively enumerable subset.

This last result motivates Post's next definition: a recursively enumerable set S of positive integers is *simple* if $\overline{S}$ is infinite, but does not contain any infinite recursively enumerable subset. A simple set cannot be recursive, for if it were, $\overline{S}$ would be an infinite recursively enumerable subset of itself. Post's next result is a clever diagonalization that proves that a simple set exists. For every basis B_i, his construction guarantees that a member of W_i (the set generated by B_i) is placed in S, provided W_i is infinite, so that this member is a witness to the fact that W_i is not a subset of $\overline{S}$. At the same time, Post constrains the number of witnesses that can be added so that $\overline{S}$ is infinite.

The existence of a simple set shows that there is a recursively enumerable set of intermediate degree in the ordering according to many-one reducibility. In fact, this result can be strengthened by defining a more general notion of reducibility. If a set A is reducible to a set B, then we can think of the reduction as being given in terms of membership queries. So, for example, if A is many-one reducible to B, we put an integer n in A if the answer to the query "Is $f(n)$ in B?" is positive, otherwise we leave it out. However, we could just as well leave n out if the answer is positive, and otherwise put it in. In other words, it is quite reasonable to say that the complement $\overline{S}$ of a set is reducible to the set S itself. This idea leads immediately to the more general notion of truth-table reducibility.

Define a *Boolean query to a set* B to be a propositional formula built out of the Boolean connectives $\wedge, \vee, \sim$, in which the atomic formulas are all of the form $(m \in B)$, where m is a positive integer; for example, the formula

$$[(5 \in B) \wedge \sim(67 \in B)] \vee [\sim(42 \in B) \wedge (87 \in B)].$$

We define a set A to be *truth-table reducible* to a set B if there is a recursive function f so that for all n, $f(n)$ is a Boolean query to B, and n is in A if and only if B satisfies the query $f(n)$. If, in addition, there is a fixed upper bound on the number of atomic formulas in the Boolean queries $f(n)$, then we say that A is *bounded truth-table reducible* to B.

Post is able to show that if S is a simple set, then a creative set cannot be reduced to S by bounded truth-table reductions. However, this result does not generalize to unbounded truth-table reductions, since he proves that if C is any creative set, then there is a simple set S so that C is reducible to S by unbounded truth-table reductions.

This last result leads Post to generalize the notion of simplicity to that of a *hyper-simple set*. A hyper-simple set H is defined to be a recursively enumerable set of positive integers so that $\overline{H}$ is infinite, but there is no infinite recursively enumerable set of mutually exclusive finite sequence of positive integers so that each sequence has at least one member in $\overline{H}$ ("mutually exclusive" means that the sequences have no elements in common). The notion of hyper-simplicity is powerful enough to show that there are intermediate degrees of unsolvability in the truth-table ordering, since Post proves, first that a hyper-simple set exists, and second that no creative set can be reduced to a hyper-simple set by a truth-table reduction.

Although the notion of truth-table reducibility is fairly broad, it is far from the most general notion of reducibility that can be defined. Let us imagine that we are trying to decide whether or not an integer belongs to a set A, while having access to information about the answers to all possible queries of the form: "Is integer n in set B?" Alan Turing, in his doctoral thesis [Turing, 1939, §4], [Davis, 1965, p. 166] suggested the picturesque metaphor of having access to an "oracle" for the set B, a terminology that is now standard. Then we say that A is *Turing-reducible* to B if we can decide whether or not an integer is in A, using a computer that has access to an oracle for B.

The definition given in the previous paragraph is not mathematically precise, but it can be made so in a variety of ways. The method usually adopted is to define a Turing machine with an auxiliary tape on which numbers can be written as queries to the oracle. When the machine enters a query of the form "Is the integer n in the set B?", the computation continues in one of two different ways, depending on the answer from the oracle (the oracle's answers are considered to be instantaneous, so that a query counts as a single computational step). Post himself described an alternative approach [Post, 1948] based on a generalization of his canonical sets to S-canonical sets, in which primitive assertions representing propositions of the form "Integer n is in S" and "Integer n is not in S" are added to a canonical system.

Post was unable to solve the problem of whether there exist intermediate degrees of unsolvability for the most general notion of reducibility, Turing-reducibility. This became known as "Post's problem," and work towards its solution drove a great deal of the research stimulated by his paper. He expressed some doubts as to whether hyper-simple sets would provide examples of intermediate degrees, but was not able to decide the question. The matter was not fully clarified until after Post's problem itself had been solved. The paper [Jockusch and Soare, 1973] shows that if H is Post's original example of a hyper-simple set, then it may or may not be Turing-complete, depending on the precise enumeration of bases $B_1, B_2, \ldots, B_n, \ldots$ adopted.

Post's program of research aimed at solving the problem of the existence of intermediate degrees was to define recursively enumerable sets with "sparse" complements, an idea implemented in the definition of simple and hypersimple sets. However, although this approach succeeded in the case of the strong reducibilities, he was not able to make it work for Turing reducibility. The solution to the problem was finally achieved by a different approach, the priority method of Friedberg and Muchnik, discussed below in §6.5.

6.3 The Post Correspondence Problem

In his brief note [Post, 1946], Post provided an addendum to [Post, 1943], in which he presents an unsolvable problem of striking simplicity. He considers strings composed of the two symbol alphabet $\{a, b\}$. An instance of the *correspondence problem* is given by a finite list $(g_1, h_1), (g_2, h_2), \ldots, (g_m, h_m)$ of pairs of non-empty strings, for example, the list $(bb, b), (ab, ba), (b, bb)$. A *solution* to such a problem consists of a finite sequence $\sigma = \sigma_1, \ldots, \sigma_k$, where $1 \leq \sigma_i \leq k$, so that $g_{\sigma_1} g_{\sigma_2} \ldots g_{\sigma_k} = h_{\sigma_1} h_{\sigma_2} \ldots h_{\sigma_k}$; a solution to the example just given is provided by the sequence 1223, since in this case $g_1 g_2 g_2 g_3 = bbababb = h_1 h_2 h_2 h_3$.

The preceding example has a very simple solution. Much more difficult instances of the correspondence problem can be constructed. As a challenge to the reader, here are two instances that are known to be solvable, but whose solutions are very difficult:

Instance 1: [Lorentz, 2001] $(abb, a), (b, abb), (a, b)$;

Instance 2: [Zhao, 2002] $(aaba, a), (baab, aa), (a, aab)$.

Post proves unsolvability of his correspondence problem by reducing the decision problem for normal systems to it. He does not give any applications of the problem. However, because of its simplicity, it later turned out to be an ideal problem for proving undecidability results in formal language theory. Bar-Hillel, Perles and Shamir [Bar-Hillel *et al.*, 1961] applied the correspondence problem to show that several problems in the theory of formal languages are undecidable. Among the problems they prove undecidable is that of deciding whether two context-free languages have a non-empty intersection (that is, given two phrase structure grammars, determining whether or not the languages generated by the grammars have a word in common). The Post correspondence problem is now a standard tool of formal language theory – see for example the textbook [Hopcroft and Ullman, 1979].

6.4 The Word Problem for Semigroups

By the 1940s, the broad significance of the concept of recursive, or computable functions was already clear. The universality and robustness of the concept of computability, and its applicability to the classical decision problems in logic was well known to the logical community. Nevertheless, the applications had remained within the realm of logic, and the fruitfulness of the idea outside this area remained to be shown. Post provided a major step in this direction by demonstrating that a decision problem that had arisen outside the area of logic and foundations was unsolvable.

The Norwegian mathematician Axel Thue in his paper [Thue, 1914] had posed the word problem for finitely presented semigroups. The word problem for such a semigroup is given as follows. There is a finite set of primitive letters, representing the generators of the semigroup, from which words can be formed using a binary associative operation. For example, if the generators of the semigroup are a, b, c, then the words $((ab)c)a$ and $a(b(ca))$ are considered to be the same, so that the words are treated as strings of letters. Then a *presentation* of a semigroup with these generators is given by a finite set of equations, for example, the equations $ab = ba$ and $abc = bcaa$. The word problem is, given such a presentation, to determine whether or not a given equation is derivable using the standard rules for reasoning with equations, where the generators are treated as constants.

Alonzo Church suggested to Post that Thue's problem, might be proved unsolvable as an application of [Post, 1946]. In fact, however, Post solved the problem not by using his correspondence problem, but by translating the formalism of Turing machines into the notation of his canonical systems. A presentation of a semigroup can be rewritten as a set of production rules of the form $P\sigma Q \Longrightarrow P\tau Q$, where σ and τ are strings over a finite alphabet. A set of production rules of this form Post defines as a *semi-Thue system*. A *Thue system*, which corresponds to a semigroup presentation, has the added property that whenever the production $P\sigma Q \Longrightarrow P\tau Q$ is in the set, so is its inverse $P\tau Q \Longrightarrow P\sigma Q$. For example, the presentation given above can be rewritten as the set of productions

$$PabQ \Longrightarrow PbaQ, PbaQ \Longrightarrow PabQ, PabcQ \Longrightarrow PbcaaQ, PbcaaQ \Longrightarrow PabcQ.$$

The decision problem for a semi-Thue system is the problem of determining whether or not a word τ can be derived from a given word σ using the production rules of the system. The word problem of Thue is equivalent to the decision problem for Thue systems. Post's strategy to prove unsolvability of this problem is to reduce the decision problem for a special class of semi-Thue systems to the decision problem for Thue systems.

This proof strategy is most easily carried out for rules that are *deterministic*, in the sense that at most one rule is applicable to any string derivable from an initial assertion. This is why canonical systems derived from Turing machines are ideal for this problem. A configuration of a Turing machine can be described as a string consisting of an endmarker h, followed by a sequence of tape symbols, followed by a symbol for an internal state of the machine, then another string, and finally another endmarker symbol h. For example, the string $h110q_301h$ represents a configuration of a machine in which the internal state of the machine is q_3, and the machine is scanning the fourth symbol of the string 11001 on an otherwise blank tape. The rules of the Turing machine are then easily rewritten as Post production rules, for example, the rule $Pq_31Q \Longrightarrow P1q_3Q$ represents the Turing machine instruction "In state q_3, if you are scanning a 1, then move right."

Post starts from an unsolvable problem in the theory of Turing machines. It is an unsolvable problem to determine whether or not a Turing machine, with internal states $q_1, \ldots, q_r$, started in its initial state q_1, reading the first symbol in a binary string σ on its input tape, halts in its highest-numbered state q_r on a blank tape. Rewriting the machine instructions as a semi-Thue system T', this decision problem becomes: Is there a derivation in the system T' of the string hq_rh from the initial string $hq_1\sigma h$? Let T'' be the system consisting of all the inverses of the productions in T'. Then hq_rh is derivable in T' from $hq_1\sigma h$ if and only if $hq_1\sigma h$ is derivable from hq_rh in T'. Now let T be the system that has all the rules of T', together with all the rules of T'' (that is, all the rules of T' and their inverses). Then because of the deterministic character of Turing machine rules, exactly the same strings are derivable from the initial string hq_rh in T as are derivable in T''. This reduces the Turing machine problem to the decision problem for a Thue system, and so shows Thue's problem of 1914 to be unsolvable.

Very simple unsolvable cases of Thue's problem are known. Building on Novikov's proof of the unsolvability of the word problem for groups (see below), [Tseitin, 1958] shows that the word problem for the semigroup with five generators $\{a, b, c, d, e\}$ and defining relations $ac = ca, ad = da, bc = cb, bd = db, eca = ce, edb = de, cca = ccae$ is unsolvable.

The formulation of Turing machines used by Post is not the original formulation of Turing, but a simplified and clarified version due to Post himself. It is in fact Post's version of the Turing machine that is the standard "Turing machine" presented in introductory texts on logic and computer science. Post includes an appendix to his paper providing a detailed critique of Turing's original paper.

The Russian logician Markov proved the same result independently and almost at the same time as Post [Markov, 1947]. These results of Post and Markov were just the beginning of a host of mathematical problems proved undecidable by the technique of reducing known unsolvable problems to them. A notable result achieved somewhat later is the unsolvability of the word problem for finitely presented groups, proved by the Russian logician P.S. Novikov in [Novikov, 1955], and independently though somewhat later by the American logician William Boone [Boone, 1957a][Boone, 1957b].

A few years later, Post showed that another concrete decision problem (though of a logical nature) was undecidable, in joint work with Samuel Linial (who later changed his name to Samuel Gulden). Linial and Post showed that there is an undecidable subsystem of classical propositional logic by showing how to encode a system of normal productions in a propositional calculus with detachment as a primitive rule. This research was published only as an abstract [Linial and Post, 1949]; an exposition of their results is to be found in [Davis, 1958, pp. 137-142].

6.5 Degrees of unsolvability

The joint paper [Kleene and Post, 1954] is an important landmark in the development of recursion theory. It had its origins in an encounter between Kleene and Post at the meeting of the American Mathematical Society at which Post presented [Post, 1944] on February 26 1944. In an interview of 1984, Kleene recalled:

> I was in the Navy when he presented his paper "Recursively Enumer-

> able Sets ..." at a meeting of the American Mathematical Society. I was teaching midshipmen in the U.S.S. John Jay. That's not a ship, it's a Columbia dormitory. But when midshipmen entered, they said "Request your permission to come aboard, sir." I only had to walk 200 yards to go to the lecture. After the lecture I had Post over to my apartment. I had an apartment just off the Columbia campus. There I presented to him – I think it was already in press, if it wasn't, it was in manuscript – my paper, "Recursive Predicates and Quantifiers," which had a very close relation to what he was doing. That's the first he knew of that [Aspray, 1984].

This paper [Kleene, 1943] is a fundamental contribution in which the arithmetical hierarchy is introduced. The bottom level of Kleene's hierarchy is the family of recursive sets of natural numbers, denoted by $\Sigma_0 = \Pi_0$. For $n > 0$, a set B of natural numbers is in Σ_n if there is a recursive relation $R(x, y_1, \ldots, y_n)$ so that for any natural number x,

$$x \in B \Longleftrightarrow \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n R(x, y_1, \ldots, y_n),$$

where Q is $\exists$ if n is odd, and $\forall$ if n is even. Similarly, a set B of natural numbers is in Π_n if there is a recursive relation $R(x, y_1, \ldots, y_n)$ so that for any natural number x,

$$x \in B \Longleftrightarrow \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n R(x, y_1, \ldots, y_n),$$

where Q is $\exists$ or $\forall$ according as n is even or odd. This hierarchy of sets generalizes the classification of sets as recursive and recursively enumerable, since the family of recursively enumerable sets is identical with Σ_1, while the complements of such sets (co-r.e. sets) constitute Π_1. By generalizing the diagonal argument that shows the existence of recursively enumerable but non-recursive sets, Kleene proves that this hierarchy is proper. That is to say, the obvious inclusions $\Sigma_n \subseteq \Sigma_{n+1}$, $\Pi_n \subseteq \Pi_{n+1}$, $\Sigma_n \subseteq \Pi_{n+1}$ and $\Pi_n \subseteq \Sigma_{n+1}$ are all proper.

Post was stimulated by Kleene's work to generalize his work on reducibilities between recursively enumerable sets to a much broader theory of degrees of unsolvability in work reported in the abstract [Post, 1948]. Recall from §4.1 Post's notion of "finite-normal-test" – a set M of integers is decidable just in case there is a finite-normal-test for it, that is, a normal system that has assertions exactly representing which integers are and are not elements of M. Post generalizes his earlier notion of canonical sets to S-canonical sets by adding to his canonical systems primitive assertions representing membership and non-membership of integers in S, where S is an arbitrary set of integers. Using this broader concept, he generalizes his earlier characterization of decidable sets by showing that a set S_1 is Turing-reducible to a set S if and only if both S_1 and its complements are S-canonical sets.

Furthermore, the existence of the complete set K generalizes to the relativized world of oracle computations. If A is an arbitrary set of integers, then there is a set $A' = K^A$ that has the property: K^A can be enumerated by a machine employing an oracle for A, and furthermore, any set of integers that can be enumerated by a machine employing an oracle for A (an A-canonical set) is reducible to K^A. If $\mathbf{a}$ is the Turing degree of the set A, define $\mathbf{a}'$ (the *jump* of $\mathbf{a}$) to be the degree of the set $A' = K^A$. As a generalization of the theorem proving the existence of recursively enumerable sets that are not recursive, we can show that $\mathbf{a}$ is reducible to $\mathbf{a}'$, but $\mathbf{a}'$ is not reducible to $\mathbf{a}$.

This definition results in a hierarchy of degrees of unsolvability, with respect to Turing reducibility. If $\mathbf{0}$ is the degree of $\emptyset$ (the degree of the recursive sets), then $\mathbf{0}'$ is the degree of the complete set K, and this is the beginning of a strictly ascending chain of degrees $\mathbf{0}, \mathbf{0}', \mathbf{0}'', \ldots$ that can be extended to transfinite levels. The results of Post just quoted imply a relationship between the arithmetical hierarchy of Kleene and Post's hierarchy of degrees, a result usually called *Post's Theorem*. Post's theorem says that $\Sigma_{n+1} \cap \Pi_{n+1}$ is exactly the sets of integers reducible to $\mathbf{0}^{(n)}$, where $\mathbf{0}^{(0)} = \mathbf{0}$, and $\mathbf{0}^{(n+1)} = (\mathbf{0}^{(n)})'$. Hence, Post's degree hierarchy is interleaved between the levels of Kleene's hierarchy.

Post's theorem is the main completed result reported in the abstract [Post, 1948]. However, he hints at further results. "Work is in progress on further equivalence proofs, further applications of the K_n-scale, incomparable degrees of unsolvability related to the K_n-scale, extension of the main theorem to Mostowski, Fund. Math. vol. 34, and extension of the K_n-scale into the constructive transfinite." These later results, though, did not appear in print, and Kleene had occasion to remonstrate with Post on this score. Kleene tells the story in a conversation (also involving Andrzej Mostowski, Gerald Sacks, Michael Morley and Anil Nerode) recorded in [Crossley, 1975, pp. 18-20].

> **Mostowski:** I must raise a protest against this habit of not publishing. That is all right here, because you meet every second month to collaborate at a conference here or there, but people like me are completely cut off.
>
> **Kleene:** Yes.
>
> **Sacks:** I agree completely.
>
> **Morley:** It should lapse.
>
> **Sacks:** And anyone who does not publish his work should be penalized.
>
> **Kleene:** This is just what I wrote to Emil Post, on construction of incomparable degrees and things like that, and he made some remarks and hinted at having some results and I said (in substance): "Well, when you leave it this way, you say you have these results, you don't publish them. The fact that you have them prevents anyone else who has heard of them from doing anything on it." So he said (in substance): "You have sort of pricked my conscience and I shall write something out," and he wrote some things out, in a very disorganized form, and he suggested that I give them to a graduate student to turn into a paper. As I recall, I think I did try them on a graduate student, and

the graduate student did not succeed in turning them into a paper, and then I got interested in them myself, and the result was eventually the Post-Kleene paper [Kleene and Post, 1954].

Morley: You mean one of your graduate students could have had the Post-X paper?

Nerode: You mean one that wanted to work hard.

Kleene: I suppose, maybe it was not good for a graduate student, because a graduate student needs a thesis he can publish under his own name, and this would have had to be joint, or maybe ...

Morley: Oh, I don't know. I think he could have borne having a Post-X paper.

Kleene: As a matter of fact, it could have – if a graduate student had picked it up. There were things that Post did not know, like that there was no least upper bound. [1] You see, Post did not know whether it was an upper semi-lattice or a lattice. I was the one who settled that thing.

Sacks: What are you talking about? The degrees of the arithmetic sets?

Kleene: No. The upper semi-lattice of degrees of unsolvability.

The paper [Kleene and Post, 1954] was written by Kleene, and its precise but very formal style forms a stark contrast with Post's own rather discursive mathematical prose. In it, the degree hierarchy is defined in terms of Kleene's own formalism of general recursive functions [Kleene, 1936], [Kleene, 1952a, Chapter XI]. The notion of reducibility is that of a function being general recursive in certain other functions, to which various earlier notions of reducibility were known to be equivalent. Post had proved in unpublished work [Post, 1948] that his own notion of canonical reducibility was equivalent to Turing reducibility [Turing, 1939], Martin Davis had proved in his Princeton doctoral thesis [Davis, 1950] that canonical reducibility is equivalent to general recursive reducibility, while Kleene had given a direct proof of the equivalence of Turing reducibility and general recursive reducibility [Kleene, 1952a, §68, §69].

The relation "A is recursive in B" is reflexive and transitive, so the relation "A is recursive in B and B is recursive in A" is an equivalence relation. Hence, this equivalence relation partitions the family of all sets of natural numbers into equivalence classes, or *degrees of unsolvability*. There are $\aleph_0$ sets in each equivalence class and hence $2^{\aleph_0}$ degrees, since there are $2^{\aleph_0}$ sets of natural numbers. The family of degrees forms an upper semi-lattice, that is, given degrees $\mathbf{a}$ and $\mathbf{b}$, there is a least upper bound $\mathbf{a} \cup \mathbf{b}$ in the degree ordering. This is because, given any two sets A and B of degrees $\mathbf{a}$ and $\mathbf{b}$, we can define a set C (for example,

[1] There seems to be an error in the transcript here. Presumably "greatest lower bound" is meant.

$\{2a : a \in A\} \cup \{2b + 1 : b \in B\}$) so that A and B are both recursive in C, while C is recursive in A, B.

The main result for which [Kleene and Post, 1954] is remembered is the proof of the existence of incomparable degrees between a degree $\mathbf{a}$ and its jump $\mathbf{a}'$. The technique employed by Kleene and Post involves a countable set of conditions, or "requirements," as they are usually called today. Given a set B, there are a countable number of Turing machines $M_0, M_1, \ldots, M_k, \ldots$ that use an oracle for B. Hence, we can give an effective list of requirements of the form: "The set C is not identical with the set computed by Turing machine M_k with an oracle for B," and a similar list with C and B interchanged. Any such condition can be fulfilled by ensuring the presence or absence of a particular integer in B or C, so Kleene and Post's construction consists in constructing B and C step by step, while ensuring that all such conditions are eventually fulfilled. Their construction of A and B, however, requires an oracle for the set K^A (where A has the degree $\mathbf{a}$) and so it does not provide an answer to Post's problem.

The paper contains a rich collection of additional results. The construction of intermediate degrees sketched in the preceding paragraph is generalized to show that between the degrees $\mathbf{a}$ and $\mathbf{a}'$, there is an infinite linearly ordered sequence of degrees that is dense, that is to say, if $\mathbf{c}, \mathbf{d}$ are degrees in the sequence, where $\mathbf{c} < \mathbf{d}$, there is a degree $\mathbf{e}$ so that $\mathbf{c} < \mathbf{e} < \mathbf{d}$. The paper also shows that the degree ordering is not a lattice, by showing that there are degrees $\mathbf{a}$ and $\mathbf{b}$ with no greatest lower bound in the degree ordering (this is the result to which Kleene alludes in the transcript above). Although the paper made considerable inroads into determining the fine structure of degrees, it also left many open problems, and provided a powerful stimulus for further work by later logicians.

It also provided an essential stepping stone in the solution of Post's problem by two young mathematicians, the first an American [Friedberg, 1957] and the second a Russian [Muchnik, 1956]. Both Friedberg and Muchnik followed Kleene and Post in constructing a pair of incomparable degrees by the technique of listing an infinite set of requirements that are all fulfilled at the end of the construction. To avoid the use of an oracle, however, requires a more subtle technique. In the original Kleene-Post method, a requirement, once fulfilled, remains fulfilled permanently and requires no further attention. In the constructions of Friedberg and Muchnik, however, a requirement can be violated, or "injured," at an intermediate stage. To ensure that a requirement is eventually fulfilled in spite of these injuries, Friedberg and Muchnik impose a priority ordering on their requirements, ensuring that only finitely many injuries can happen to a given requirement, so that it is eventually satisfied. Consequently, this new technique, which is one of the fundamental methods in recursion theory, is now known as the "priority method."

Since the original solution of Post's problem, recursion theory has become a discipline of extraordinary depth and richness, and the priority method has been used in more and more complex settings. Excellent expositions of the resulting theory are available in the books [Rogers, 1967], [Shoenfield, 1971], [Soare, 1987], [Odifreddi, 1989] and [Odifreddi, 1999].

7 PROVABILITY AND DEFINABILITY

The last major project of Post's career grew out of his early interest in the question of absolutely undecidable propositions. Logical developments in the early twentieth century had apparently shown that notions of provability and definability are not absolute. In the case of provability, the incompleteness theorems demonstrate that any consistent formal axiomatic theory for elementary number theory can be consistently extended by the addition of an unprovable sentence, so that it seems that we can not identify any fixed or absolute notion of provability. Similarly, it would appear that the diagonal method, applied to any constructively defined collection of definable notions, allows us to find a new definable concept not included in the original list.

Post was very impressed by the absolute character of the concept of computable function, since the extent of the computable functions does not depend on the formalism chosen, a point that he emphasized repeatedly. This led him to hope that a similar absolute notion of provability might be attainable, leading to absolutely undecidable propositions.

Already in the early 1920s, Post was interested in the question of the existence of propositions that are absolutely undecidable in some sense. He describes "A Probably Fallacious Suggestion for a Non-Provable Theorem" in the Appendix to his unpublished manuscript of 1941 [Post, 1965, pp. 421-422] [Post, 1994, p. 432]. He considers the enunciation: "For each deductive system of the normal form and enunciation in it there exists a finite method of proving or disproving the derivability of the enunciation in the system," and attempts to argue that if it were provable or disprovable, this would contradict the unsolvability of the decision problem for normal systems.

The preceding attempt proved abortive; Tarski pointed out to Post that his attempted argument was invalid because the supposed proof might be non-constructive. However, he continued his attempts along these lines, and in a footnote to the 1941 manuscript [Post, 1965, pp. 341-342][Post, 1994, p. 376] says that "since February 1938 we have given an occasional week to a continuation of this work, and largely in the spirit of the Appendix. Our goal, however, is now an analysis of proof, perhaps leading to an absolutely undecidable proposition, rather than an analysis of finite process."

The question of the existence of absolutely undecidable propositions was one of the topics that he discussed with Gödel at their meeting in October 1938, as appears from an entry of 4 November 1940 in one of Post's notebooks:

> In first meeting with Gödel (about a year ago or more [probably spring 1939]) [2] suggested to him abs. undec. prob. He said perhaps Cont. Hyp. I said this more like parallel axiom, i.e. would merely mean different theories of classes. Wanted rather abs. undec. arith. prop. where analyse all possible methods of proof & perhaps find a property

[2] In fact the meeting was on October 29 1938, as described above.

> of all such which would lead to abs. und. arith. prob. Gödel poo-poohed idea. Said (roughly) it was absurd [Grattan-Guinness, 1990, pp. 82-83].

It appears that Post continued this discussion with Gödel at a second meeting, for his diary entry continues:

> In second meeting, Oct. 1940, I asked him what he was working on now ... Said to prove negative of con. hyp. consistent with rest of set theory. Said he hadn't succeeded yet, but if so would be abs. undec. prob. Raised ‖ axiom analogy again. But he said no, that axioms of set theory categorical for all models [Grattan-Guinness, 1990, p. 83].

Gödel expressed an opinion in print around this time related to those reported by Post (though considerably more cautious). In his first announcement of his consistency proof [Gödel, 1938], he wrote about the axiom of constructibility ("proposition A"), from which the generalized continuum hypothesis can be deduced within the system T of set theory without the axiom of choice:

> The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency proof for A does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase.

Later, of course, he changed his mind on this point, and in [Gödel, 1947] espoused the view that new axioms of set theory would be found that would decide the continuum hypothesis (most likely in a negative sense).

Post, however, was not looking for absolutely undecidable propositions in the realm of set theory, where he appears to have thought that alternative set theories were possible, in analogy with alternative theories of geometry. His own goal was to find such problems in the realm of arithmetic. In his paper of 1943, he refers again to the problem in a footnote to a passage in which he states that the decision problem for normal systems is unsolvable:

> Absolutely unsolvable, that is, to use a phrase due to Church. By contrast, the undecidable propositions of Gödel's epoch making paper of 1931 ... are but relatively undecidable, the very proof of their undecidability in the given logic leading to an extension of that logic in which they are, indeed, proved to be true. A fundamental problem is the question of the existence of absolutely undecidable propositions, that is, propositions which in some *a priori* fashion can be said to have a determined truth-value, and yet cannot be proved or disproved by any valid logic [Post, 1943, p. 200], [Post, 1994, p. 445].

Even though Post reported that Gödel had "poo-poohed" his idea of absolutely undecidable propositions in 1938, it is significant that in his Gibbs lecture of 1951, Gödel asserted the possibility of absolutely undecidable arithmetical propositions, though only as one half of a disjunction. The lecture is devoted to the philosophical consequences of the incompleteness theorems. Gödel considers arguments somewhat resembling those given in [Lucas, 1961] and [Penrose, 1989], but (unlike these later writers) drawing not an absolute but a disjunctive conclusion:

> Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified [Gödel, 1995, p. 310].

Post continued to work along the lines suggested by the footnote to his 1943 paper, searching for a concept of provability in arithmetic that could play a similar role in the theory of numbers as recursivity in the theory of algorithms. The only remaining published record of his thought on this subject in the late 1940s is in the brief abstract [Post, 1953b], in which he says:

> In 1947, in a lost letter to Tarski via Church, the author proposed a formulation of "Primitive Inductive-Reflective Proof" involving, besides the elementary, only mathematical induction and "Gödelization." But representation theory for Gödel's system P (1931) failed to materialize due to P's Axiom of Reducibility. Indeed, Kleene's Example (Proceedings of the International Congress of Mathematicians, 1950, vol. II, p. 683) offers hope of an impossibility proof.

In his Princeton doctoral thesis [Turing, 1939] [Feferman, 1995] Turing studied the possibility of overcoming arithmetical incompleteness by the successive adjunction of unprovable sentences, given by Gödel's construction – this is presumably the process that Post describes as "Gödelization." Post seems to have hoped to show that the adjunction of axioms involving higher types was in some sense conservative with respect to first-order arithmetical sentences, but encountered insuperable difficulties in connection with the type-theoretical comprehension axiom, or "Axiom of Reducibility," as he calls it. The result of Kleene to which Post alludes is a theorem in [Kleene, 1952b] showing that Brouwer's fan theorem [Brouwer, 1924] fails if "choice sequence" is interpreted to mean "recursive function." In classical terms, he shows that there is an infinite recursive binary tree with no infinite recursive branch [Odifreddi, 1989, p. 506].

The conclusion that Post drew from his abortive attempts towards constructing an absolute notion of provability is that he was attacking the problem in the wrong order. The right order, he now thought, was: solvability, definability, provability. The immediately succeeding abstract [Post, 1953a] reports his success in finding a candidate for the role of an absolute notion of definability.

An obvious difficulty that arises in attempting to give an absolute notion of definability is this: if not all ordinals are definable, then there must be a least indefinable ordinal (this will certainly be true if we have only countably many definitions). But if our notion of definability is itself definable, then a contradiction (Richard's paradox) results immediately, since the least indefinable ordinal would then be definable.

A possible way out of this impasse is to take the notion of definability to be highly non-constructive, so that the definable sets form a proper class, and all ordinal numbers are definable. Post carries out this idea by extending the simple theory of types of order ω, as formulated in [Gödel, 1931], to transfinite levels, with a type level for all ordinals α. Thus Gödel's system P is extended to a system $P^{(\alpha)}$, with variables $x_0^{(\alpha)}, x_1^{(\alpha)}, \ldots$ ranging over sets of type α in the type hierarchy, and the usual apparatus of connectives and quantifiers. Post remarks: "The present thesis is that every definable set is given by some α-formula" (his intention was to give a necessary condition for definability, rather than a full-fledged explication, as in the case of computability).

Post was unaware that a closely related absolute concept of definability had been proposed earlier by Gödel, in his remarks at the Princeton Bicentennial Conference on Problems in Mathematics held in 1946. Gödel's paper was not published at the time, and the full text of his address first appeared in print in the collection edited by Martin Davis [Gödel, 1965, pp. 84-88].

Gödel's remarks strikingly parallel Post's own thinking on these topics. Commenting on an earlier lecture by Tarski, he remarks that the great importance of the concept of general recursiveness is largely due to the fact that "with this concept one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen." Contrasting the situation with that for provability and definability, he says:

> By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion. This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability) [Gödel, 1965, p. 84].

Gödel goes on to make a very tentative suggestion about an absolute concept of provability, conjecturing that any set-theoretical assertion can be decided by a sufficiently powerful axiom of infinity, so that "every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets."

He continues with "somewhat more definite suggestions," proposing definability in terms of ordinals as an absolute notion of definability in set theory. His proposal is to extend the usual language of Zermelo-Fraenkel set theory by including primitive names for all ordinal numbers (so that the formulas of the extended language form a proper class); ordinal definable sets are those specified by a formula from this extended language. This concept of definable set is broader than his earlier notion of constructible set, used in proving the consistency of the generalized continuum hypothesis with the axioms of set theory, since the quantifiers in the extended language range over all sets, whereas constructible sets are defined by quantifiers restricted to the constructible universe.

Post's own notion is slightly more restrictive than that of Gödel, and is now known as that of "hereditary ordinal definability"; a set is hereditarily ordinal definable if it is not only ordinal definable, but its members, members of members ...etc. are also ordinal definable. The notion of ordinal definability was rediscovered again independently during the 1960s [Takeuti, 1961], [Myhill and Scott, 1971].

Because of Post's untimely death, there is no other published record of his last logical project. However, two notebooks on "Definability" written in 1952-1954 remain in the archives of the American Philosophical Society in Philadelphia, and may still repay investigation.

8 POST'S LEGACY

The remarkable fertility of Post's ideas, particularly in the areas of recursion theory and computer science, is attested by the number of concepts and results named after him: "Post-completeness," "Post's problem," "Post productions," "Post correspondence problem," "Post's classification theorem" are some of the more prominent examples. His influence on computer science is most remarkable, considering that he seemingly never had any contact with computing machinery or computer programming.

Post's ideas also bore fruit in the area of formal linguistics. The early work of Noam Chomsky on phrase structure grammars was influenced by the ideas of Post; Chomsky formulates basic notions such as that of a context-free language in terms of rules closely akin to Post productions [Chomsky, 1956]. He seems to have known of these ideas indirectly through the unusual textbook of Paul Rosenbloom [Rosenbloom, 1950], in which Post's production systems are given a starring role, particularly in the last chapter. Seymour Ginsburg, an important early contributor to formal language theory, and author of a well known monograph giving a comprehensive overview of early work on context-free languages [Ginsburg, 1966], was a student with Martin Davis in an honors mathematics class taught by Post [Gleiser, 1980]. All five of the students in this course, described by Davis as a "pressure cooker experience," became mathematicians.

Post also exerted influence through the students who were inspired by him and his remarkable classes. Martin Davis has described Post's teaching methods:

> Post's classes were tautly organized tense affairs. Each period would begin with student recitations covering problems and proofs of theorems from the day's assignment. These were handed out apparently at random and had to be put on the blackboard without the aid of

> textbook or notes. Woe betide the hapless student who was unprepared. He (or rarely she) would have to face Post's "more in sorrow than anger" look. In turn, the students would recite on their work. Afterwards, Post would get out his 3 by 5 cards and explain various fine points. The class would be a success if he completed his last card just as the bell rang. Questions from the class were discouraged: there was no time. Surprisingly, these inelastic pedagogic methods were extremely successful, and Post was a very popular teacher [Davis, 1994, p. xxv].

David Hilbert, in his address before the International Congress of Mathematicians in Paris in 1900, had posed a series of problems to challenge mathematicians in the new century. Hilbert's tenth problem reads as follows:

> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers* [Hilbert, 1902].

A diophantine equation is any algebraic polynomial equation with whole number coefficients, for example: $31x^2 + 41y^3 - 17x = 0$. Hilbert's problem asks for an algorithm to determine whether or not an arbitrary diophantine equation is solvable (in line with his rationalistic and optimistic outlook, he seems to have expected a positive solution). In his address of 1944, Post remarked that Hilbert's problem "begs for an unsolvability proof" [Post, 1944, p. 289]. The challenge was taken up by Post's student Martin Davis, who was one of the key contributors to the proof of unsolvability, together with Hilary Putnam and Julia Robinson. The final step completing the proof was provided by the young Russian mathematician Yuri Matiyasevich [Matiyasevich, 1970]. The reader will find excellent and very readable accounts of the history of this problem in [Reid, 1996] and [Yandell, 2002]. A fully detailed account of the proof, together with much else, is in the beautifully written book [Matiyasevich, 1993].

8.1 Post's Papers

Post's papers are housed at the American Philosophical Society in Philadelphia. They include correspondence, unpublished manuscripts, research notes, and photographs documenting Post's career. The notebooks were donated by Martin Davis in 1986; the remaining papers were donated by Phyllis Post Goodman in 1992. The correspondence includes letters from Paul Bernays, Alonzo Church, Martin Davis, Frederic Fitch, Abraham Fraenkel, Kurt Gödel, Stephen Kleene, W.V. Quine and Alfred Tarski. Ivor Grattan-Guinness has given a description of the papers [Grattan-Guinness, 1990], and has transcribed an unpublished manuscript from 1935 [Post, 1990]. This manuscript is a somewhat sketchy essay on Russell's paradox, in which Post works out the idea that the class of existing classes may not be fixed, but rather can expand through the process of thinking about existing objects. This vision of mathematical objects as processes that evolve through time is of course one of the fundamental ideas that runs through all of Post's work, starting from his dissertation, where he introduces the class of elementary propositions by describing "the vision of the totality of these functions streaming out from the unmodified variable p through forms of ever growing complexity ..." [Post, 1921a, p. 165].

This obsession with time also manifests itself throughout the surviving notebooks. There are four major numbered series, five volumes on "Closed Truth Systems: Towards a new presentation," written in 1929-1931 when Post was working out the new version of his results on closed system of truth functions published as [Post, 1941], a series of eighteen volumes on "Creative Logic" written 1938-1952, a series of seventeen volumes on "Theory of Finite Processes" (Volume IV is missing), written 1944-1951, and two volumes on "Definability," written 1952-1954. In addition, there are notebooks on "Calculus of Finite Processes" 1944, "Complete Equivalence of Normal Set and Recursive Function Development" 1942-1945 and "The Logic of Mathematics."

In his notebooks, Post frequently makes a meticulous record of clock times and of time spent working. An example quoted by Grattan-Guinness is of a manuscript on the Laplace transform. Post noted that he was writing the paper on 19 September 1923, starting at 10:08 AM, and ending at 12:28 PM, with three running times noted within the text [Grattan-Guinness, 1990, p. 79]. Post's original attack of mania occurred when he was enormously excited about his new ideas in logic in 1921, and subsequently he attempted to avoid periods of great mental excitement both by severely restricting the time he spent on research, and by alternating between one research project and another. Martin Davis has described the onset of one of his manic episodes in 1947:

> With another student, I had begun an "honors" reading course on mathematical logic under Post's tutelage. We had just reached the "deduction theorem" in propositional calculus, when Post met us full of excitement about his new work. We did not meet again with Post that semester; we had witnessed the beginning of one of his manic episodes. Apparently it was precisely excitement that had the potential to push him over the edge [Davis, 1994, p. xxiv].

It is impossible not to feel deep admiration for Post's remarkable achievements in mathematical logic, accomplished while labouring under such a severe handicap.

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also a reliable source of biographical information. The online MacTutor History of Mathematics archive at the University of St Andrews, Scotland was also helpful. The Aspray interview with Kleene was obtained through the online version of the transcripts of the Princeton mathematical oral history project, obtainable at the web site of the Seeley G. Mudd Manuscript Library.

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