Problem Corner:

Seventy-Five Problems for Testing Automatic Theorem Provers

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The purpose of this note is to provide a graduated selection of problems for use in testing an automatic theorem proving (ATP) system. Included in these problems were the ones I have used in testing my own ATP system (Pelletier 1982, and more recent updates to the system). Some of the problems, especially some of the more difficult ones described at the end of this note, are due to discussions with Len Schubert (Computing, Univ. Alberta), Alasdair Urquhart (Philosophy, Univ. Toronto), and Charles Morgan (Philosophy, Univ. Victoria).

People who have tried to compile lists of problems for ATPs in the past (for example, the Association for Automated Reasoning) have discovered that the production of such a list is difficult because (a) what is 'easy' for one system might not be for another, (b) researchers are understandably shy about saying what problems their ATP might have, unless they know that it is so difficult that any ATP will have trouble with it, (c) especially with problems at the 'easy end' of the scale, researchers are prone to think that any ATP system can prove it and so there is no call to write them up, (d) the goals of 'natural' system ATPs and resolution-based ATPs are different: the former tries to produce a 'natural' proof usually without prior conversion to clause form. This means that some problems, especially at the 'easy end' of the scale, will be trivial for resolution systems but difficult for 'natural' systems. On the other hand, proponents of 'natural' systems think that at the 'difficult end' of the scale, there will be problems within the grasp of the 'natural' systems which are beyond the reach of resolution systems. It is therefore difficult to even give a graduated scale of problems. All this has led to publication of difficult problems, but not to publication of lists of problems suitable for developing an ATP. The locus classicus of problems is McCharen et al. (1976) which contains a wide range of problems all in clause form. More recently, the Journal of Automated Reasoning has instituted its 'Problem Corner'. A notable item herein is Lusk and Overbeek (1985). All of us involved in ATP wish to encourage others – graduate students, for example – to look into the field. But where is such a person to start? It is with such neophyte
ATPers in mind that the following list is offered. None of these problems will be the sort whose solution is, of itself, of any mathematical or logical interest. Such 'open problems' are regularly published in the *Newsletter* of the Association for Automated Reasoning. Most (but not all) of my problems can be found in elementary logic textbooks – but they have been chosen either because logic students find them difficult, or because previous ATP systems have reported difficulties in establishing them, or because they have some interesting connection with other areas of mathematics (such as set theory). The judgements of difficulty are my own, and are based primarily on my years of teaching elementary logic (so the judgements reflect how difficult beginning students find the problems). The scales are from 1 (easiest) to 10, and are relativized to the kind of problem involved. Thus; a '9' in the propositional logic section might in fact be easier than a '4' in the monadic logic section. I give problems in both 'natural' and clause form. The negated-conclusion clause form has eliminated tautologous clauses (but see the remark below in the 'Acknowledgement' section). As mentioned, many of the 'easier' problems – especially in propositional calculus – are trivial in resolution systems. Still, 'natural' systems might find them diverting. In either case, however, I would think that any ATP ought to produce an explicit proof, detailing what preceding formulas/clauses were invoked and (in the case of 'natural' systems) what rules of inference were used. When comparing two ATPs for efficiency (say by measuring CPU time on identical machines) it is of course mandatory that the systems should include the cost of conversion to clause form since it is only very rarely that problems 'in the real world' are presented in clause form.

* A Word on Notation

→: if-then
¬: not
+: or
&: and
↔: if and only if
A: for all
E: there exists
p, q, r, s . . . (perhaps with subscripts): propositional (sentence) letters; that is, 0-place predicate letters.
F, G, . . . P, Q, . . . (perhaps with subscripts): predicate letters; context is used to determine the adicity.
x, y, z, w, . . . (perhaps with subscripts): (individual) variables.
a, b, c . . . (perhaps with subscripts): individual constants; that is, 0-place function symbols.
f, g, h . . . (perhaps with subscripts): function symbols; context is used to determine the adicity.
**Precedence**
\[-\lor A, E \text{ highest} \\
\&\lor + \text{ medium} \\
\rightarrow\leftrightarrow \text{ lowest}\]

**Associativity**
\& and + are allowed to take arbitrarily many conjuncts/disjuncts. For the most part, the notation is in common use and should provide no difficulties.

**Propositional Logic**
What follows here are some propositional logic theorems and arguments. They are given both in 'natural' form and in negated-conclusion clause form. Some are of historical interest (mostly having to do with the logical theorist), while others illustrate various 'tricks'.

<table>
<thead>
<tr>
<th>'Natural' form</th>
<th>Negated-conclusion clause form</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (2pts) A biconditional version of the 'most difficult' theorem proved by the original Logic Theorist (Newell et al. 1957)</td>
<td>$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$</td>
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<tr>
<td></td>
<td>$\neg p + q$</td>
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<td></td>
<td>$\neg q + p$</td>
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<td></td>
<td>$\neg q$</td>
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<tr>
<td></td>
<td>$p$</td>
</tr>
<tr>
<td>2. (2pts) A biconditional version of the 'most difficult' theorem proved by the new logic theorist (Newell and Simon 1972)</td>
<td>$\neg \neg p \leftrightarrow p$</td>
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<tr>
<td></td>
<td>$p$</td>
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<tr>
<td></td>
<td>$\neg p$</td>
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<tr>
<td>3. (1pt) The 'hardest' theorem proved by a breadth-first logic theorist (Siklóssy et al. 1973)</td>
<td>$\neg (p \rightarrow q) \rightarrow (q \rightarrow p)$</td>
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<tr>
<td></td>
<td>$p$</td>
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<tr>
<td></td>
<td>$\neg q$</td>
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<td></td>
<td>$q$</td>
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<tr>
<td></td>
<td>$\neg p$</td>
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<tr>
<td>4. (2pts) Judged by Siklóssy et al. (1973) to be 'hardest' of first 52 theorems of Whitehead and Russell (1910)</td>
<td>$(\neg p \rightarrow q) \leftrightarrow (\neg q \rightarrow p)$</td>
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<tr>
<td></td>
<td>$p + q$</td>
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<tr>
<td></td>
<td>$\neg q + \neg p$</td>
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<tr>
<td></td>
<td>$\neg q$</td>
</tr>
<tr>
<td></td>
<td>$\neg p$</td>
</tr>
</tbody>
</table>
5. (4pts) Judged by Siklóssy et al. to be ‘hardest’ of first 67 theorems of Whitehead and Russell (1910)

\[
\begin{align*}
((p + q) \rightarrow (p + r)) \rightarrow & 
\neg q + p + r \\
(p + (q \rightarrow r)) \rightarrow & 
\neg \neg p \\
& 
q \\
& 
\neg \neg r
\end{align*}
\]

6. (2pts) The Law of Excluded Middle: can be quite difficult for ‘natural’ systems

\[
\begin{align*}
(p + \neg p) \rightarrow & 
\neg p \\
& 
p
\end{align*}
\]

7. (3pts) Expanded Law of Excluded Middle. The strategies of the original Logic Theorist cannot prove this

\[
\begin{align*}
(p + \neg \neg \neg \neg p) \rightarrow & 
\neg p \\
& 
p
\end{align*}
\]

8. (5pts) Pierce’s Law. Unprovable by Logic Theorist, and tricky for ‘natural’ systems.

\[
\begin{align*}
((p \rightarrow q) \rightarrow p) \rightarrow p \\
& 
\neg \neg p
\end{align*}
\]

9. (6pts) A problem not solvable by unit resolution nor by ‘pure’ input resolution.

\[
\begin{align*}
[(p + q) \& (\neg p + q) \& (p + \neg q)] \rightarrow & 
\neg (\neg p + \neg q) \\
p + q \\
\neg \neg p + q \\
p + \neg q \\
\neg p + \neg q
\end{align*}
\]

10. (4pts) A reasonably simple problem with premises, designed to see whether ‘natural’ systems correctly manipulate premises.

\[
\begin{align*}
q \rightarrow r & \rightarrow \neg q + r \\
r \rightarrow (p \& q) & \rightarrow \neg r + p \\
& \rightarrow \neg r + q \\
p \rightarrow (q + r) & \rightarrow \neg p + q + r \\
p \leftrightarrow q & \rightarrow p + q
\end{align*}
\]

11. (1pt) A simple problem designed to see whether ‘natural’ systems can do it efficiently (or whether they incorrectly try to prove the → each way)

\[
\begin{align*}
(p \leftrightarrow p) \rightarrow & 
p \\
& \neg p
\end{align*}
\]
12. (7pts) The 'hardest' propositional problem found in Kalish and Montague (1964), according to Pelletier (1982)

\[
[(p \leftrightarrow q) \leftrightarrow r] \leftrightarrow [p \leftrightarrow (q \leftrightarrow r)]
\]

\[
p + q + r \\
\neg q + \neg p + r \\
\neg r + \neg p + q \\
\neg r + \neg q + p \\
p + \neg q + r \\
p + \neg r + q \\
q + r + \neg p \\
\neg r + \neg q + \neg p
\]

Distribution Laws can be very tricky for 'natural' systems. (They are assumed by resolution systems in the conversion to clause form). The next few problems list some

13. (5pts)

\[
[p + (q \& r)] \leftrightarrow [(p + q) \& (p + r)]
\]

\[
p + q \\
p + r \\
\neg p \\
\neg q + \neg r
\]

14. (6pts)

\[
(p \leftrightarrow q) \leftrightarrow ((q + \neg p) \& (\neg q + p))
\]

\[
\neg p + q \\
\neg q + p \\
\neg q + \neg p \\
p + q
\]

15. (5pts)

\[
(p \rightarrow q) \leftrightarrow (\neg q + q)
\]

\[
\neg p + q \\
p \\
\neg q
\]

16. (4pts) A surprising theorem of propositional logic

\[
(p \rightarrow q) + (q \rightarrow p)
\]

\[
p \\
\neg q \\
q \\
\neg p
\]

17. (6pts) A problem which appears not to be provable by Bledsoe et al. (1972). (For details of why not, see Pelletier (1982), p. 135f).

\[
((p \& (q \rightarrow r)) \rightarrow s) \leftrightarrow
\]

\[
((\neg p + q + s) \& (\neg p + \neg r + s))
\]

\[
\neg p + q + s
\]

\[
\neg p + q + s
\]

\[
\neg p + q + s
\]

\[
\neg p + q + s
\]
Monadic Predicate Logic

Problems in the monadic predicate logic are not much more difficult than those in the propositional logic. All that’s required is a method of handling quantifiers correctly (by finding appropriate instances or substitutions). The problems in this section are designed to test whether this happens.

18. (1pt)
\[(Ey)(Ax)(Fy \rightarrow Fx) \quad Fx \quad \neg Ff(x)\]

19. (3pts)
\[(Ex)(Ay)(Az)((Py \rightarrow Qz) \rightarrow (Px \rightarrow Qx)) \quad \neg Pf(x) + Qg(x) \quad Px \quad \neg Qx\]

20. (4pts)
\[[(Ax)(Ay)(Az)(Aw)((Px & Qy) \rightarrow (Rz & Sw)) \rightarrow ((Ex)(Ey)(Px & Qy) \rightarrow (Ez) Rz)] \quad Pa \quad Qb \quad \neg Rw\]

21. (5pts) A moderately tricky problem, especially for ‘natural’ systems with ‘strong’ restrictions on variables generated from existential quantifiers.

\[(Ex)(p \rightarrow Fx) \quad \neg p + Fa \quad (Ex)(Fx \rightarrow p) \quad \neg Fb + p \quad (Ex)(p \leftrightarrow Fx) \quad p + Fx \quad \neg Fx + \neg p\]

Some problems having to do with ‘confinement’ of quantifiers. These are often trivial in resolution systems because they are assumed in conversion to clause form.

22. (3pts)
\[(Ax)(p \leftrightarrow Fx) \rightarrow (p \leftrightarrow (Ax) Fx) \quad p + \neg Fx \quad Fx + \neg p\]
23. (4pts)

\[(Ax)(p + Fx) \leftrightarrow (p + (Ax) Fx)\]

\[p + Fx + Fy\]
\[p + Fx + Fb\]
\[\neg p\]
\[\neg Fa + p + Fy\]
\[\neg Fa + \neg Fb\]

The following are some more tedious monadic logic problems from Kalish and Montague (1964).

24. (6pts)

\[\neg (Ex)(Sx \& Qx)\]
\[(Ax)(Px \rightarrow (Qx + Rx))\]
\[\neg (Ex)Px \rightarrow (Ex)Qx\]
\[(Ax)(Qx + Rx \rightarrow Sx)\]
\[(Ex)(Px & Rx)\]

\[\neg Sx + \neg Qx\]
\[\neg Px + Qx + Rx\]
\[Pa + Qb\]
\[\neg Qx + Sx\]
\[\neg Rx + Sx\]
\[\neg Px + \neg Rx\]

25. (7pts)

\[(Ex)Px\]
\[(Ax)(Fx \rightarrow (\neg Gx & Rx))\]
\[(Ax)(Fx \rightarrow (Gx & Fx))\]

\[Pa\]
\[\neg Fx + \neg Gx + \neg Rx\]
\[\neg Px + Fx\]
\[\neg Px + Gx\]

\[\neg Px + Qx + Pb\]

\[(Ex)(Qx & Px)\]

\[\neg Px + Qx + Rb\]
\[\neg Qx + \neg Px\]

26. (7pts)

\[(Ex)Px \leftrightarrow (Ex) Qx\]
\[(Ax)(Ay)(Px & Qy \rightarrow (Rx \leftrightarrow Sy))\]

\[\neg Px + Qa\]
\[\neg Qx + Pb\]

\[\neg Px + \neg Qy + \neg Rx + Sy\]

\[\neg Px + \neg Qx + Sx\]
\[\neg Px + Rx + Pc\]
\[\neg Px + Rx + \neg Rc\]
\[Qd + \neg Qx + Sx\]
\[Qd + Pc\]
27. (6pts)

\[(\exists x)(F_x \land \lnot G_x)\]
\[(A x)(F_x \rightarrow H_x)\]
\[(A x)(J_x \land I_x \rightarrow F_x)\]
\[
\frac{(\exists x)(H_x \land \lnot G_x) + (A x)(Z_x \land \lnot H_x)}{(A x)(J_x \rightarrow \lnot I_x)}
\]

\[\neg H_x + G_x + \lnot I_y \land \lnot H_y\]
\[J_b\]
\[I_b\]

28. (8pts)

\[\neg Pa + Qx\]
\[\neg Qb + Qc\]
\[\neg Rb + Qc\]
\[\neg Rb + Sc\]
\[\neg S_x + \lnot F_y + G_y\]
\[P_d\]
\[F_d\]
\[\neg G_d\]

29. (7pts)

\[(\exists x)F_x \land (\exists x)G_x\]
\[(A x)(F_x \rightarrow H_x) \land (A x)(G_x \rightarrow J_x)\]
\[
\frac{(A x)(A y)(F_x \land G_y \rightarrow H_x \land J_y)}{(\exists x)(A x)(F_x \land G_x \rightarrow H_x \land J_y)}
\]

\[-F_x + H_x + \lnot F_y + \lnot G_z + H_y\]
\[-F_x + H_x + \lnot F_y + \lnot G_z + J_z\]
\[-F_x + H_x + F_e + G_f\]
\[-F_x + H_x + \lnot H_e + G_f\]
\[-F_x + H_x + \lnot H_e + J_f\]
\[-G_x + J_x + \lnot F_y + \lnot G_z + H_y\]
\[-G_x + J_x + \lnot F_y + \lnot G_z + J_z\]
\[-G_x + J_x + F_e + G_j\]
\[-G_x + J_x + F_e + \lnot J_f\]
\[-G_x + J_x + \lnot H_e + G_f\]
\[-G_x + J_x + \lnot H_e + \lnot J_f\]
\[-F_c + \lnot F_x + \lnot G_y + H_x\]
30. (6pts)

\((Ax)(Fx + Gx \rightarrow \neg Hx)\)

\((Ax)((Gx \rightarrow \neg Ix) \rightarrow Fx & Hx)\)

\((Ax)Ix\)

\(Fc + \neg Fx + \neg Gy + Jy\)

\(Fc + Fe + Gf\)

\(Fc + Fe + \neg Jf\)

\(Fc + \neg He + Gf\)

\(Fc + \neg He + \neg Jf\)

\(Gd + \neg Fx + \neg Gy + Hx\)

\(Gd + \neg Fx + \neg Gy + Jy\)

\(Gd + Fe + Gf\)

\(Gd + Fe + \neg Jf\)

\(Gd + \neg He + Gf\)

\(Gd + \neg He + \neg Jf\)

\(\neg Hc + \neg Jd + \neg Fx + \neg Gy + Hx\)

\(\neg Hc + \neg Jd + \neg Fx + \neg Gy + Jy\)

\(\neg Hc + \neg Jd + Fe + Gf\)

\(\neg Hc + \neg Jd + Fe + \neg Jf\)

\(\neg Hc + Jd + \neg He + Gf\)

\(\neg Hc + \neg Jd + \neg He + \neg Jf\)

31. (5pts)

\(\neg (Ex)(Fx & (Gx + Hx))\)

\(Ex)(Ix & Fx)\)

\((Ax)(\neg Hx \rightarrow Jx)\)

\((Ex)(Ix & Jx)\)

\(\neg Ia\)

32. (6pts)

\((Ax)(Fx & (Gx + Hx) \rightarrow Ix)\)

\((Ax)(Ix & Hx \rightarrow Jx)\)

\((Ax)(Kx \rightarrow Hx)\)

\((Ax)(Fx & Kx \rightarrow Jx)\)

\(Fa\)

\(Ka\)

\(\neg Ja\)
33. (4pts) This is a monadic predicate logic formulation of problem (17) above.

\[(Ax)(Pa \& (Px \rightarrow Pb) \rightarrow Pc) \iff \neg Pa + Px + Pc + Py\]
\[(Ax)((\neg Pa + (Px + Pc)) \& (\neg Pa + (\neg Pb + Pc)))\]
\[\neg Pa + P_a + Pb + Pc\]

34. (10pts) Andrew’s Challenge (cf. de Champeaux (1979)). The problem is logically simple, but its size makes it difficult. (The conversion to clause form is left as an exercise for the reader, about 1600 clauses.)

\[\{(Ex)(Ay)(Px \leftrightarrow Py) \iff ((Ex)Qx \leftrightarrow (Ay) Py)\} \iff \]
\[\{(Ex)(Ay)(Qx \leftrightarrow Qy) \iff ((Ex)Px \leftrightarrow (Ay)Py)\}\]

### Full Predicate Logic (without Identity and Functions)

Once again we start with some problems to determine whether the quantifiers are being handled properly.

35. (2pts)

\[(Ex)(Ey)(Pxy \rightarrow (Ax)(Ay)Pxy) \iff Pxy \iff Pf(x, y)g(x, y)\]

36. (3pts)

\[(Ax)(Ey)Fxy \iff Fxf(x)\]
\[(Ax)(Ey)Gxy \iff Gxg(x)\]
\[(Ax)(Ay)(Fxy + Gxy \rightarrow (Az)(Fyz + Gyz \rightarrow Hxz)) \iff \neg Fxy + \neg Fyz + Hxz\]
\[(Ax)(Ey)Hxy \iff \neg Fxy + \neg Gyz + Hxz\]
\[\neg G_{xy} + \neg Fyz + H_{xz}\]
\[\neg Gxy + \neg Gyz + H_{xz}\]
\[\neg Hax\]

37. (3pts)

\[(Az)(Ew)(Ax)(Ey)[(Pzx \rightarrow Pyw) \& Pyz \& (Pyw \rightarrow (Eu)Quw)] \iff \neg Pyx + Pf(x, y)g(x)\]
\[(Ax)(Az)[\neg Pzx \rightarrow (Ey)Qyz] \iff \neg Pf(x, y), g(x) + Qh(x, y), g(x)\]
\[(Ex)(Ey)Qxy \rightarrow (Ax)Rxz \iff Px, y + Qi(x, y), x\]
\[(Ax)(Ey)Rxy \iff \neg Qx, y + Rz, z\]
\[\neg Ra, x\]
38. (4pts) Here is a full predicate logic version of problems (17) and (33). Conversion to clause form left as an exercise.

\[
\{ (Ax)[ (Pa \land (Px \to (Ey)(Py \land Rxy))) \to (Ez)(Ew)(Pz \land Rxw \land Rwz)] \leftrightarrow \\
(Ax)[(\neg Pa + Px + (Ez)(Ew)Pz \land Rxw \land Rwz) \land \\
(\neg Pa + \neg (Ey)(Py \land Rxy) + (Ez)(Ew)(Pz \land Rxw \land Rwz))] \}
\]

Some problems in set theory can be represented in first order logic using the predicate 'F' to stand for 'is and element of'. Here are some reasonably simple ones.

39. (3pts) Russell's paradox: there is no 'Russell set' (a set which contains exactly those sets which are not members of themselves)

\[-(Ex)(Ay)(Fyx \leftrightarrow \neg Fyy) \quad \neg Fxa + \neg Fxx \\
Fxx + Fxa \]

40. (5pts) If there were an 'anti-Russell set' (a set which contains exactly those sets which are members of themselves), then not every set has a complement.

\[-(Ey)(Ax)(Fxy \leftrightarrow Fxx) \to \\
(\neg (Ax)(Ey)(Az)(Fxy \leftrightarrow \neg Fzx) \to \\
\neg (Ax)(Ey)(Fxy \leftrightarrow Fxx) \to \neg Fxa + Fxx \\
\neg Fxx + Fxa \\
\neg Fx, f(x) + \neg Fyx \\
Fyx + Fx, f(x) \]

41. (6pts) The 'restricted comprehension axiom' says: given a set z, there is a set all of whose members are drawn from z and which satisfy some property. If there were a universal set then the Russell set could be formed, using this axiom. So given the appropriate instance of this axiom, there is no universal set.

\[ (Az)(Ey)(Ax)(Fxy \leftrightarrow Fxz) \to \neg Fx, f(y) + Fxy \\
\neg (Ez)(Ax)Fxz \to \neg Fx, f(y) + \neg Fxx \\
(\neg Fx, f(x) + Fxx + Fx, f(y) \\
Fxa \]

42. (6pts) A set is 'circular' if it is a member of another set which in turn is a member of the original. Intuitively all sets are non-circular. Prove there is no set of noncircular sets.

\[ (Ey)(Ax)(Fxy \leftrightarrow Fzx) \to \neg Fxa + Fxy + \neg Fyx \\
\neg (Ez)(Fxz & Fzx) \to \neg Fzf(x) + Fxa \\
Ff(x)x + Fxa \]

43. (5pts) de Champeaux (1979). Define set equality ('Q') as having exactly the same members. Prove set equality is symmetric.

\[ (Ax)(Ay)(Qxy \leftrightarrow (Az)(Fzx \leftrightarrow Fzy)) \\
(\neg Qxy + \neg Fzx + Fzy \to \\
(\neg Qxy + \neg Fzy + Fzx \to \\
Ff(x, y), x + Ff(x, y), y + Qxy \\
\neg Ff(x, y), y + \neg Ff(x, y), x + Qxy \]
Here are some problems taken from Kalish & Montague (1964)

44. (3pts)

\[(Ax)[Fx \to (Ey)(Gy & Hxy) \& (Ey)(Gy \& \neg Hxy)]\]
\[(Ex)[Jx & (Ay)[Gy \to Hxy]]\]
\[(Ex)(Jx & \neg Fx)\]

\[\neg Fx + Gf(x)\]
\[\neg Fx + Hx, f(x)\]
\[\neg Fx + \neg Hx, g(x)\]
\[Ja\]
\[\neg Gx + Hax\]
\[\neg Jx + Fx\]

45. (5pts)

\[(Ax)(Fx \& (Ay)[Gy & Hxy \to Jxy] \to (Ay)(Gy & Hxy \to Ky))\]
\[\neg (Ey)(Ly & Ky)\]
\[(Ex)[Fx \& (Ay)(Hxy \to Ly) \& (Ay)(Gy & Hxy \to Jxy)]\]
\[(Ex)(Fx & \neg (Ey)(Gy & Hxy))\]

\[\neg Lx + \neg Kx\]
\[Fa\]
\[\neg Hax + Lx\]
\[\neg Gx + \neg Hax + Jax\]
\[\neg Fx + Gg(x)\]
\[\neg Fx + Hx, g(x)\]

46. (6pts)

\[(Ax)(Fx \& (Ay)[Fy & Hxy \to Gxy] \to Gx)\]
\[{(Ex)(Fx & \neg Gx) \to (Ex)(Fx & \neg Gx & (Ay)(Fy & \neg Gy \to Jxy))}\]
\[(Ax)(Ay)(Fx & Fy & Hxy \to \neg Jyx)\]
\[(Ax)(Fx \to Gx)\]

\[\neg Fx + Ff(x) + Gx\]
\[\neg Fx + Hf(x) + Gx\]
\[\neg Fx + \neg Gf(x) + Gx\]
\[\neg Fx + Gx + Fa\]
\[\neg Fx + Gx + \neg Ga\]
\[\neg Fx + Gx + \neg Fy + Gy + Jay\]
\[\neg Fx + \neg Fy + \neg Hxy + Jyx\]
\[Fb\]
\[\neg Gb\]
A problem which has gotten some considerable play in the literature is 'Schubert’s Steamroller', after Len Schubert. (See Pelletier (1982), Walther (1985), McCune (1985), Stickel (1986)). The problem presented in English is this:

47. (10pts) Wolves, foxes, birds, caterpillars, and snails are animals, and there are some of each of them. Also there are some grains, and grains are plants. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves. Wolves do not like to eat foxes or grains, while birds like to eat caterpillars but not snails. Caterpillars and snails like to eat some plants. Therefore there is an animal that likes to eat a grain-eating animal.

The previously mentioned authors symbolize the problem (especially the conclusion) differently (see Stickel (1986) for details). In its original form it is as follows: Let

\[ P_0: a \text{ is an animal} \]
\[ P_1: a \text{ is a wolf} \]
\[ P_2: a \text{ is a fox} \]
\[ P_3: a \text{ is a bird} \]
\[ P_4: a \text{ is a caterpillar} \]
\[ Q_0: a \text{ is a plant} \]
\[ Q_1: a \text{ is a grain} \]
\[ S: a \text{ is much smaller than } b \]
\[ R: a \text{ likes to eat } b \]
In negated conclusion clause form, the problem becomes:

\[ \neg P_a \]
\[ \neg P_b \]
\[ \neg P_c \]
\[ \neg P_d \]
\[ \neg P_e \]
\[ Qf \]
\[ \neg P_1 x + P_0 x \]
\[ \neg P_2 x + P_0 x \]
\[ \neg P_3 x + P_0 x \]
\[ \neg P_4 x + P_0 x \]
\[ \neg P_5 x + P_0 x \]
\[ \neg Q_1 x + Q_0 x \]
\[ \neg P_0 x + \neg P_1 y + \neg Rxy \]

**Full Predicate Logic with Identity (without Functions)**

48. (3pts) 'A problem to test identity ATPs' – Piotr Rudnicki (Computing, Univ. Alberta).

\[ a = b + c = d \]
\[ a = c + b = d \]
\[ a = d + b = c \]
\[ a \neq d \]
\[ b \neq c \]

Here are some problems which straightforwardly test the identity components of ATPs without putting much strain on the rest of the system.

49. (5pts)

\[(Ex)(Ey)(Az)(z = x + z = y)\]
\[Pa \land Pb\]
\[a \neq b\]
\[(Ax)\neg P x\]

50. (4pts)

\[(Ax)[Fax + (Ay)Fxy] \rightarrow Fax + Fxy\]
\[(Ex)(Ay)Fxy\]
\[\neg Fxy(x)\]

51. (5pts)

\[(Ez)(Ew)(Ax)(Ay)[Fxy \leftrightarrow (x = z \& y = w)]\]
\[\neg Fxy + x = a\]
\[
\begin{align*}
\neg Fxy + y &= b \\
x \neq a + y \neq b + Fxy \\
\neg Ff(x), y + y &= g(x) + f(x) = x \\
\neg Ff(x), y + y &= g(x) + \\
Ff(x), h(x, z) + h(x, z) &= z \\
\neg Ff(x), y + y &= g(x) + \\
h(x, z) \neq z + \neg Ff(x), h, (x, z) \\
y \neq g(x) + Ff(x), y + \\
Ff(x), h(x, z) + h(x, z) &= z \\
y \neq g(x) + Ff(x), y + f(x) = x \\
y \neq g(x) + Ff(x), y + h(x, z) \neq z + \neg Ff(x), h(h, z) \\
f(x) \neq x + Ff(x), h(x, z) + \\
h(x, z) &= z \\
f(x) \neq x + h(x, z) \neq z + \\
\neg Ff(x), h(x, z) &
\end{align*}
\]

52. (5pts)
\[
\begin{align*}
(Ez)(Ew)(Ax)(Ay)[Fxy \leftrightarrow \\
(x = z \& y = w)] \\
(Ew)(Ay)[(Ez)(Ax)(Fxy \leftrightarrow \\
x = z) \leftrightarrow y = w]
\end{align*}
\]

\[
\begin{align*}
\neg Fxy + x &= y \\
\neg Fxy + y &= b \\
x \neq y + y \neq b + Fxy \\
\neg Fy, f(x) + y \neq g(x) + \\
f(x) &= x \\
\neg Fy, f(x) + y &= g(x) + \\
Fh(x, z)f(x) + \neg Fh(x, z), f(x) \\
y \neq g(x) + Fy, f(x) + f(x) &= x \\
y \neq g(x) + Fy, f(x) + \\
Fh(x, z), f(x) + h(x, z) &= z \\
y \neq g(x) + Fy, f(x) + h(x, z) \neq z + \neg Fh(x, z), f(x) \\
f(x) \neq x + Fh(x, z), f(x) + \\
h(x, z) &= z \\
f(x) \neq x + h(x, z) \neq z + \\
\neg Fh(x, z)f(x) &
\end{align*}
\]

53. (7pts) [Test your converter-to-clause-form: about 146 clauses.]
\[
(Ex)(Ey)[x \neq y \& (Az)(z = x + \\
z = y)]
\]
54. (9pts) Montague’s (1955) paradox of grounded classes.

\[ (Ay)(Ey)(Fxz \leftrightarrow x = y) \]
\[ \neg (Ey)(Fxw \leftrightarrow (Au)(Fxu \rightarrow (Fxu \rightarrow Fyu \& \neg (Eu)(Fxu \& Fuz))) \]
\[ (Ax)(x = 2) 4-b y = w \}

55. (8pts) The following problem was given by Len Schubert. For ATPs with an 'answer extraction' mechanism, the conclusion might replaced with the query ‘who killed Aunt Agatha?’

**English:** Someone who lives in Dreadsbury Mansion killed Aunt Agatha. Agatha, the butler, and Charles live in Dreadsbury Mansion, and are the only people who live therein. A killer always hates his victim, and is never richer than his victim. Charles hates no one that Aunt Agatha hates. Agatha hates everyone except the butler. The butler hates everyone not richer than Aunt Agatha. The butler hates everyone Agatha hates. No one hates everyone. Agatha is not the butler. Therefore: Agatha killed herself.

\[ (Ex)(Lx \& Kxa) \]
\[ La \& Lb \& Lc \]
\[ (Ax)(Lx \rightarrow x = a + x = b + x = c) \]
\[ (Ay)(Ax)(Kxy \rightarrow Hxy) \]
\[ (Ax)(Ay)(Kxy \rightarrow \neg Rxy) \]
\[ (Ax)(Hax \rightarrow \neg Hcx) \]
\[ (Ax)(\neg Rxa \rightarrow Hax) \]
\[ (Ax)(Hax \rightarrow Hbx) \]
\[ (Ax)(Ey) \neg Hxy \]
\[ a \neq b \]

\[ \neg Lx + x = a + x = b + x = c \]
\[ \neg Kxy + Hxy \]
\[ \neg Kxy + \neg Rxy \]
\[ \neg Hax + \neg Hcx \]
\[ x = b + Hax \]
\[ Rxa + Hbx \]
\[ \neg Hax + Hbx \]
\[ \neg Hx, f(x) \]
\[ a \neq b \]

\[ \neg Kaa \]
57. (2pts)

\[(Ax)((Ey)[Fy \& x = f(y)] \rightarrow Fx) \iff \neg Fx + y \neq f(y) + Fy + \neg Fz + Ff(z)\]

\[(Ax)[Fx \rightarrow Ff(x)]\]

\[\neg Fx + y \neq f(y) + Fy + Fa\]

\[\neg Fx + y \neq f(y) + Fy + \neg Fb\]

\[Fc + \neg Fx + Ff(x)\]

\[Fc + Fa\]

\[Fc + b = f(b)\]

\[Fc + \neg Fb\]

\[\neg Ff(c) + \neg Fx + Ff(x)\]

\[\neg Ff(c) + Fa\]

\[\neg Ff(c) + b = f(b)\]

\[\neg Ff(c) + \neg Fb\]

58. (3pts)

\[Ff(a, b), f(b, c)\]

\[Ff(a, b), f(b, c)\]

\[Ff(b, c), f(a, c)\]

\[Ff(b, c), f(a, c)\]

\[(Ax)(Ay)(Az)[Fxy \& Fyz \rightarrow Fxz]\]

\[Ff(a, b), f(a, c)\]

\[\neg Fxy + \neg Fyz + Fxz\]

59. (3pts)

\[(Ax)(Ay)f(x) = g(y)\]

\[f(x) = g(y)\]

\[f(x) = g(y)\]

\[f(f(a)) \neq f(g(b))\]

60. (4pts)

\[(Ax)[Fx, f(x) \iff (Ey)[(Az)(Fzy \rightarrow Fz, f(x)) \& Fxy}]\]

\[Fa, f(a) + \neg Fby + Fy, f(a)\]

\[Fa, f(a) + Fa, b\]

\[Fg(x), x + \neg Fax + \neg Fyb + Fy, f(a)\]

\[Fg(x), x + \neg Fax + \neg Fyb\]

\[Fg(x), x + \neg Fax + Fab\]

\[Fg(x), x + \neg Fax + \neg Fa, f(a)\]

\[\neg Fg(x), f(a) + \neg Fax + \neg Fby + Fy, f(a)\]

\[\neg Fg(x), f(a) + \neg Fax + Fab\]

\[\neg Fg(x), f(a) + \neg Fax + \neg Fa, f(a)\]

\[\neg Fg(x), f(a) + \neg Fax + \neg Fa, f(a)\]
Having warmed up with some straightforward examples, let’s turn our attention to some more difficult ones.

61. (6pts)

\[
\begin{align*}
(Ax)(Ay)(Az)f(x, f(y, z)) &= f(x, f(y, z)) = f(f(x, y) z) \\
(Ax)(Ay)(Az)(Aw) f(x, f(y, f(z, w))) &= f(f(f(x, y), f(z, w))) = f(f(f(x, y), f(z, w)))
\end{align*}
\]

62. (5pts) Here is the original formulation in Bledsoe et al. (1972), p. 59 of the problem mentioned in (17), (33), and (38).

\[
\begin{align*}
[Fa & \land (Ax)(Fx \rightarrow Ff(x))] \iff \\
\neg Fa + Fb + Ff(x) + Fy + Ff(y) \\
(Ax)[[\neg Fa + Fx + Ff(x)] & \land \\
\neg Fa + Fb + Ff(x) + \\
\neg Ff(y) + Ff(y) \\
\neg Fa + \neg Ff(b) + Ff(x) + Fy + Ff(y) \\
\neg Fa + \neg Ff(b) + Ff(x) + \\
\neg Ff(y) + Ff(y) \\
\neg Fa + \neg Ff(b) + Ff(x) + Fz + Ff(z) \\
\neg Fa + \neg Ff(b) + Ff(x) + \\
\neg Ff(d) \\
Fa \\
\neg Fc + Ff(c) + \neg Fa + Fy + Ff(y) \\
\neg Fc + Ff(c) + \neg Fa + \\
\neg Ff(y) + Ff(y) \\
\neg Fc + Ff(c) + \neg Fz + Ff(z) \\
\neg Fc + Ff(c) + \neg Ff(d) \\
\neg Ff(f(c)) + \neg Fa + Fy + Ff(y) \\
\neg Ff(f(c)) + \neg Fa + Fy + Ff(y) \\
\neg Ff(f(c)) + \neg Fa + \neg Ff(y) + Ff(y) \\
\neg Ff(f(c)) + \neg Fa + \neg Ff(y) + Ff(y) \\
\neg Ff(f(c)) + \neg Fz + Ff(z) \\
\neg Ff(f(c)) + \neg Fz + Ff(z) \\
\neg Ff(f(c)) + \neg Ff(f(d)) \\
\neg Ff(f(c)) + \neg Ff(f(d))
\end{align*}
\]

Here are three group theory problems, published I think, by Larry Wos (but I cannot locate where). Consider
(a) \((Ax)(Ay)(Az)f(f(x,y), z) = f(x, f(y, z))\)
\(f(f(x,y), z) = f(x, f(y, z))\)

(b) \((Ax)f(a, x) = x\)
\(f(a, x) = x\)

(c) \((Ax)(Ey)f(y, x) = a\)
\(f(g(x), x) = a\)

63. (6pts) Show that (a), (b), (c) entail
\[(Ax)(Ay)(Az)[f(x, y) = f(z, y) \rightarrow x = z]\]
\(b \neq d\)

64. (6pts) Show that (a), (b), (c) entail
\[(Ax)(Ay)[f(y, x) = a \rightarrow f(x, y) = a]\]
\(f(c, b) = a\)

65. (8pts) Show that (a) and (b) entail
\[[(Ax)f(x, x) = a \rightarrow f(x, x) = a]\]
\(f(b, c) \neq a\)

Charles Morgan (see AAR Newsletter #3) has suggested a method of constructing difficult function-problems out of easy propositional logic ones. To do this, take a propositional logic theorem, eg \((\neg \neg P \rightarrow P)\), treat the propositional letters as objects which can be quantified over, encode the sentence operators \((\neg\) and \(\rightarrow\), here\) as functions, and treat 'is a theorem' as a (monadic) predicate 'T'. Thus, this theorem would become \((Ax)Ti(n(n(x)), x)\). Do this for every axiom of the logic under consideration, and treat the results as premises for every argument. The rules of inference of the logic are treated in this manner by saying that if the premise of the rule has 'T' applied to it then the conclusion of the argument has 'T' applied to it. So for the rule modus ponens: if \(P\) and \((P \rightarrow Q)\) are derivable, then so is \(Q\), we would convert this to \((Ax)(Ay)(Ti(x, y) \& Tx \rightarrow Ty)\), and treat it as a premise of every argument. That this can generate quite difficult problems out of very simple ones is demonstrated by the following examples of Morgan's. The initial propositional logic has three axioms plus modus ponens:

(a) \((Ax)(Ay)T(i(x, i(y, x)))\)
(b) \((Ax)(Ay)(Az)T(i(i(x, i(y, z)), i(i(x, y), i(x, z)))\)
(c) \((Ax)(Ay)T(i(i(n(x), n(y)), i(y, x)))\)
(d) \((Ax)(Ay)(T(i(x, y)) \& T(x) \rightarrow T(y))\)

66. (7pts) From (a)–(d) prove
\((Ax)T(i(x, n(n(x))))\)

67. (7pts) From (a)–(d) prove
\((Ax)T(i(n(n(x)), x))\)
68. (8pts) Replace (c) with \((Ax)(Ay)T(i(i(y, x), i(n(x), n(y))))\) and prove \((Ax)T(i(x, n(n(x))))\)

Morgan’s method is completely generalizable. The examples above used as an ‘object language’ only \(\rightarrow\) and \(\neg\), but we could have added on other connectives together with appropriate definitions or axioms. For example, we might add on \(\leftrightarrow\) together with one or more of the definitions of \(\leftrightarrow\) in terms of \(\rightarrow\) and \(\neg\) and perhaps with new axioms for \(\leftrightarrow\) and perhaps with new rules of inference for \(\leftrightarrow\). For instance, we might add on

\[
(Ax)(Ay)(T(i(x, y)) \& T(i(y, x)) \leftrightarrow T(e(x, y)))
\]

and/or

\[
(Ax)(Ay)e(x, y) = n(i(i(x, y), n(i(y, x))))
\]

and/or

\[
(Ax)(Ay)(T(e(x, y)) \& T(x) \rightarrow T(y))
\]

and/or

\[
(Ax)(Ay)e(x, y) = e(n(x, n(y))
\]

and/or

\[
(Ax)(Ay)e(x, y) = e(y, x)
\]

and/or

\[
(Ax)(Ay)n(e(x, y)) = e(x, n(y))
\]

and the like. With sufficient of these, you can try to prove (the translations of) arguments using \(\rightarrow, \neg, \leftrightarrow\). They are very difficult. But the method can be generalized even further and extended to other logics which include propositional logic as a part.

Let us suppose we have a characterization of the propositional logic using Morgan’s method. That is, suppose we have translated a complete set of propositional axioms and rules of inference, such as the ones mentioned before Problem 66, into the function-notation just mentioned. Now suppose we wish to consider modal propositional logics. These logics introduce one new propositional operator, \(L\) (logical necessity). The modal system \(T\) can be described as

(a) propositional logic axioms
(b) Modus Ponens
(c) \(L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq)\)
(d) \(Lp \rightarrow p\)
(e) if \(p\) is a theorem, then \(Lp\) is a theorem.

Morgan’s method as described shows how to get the translation of (a) and (b). Thus the function-notation version of system \(T\) would be arrived at by introducing a new function symbols for \(L\), say ‘\(b_1\)’, and an axiomatization of system \(T\) would be

(a) (the translations of the propositional logic axioms)
(b) (the translation of Modus Ponens)
(c) \((Ax)(Ay)T(i(b_1(i(x, y)), i(b_1(x), b_1(y))))\)
(d) \((Ax)T(i(b_1(x), x))\)
(e) \((Ax)(T(x) \rightarrow T(b_1(x)))\)

69. (9pts) Using the (a)–(e) just given as premises, translate simple theorems of the modal system \(T\) and prove them. For example:

\((Ax)T(i(b_1(x), n(b_1(n(x))))))\)
70. (open problem) The example in (69) used the modal system \( T \), and \( b_1 \) was the function corresponding to the \( L \) of system \( T \). Now let's consider the modal system \( K \), and let's represent its \( L \) by \( b_2 \). The axioms and rules of inference for \( K \) are translated as

(a) (translations of the propositional logic axioms)
(b) (translation of Modus Ponens)
(f) \((Ax)(Ay)T(i(b_2(i(x, y)), i(b_2(x), b_2(y))))\)
(g) \((Ax)(T(x) \rightarrow T(b_2(x)))\)

In Pelletier (1985) I posed the question of whether there might not be a way to 'define' the \( b_1 \) function in terms of the other functions (including \( b_2 \)), and a way to 'define' the \( b_2 \) function in terms of the other functions (including \( b_1 \)), so that if \( X \) was a theorem which mentioned \( 'b_1' \) but not \( 'b_2' \), then the result of replacing \( 'b_1' \) (and its argument) by its 'definition' would result in a theorem. [And conversely for a theorem \( Y \) which mentioned \( 'b_2' \) but not \( 'b_1' \).] These 'definitions' were also to have the following feature: let \( f_1 \) be the 'definition' of \( 'b_1' \) in terms of \( 'b_2' \) and \( f_2 \) be the 'definition' of \( 'b_2' \) in terms of \( 'b_1' \). Then

(h) \((Ax)T(e(x, f_1(f_2(x))))\)
(i) \((Ax)T(e(x, f_2(f_1(x))))\)

were also to be true. That is, the result of 'translating' any sentence purely of one of the sublanguages into the other sublanguage can be 'translated' back into the first sublanguage and the result will be provably equivalent to the original sentence. Therefore, given (a)–(i) the problem is to first determine whether there are such \( f \)'s and second find out what they are.

Some Problems for Studying the Computational Complexity of ATP's

The difficulty in constructing problems for studying the complexity of the proof system of an ATP is to describe a set of problems whose complexity can independently be characterized in terms of some metric which can be varied and which does not introduce any 'side effects' into the resulting proofs. Various attempts to state such a set of problems have usually focussed on (a) number of clauses, (b) number of symbols, (c) number of distinct symbols. It is extremely difficult to guarantee in advance that the increase in proof size which is observed when (say) the number of clauses is increased is due solely to the increase in number of clauses, as opposed to being also influenced by some hidden increase in 'number of tricks required' (say). The following problem-types are designed to give a measure of complexity which does not involve any other types of difficulty as the problems get more complex.

71. (U-problems, after Alasdair Urquhart).

Consider the following problems (the sub-problem of conversion to clause form is left to the reader).
The number of distinct sentence letters in $U_n$ is $n$, the number of occurrences of sentence letters is $2n$. The number of embedded $\leftrightarrow$'s is $(2n-1)$. The number of clauses goes up dramatically as $U_n$ increases, but I don't think it shows that the problems are dramatically more difficult as we go from $U_2$ to $U_3$, say. Rather, it's that the awkward clause form representation comes to the fore most dramatically with embedded biconditionals. On all other measure of increase of complexity from $U_1$ to $U_n$, one should say that the problems increase linearly in difficulty. So, given that the $U$-series of problems increases linearly in difficulty, compare the increase of your ATP along the dimensions of CPU time, size of proof, number of program statements executed, etc. [Alasdair Urquhart informs me that the proof size of any resolution system increases exponentially with increase in $n$].

Suppose there are $n$ holes and $(n + 1)$ objects to put in the holes. Every object is in a hole and no hole contains more than one object. Pictorially, we can represent this (for the 4 object, 3 hole problem) as:

```
   A   B   C
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
  2  |   |   |
  |   |   |   |
  |   |   |   |
  |   |   |
  3  |   |
  |   |
  4  |
```

Each cell $(i, j)$ says that the $i$th object is in the $j$th hole. For each cell use a different sentence letter ($P_1$, $P_2$, ) for the 4-object, 3 hole problem we might assign the letters in this fashion:
Each object is in a hole’ becomes (for this example)

(a) \( P_1 + P_2 + P_3 \)
(b) \( P_4 + P_5 + P_6 \)
(c) \( P_7 + P_8 + P_9 \)
(d) \( P_{10} + P_{11} + P_{12} \)

‘No hole has more than one object in it’ becomes

\[ (e) \neg P_1 + \neg P_4 \]
\[ (f) \neg P_1 + \neg P_7 \]
\[ (g) \neg P_1 + \neg P_{10} \]
\[ (h) \neg P_4 + \neg P_7 \]
\[ (i) \neg P_4 + \neg P_{10} \]
\[ (j) \neg P_7 + \neg P_{10} \]
\[ (k) \neg P_2 + \neg P_5 \]
\[ (l) \neg P_2 + \neg P_8 \]
\[ (m) \neg P_2 + \neg P_{11} \]
\[ (n) \neg P_5 + \neg P_8 \]
\[ (o) \neg P_5 + \neg P_{11} \]
\[ (p) \neg P_8 + \neg P_{11} \]
\[ (q) \neg P_3 + \neg P_6 \]
\[ (r) \neg P_3 + \neg P_9 \]
\[ (s) \neg P_3 + \neg P_{12} \]
\[ (t) \neg P_6 + \neg P_9 \]
\[ (u) \neg P_6 + \neg P_{17} \]
\[ (v) \neg P_9 + \neg P_{12} \]

The set of clauses (a)–(v) are inconsistent, as will any set be which is generated this way when the number of holes is less than the number of objects. Picking the number of objects to be one greater than the number of holes will yield the
'hardest' problem (for that number of holes), so we will just consider the 'n-hole problems', assuming that the number of objects is \((n + 1)\). It will be noticed that, for an \(n\)-holes problem, the number of distinct sentence letters is \(n^2 + n\) and the number of clauses is

\[
\frac{n^3 + n^2}{2} + (n + 1)
\]

Thus the number of sentence letters increases quadratically and the number of clauses increases cubically. In any case, it seems that the 'difficulty' of the \(n\)-hole problems increases polynomially with \(n\). See if your ATP can emulate that.

73. (Predicate logic pigeon hole problems). Problem (72) represented the \(n\)-hole problems by distinct sentence letters. This problem can also be represented by selecting a predicate '0' meaning that \(x\) is an object, and a predicate 'H' meaning that \(x\) is a hole. Let \(Ixy\) be a 2-place relation saying that (object) \(x\) is in hole \(y\). The 3-hole problem then becomes

\[
\begin{align*}
(a) \ & (Ex)(Ey)(Ez)(Ew)[Ox & Oy & Oz & Ow & x \neq y & x \neq z & x \neq w & y \neq z & y \neq w & z \neq w] \\
(b) \ & (Ex)(Ey)(Ez)[Hx & Hy & Hz & x \neq y & x \neq z & y \neq z & \\
& (Ax)(Hy \rightarrow (Ey)(Hy & Ixy)) \\
& (Ax(Hy \rightarrow (Ay)(Az)(Oy & Oz & Ixy & Izx \rightarrow y = z))
\end{align*}
\]

These four formulas (a)–(d) are inconsistent, as are any generated in this manner. Test how your ATP increases in effort spent as the number of holes increases.

74. (Arbitrary graph problems. Due to Tseitin (1968), see Galil (1977) for an expository version. See also Urquhart (unpublished a, b). My thanks to Urquhart for explaining them to me.) Consider a graph with the edges labelled. For example

\[
\begin{array}{c}
A \\
\bullet \\
B \\
C \\
D \\
E
\end{array}
\]

Assign 0 or 1 arbitrarily to nodes of the graph. For each node of the graph, we associate a set of clauses as follows:

1. every label of an edge emanating from that node will occur in each clause (of the set of clauses generated from that node)
2. if the node is assigned 0, then the number of negated literals in each of the generated clauses is to be odd. Generate all such clauses for that node.
(3) if the node is assigned 1, then the number of negated literals in each of the generated clauses is to be even. Generate all such clauses for that node.

Tseitin's result is this: the sum (mod 2) of the 0's and 1's assigned to the nodes of the graph equals 1 if and only if the set of all generated clauses is inconsistent. For example, if we assign the node at the top of the above graph a 1 and all others 0, then the set of all generated clauses will be inconsistent. The clauses generated are:

(a) \( A + B \)  
(b) \( \neg A + \neg B \) from top node, which was assigned 1, so the number of negated literals is even in each clause.
(c) \( A + C + \neg D \)  
(d) \( A + \neg C + D \) from left node, which was assigned 0, so the number of negated literals is odd in each clause
(e) \( \neg A + C + D \) (generate all possible such clauses)
(f) \( A + \neg C + \neg D \)
(g) \( B + C + \neg E \)  
(h) \( B + \neg C + E \) (which was assigned 0)
(i) \( \neg B + C + E \)  
(j) \( \neg B + \neg C + \neg E \)
(k) \( D + \neg E \)  
(l) \( \neg D + E \) generated from bottom node

Clauses (a)–(l) are inconsistent: prove that with your ATP. Now, we can increase the complexity of the graph in a number of different ways. Pick a 'natural' way and see how your ATP's proof increases CPU or a number of resolvants generated, etc.

75. According to Alasdair Urquhart, the \( U \)-problems of problem (71) can be graphically represented by the method of problem (74). The relevant graph is

\[
\text{where the double-circled node is assigned 1 and all others are assigned 0. Have your ATP prove this. (Number of vertical lines is number of distinct sentence letters).}
\]

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