Non-Normal Propositional Modal Logics and their Tableaux Rules

Before Kripke put some order into the study of modal logics, there were a number of studies from the purely syntactic (axiomatic) point of view that identified many different modal systems. The first and most influential of these studies was by C.I. Lewis (jointly authored with C.H. Langford, 1932, "Symbolic Logic"). This work identified the modal systems S_1, S_2, S_3, S_4, and S_5. (They were not developed as we nowadays do it, with unary modal operators $\Box$ and $\Diamond$, but rather with an 'if-then' connective that was symbolized $⥽$, and often called "the fishhook" or "strict implication"). Lewis' idea was to start out with some minimal notion of a conditional, and then add on to it, little-by-little, more constraints; and that was how he developed the five systems. Nowadays, we like to define the fishhook away by the definition $\Phi ⥽ \Psi =_{df} \Box(\Phi \supset \Psi)$... which is also equivalent to $\neg\Diamond(\Phi \wedge \neg\Psi)$. [In fact, Lewis' favorite contender for "if–then" was in S_2].

S_4 and S_5 are normal logics (they are called KT4 and KT5 (also KT4B) in my list of systems as developed by the axioms... Priest characterizes them semantically as $K\rho\tau$ and $K\upsilon$), respectively. The non-normal logics S_1, S_2, and S_3 can't be given a possible worlds analysis along the same lines as normal ones–with a binary accessibility relation on possible worlds and the truth of a $\Box\Phi$ formula being defined as Φ being true at all accessible worlds. However, in 1965 Kripke gave an account using the notions of normal vs. non-normal possible worlds.

In a non-normal world, $\Box\Phi$ is false, no matter what Φ is. So, even $\Box(p \vee \neg p)$ and $\Box(p \supset p)$ are false at such worlds. (In turn, this means that the rule of necessitation can not be universally applied, since $\models (p \vee \neg p)$ and $\models (p \supset p)$ – so these formulas are true at every world, even the non-normal ones. So necessitation can't be done on them, at least not at every world.) Priest says that, although non-normal worlds were introduced as a purely technical device to give a semantics for S_2 and S_3, they "have a perfectly good philosophical meaning."

Using these non-normal worlds, we evaluate formulas to be theorems (and arguments to be valid) if they are true (valid) in every normal world. So, although non-normal worlds are employed in evaluating modal sentences (e.g., a $\Diamond\Phi$, i sentence requires that there be a world accessible to i where Φ is true, this world need not be normal), they do not play a role in determining whether a sentence or argument is semantically valid.

Priest calls these non-normal systems N-systems. The basic system N is very much like the system K, except that it has these non-normal worlds. The system N can be extended. If we add the axiom $(\Box p \supset p)$, then we get system NT... a non-normal-worlds version of KT. And just like the normal-worlds case, where we added ρ to the accessibility relation to get $K\rho$, here we add it to the N-version to get $N\rho$. This turns out to be the semantics required for Lewis' system S_2. Semantically, we can add Priest's τ to N, yielding $N\tau$. And if we add the τ requirement on top of $N\rho$, we get $N\rho\tau$ ...which turns out to be the semantics required for Lewis' system S_3. If we add σ on top of that, to get $N\rho\sigma\tau$, this turns out to be a system that others (not Lewis) had called $S_{3.5}$.

The tableaux rules for the N-systems are pretty straightforward, although they require the notion of a world being $\Box$-inhabited on a branch of a tableau:

> A world i is $\Box$-inhabited on a branch if there is a formula of the form $\Box\Phi,\ i$ (for any Φ) on that branch.

Note that if a world i is $\Box$-inhabited, then it must be a normal world, because only normal worlds can have $\Box$-formulas be true. Finally, note that the actual world–world 0–is considered a normal world.

To construct a tableau for the N-systems, the only difference from the K-systems is the following:

The rule for $\Diamond\Phi,\ i$ only is employed if $i = 0$ or if i is $\Box$-inhabited.

That is, the $\Diamond\Phi,\ i$-rule only applies in normal worlds. All the other features we discussed for tableaux in normal modal logics hold here: the $\Box\Phi,\ i$ rule works without any restriction, and the ρ, σ, τ rules work as before.

Here are a couple of problems to test your non-normal modal logic tableaux abilities with. (For problems with $\prec$ in them, replace it by its definition before starting the tableaux.) Start with system N. For problems that are not valid in N, check them out in Nρ, Nτ, N$\rho\tau$, and N$\rho\sigma\tau$. (Well. . . if one were valid in Nρ, you needn't check it in N$\rho\tau$, etc.)

1. $\models (p \prec q) \supset (\Box p \supset \Box q)$
2. $\models p \prec \Box(q \supset q)$
3. $\models (p \prec q) \prec (\neg q \prec \neg p)$
4. $\models \Diamond\Diamond p$