1a) first order model:

$$y_k = a_1 y_{k-1} + b_1 u_{k-1} + \varepsilon_t \quad \text{— (1)}$$

where: y : heat rate per minute (deviation)
u : power output (deviation)
a_1, b_1 : unknown parameters
N : # of measurements

To estimate the parameters of (1) we construct the following matrices

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} ; \quad X = \begin{bmatrix} y_0 & u_0 \\ \vdots & \vdots \\ y_{N-1} & u_{N-1} \end{bmatrix} ; \quad \theta = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}.$$

$$\min \ (y-\hat{y})^T (y-\hat{y})$$
$$\text{s.t.} \quad \hat{y} = \underline{\underline{X}}\,\underline{\theta}$$

Sol$^{\underline{n}}$ to the above optimization problem is

$$\hat{\theta} = (\underline{X}^T \underline{\underline{X}})^{-1} \underline{\underline{X}}^T \underline{Y}.$$

for January dataset:

$$\hat{\theta} = \begin{bmatrix} 0.9897 \\ 0.0021 \end{bmatrix} ; \quad \mathrm{cov}(\hat{\theta}) = \begin{bmatrix} 0.572 & -0.1023 \\ -0.1023 & 0.0207 \end{bmatrix} 10^{-6}$$

for April data set

$$\hat{\theta} = \begin{bmatrix} 0.9845 \\ 0.0031 \end{bmatrix} ; \quad \mathrm{cov}(\hat{\theta}) = \begin{bmatrix} 0.100 & -0.0181 \\ -0.0187 & 0.0036 \end{bmatrix} 10^{-5}$$

1b) Eq (1) can be written as (by introducing z-transform)

$$y_k = \frac{b_1 z^{-1} u_k}{1 - a_1 z^{-1}}$$

or;

$$y_k = \frac{b_1}{z - a_1} u_k$$

Now, the gain is computed as: $\lim_{z \to 1} (z-1) GH(z) \frac{z}{z-1}$

$$\lim_{z \to 1} (z-1) \frac{b_1}{z-a_1} \frac{z}{z-1}$$

$$K_p = \frac{b_1}{1-a_1}$$

- The time constant is computed as: $a_1 = e^{-T_s/\tau}$

$$\Rightarrow a_1 = e^{-1/\tau} \quad (\because T_s = 1)$$

$$\Rightarrow \tau = -1/\ln(a_1)$$

for January dataset

$K_p = 0.2038$

$\tau = 96.58$ sec

for April dataset

$K_p = 0.2$

$\tau = 64.01$

Gain for the datasets are approximately the same; however, the time constants are different.

Solution 1:

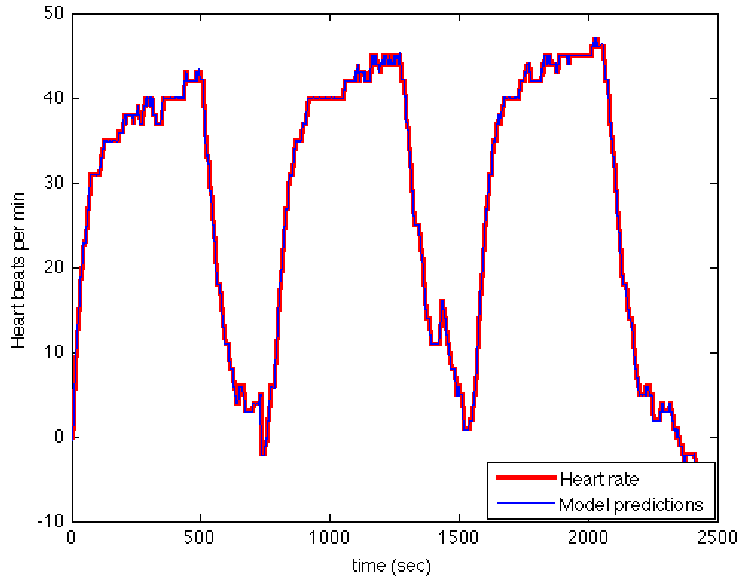

Figure 1: Model predictions for January session

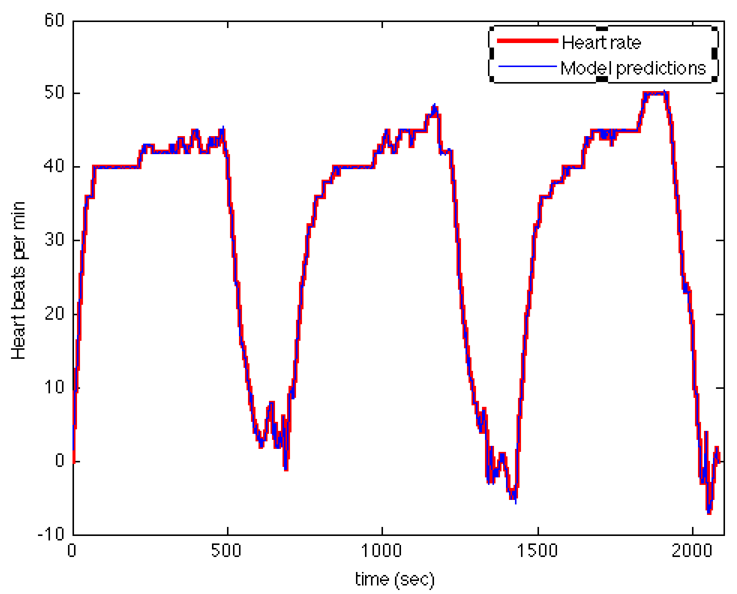

Figure 2: Model predictions for April session

1c) Process gains for the January and April dataset are approximately the same; however the time constants are different. Small time constant for the April session suggests that the heart is quick in responding to the change in the training load. Intuitively, with proper training, we would expect to see a slow or gradual change in the heart rate (in control parlance: larger time constant) for the same training load. The time constants suggest that the performance of the athlete has degraded.

Side Note: In this problem, we are trying to model the cardiovascular response of an athlete to the change in the training load. Note that the dynamics of a biological system keeps changing continuously; therefore any attempt to capture or model long-term effect of training with a single time-invariant model (as in this problem) is not only inaccurate, but also misleading. In such situation, probably analysis or model building based on a single step test is more meaningful. The results or model developed based on a single step change or a single training interval is not provided; however, the students are encouraged to do the analysis themselves and see how the gain and time constant compare between the training intervals.

1d) The choice of the sampling time (Ts=1 sec) is not appropriate, since the time constants (tau) for the January and April dataset are 97 and 64 sec, respectively. From our basic understanding of modeling, we know that it suffices to pick Ts roughly in the interval: tau/5<Ts<tau/10. Note that this is just a rule of thumb and shouldn't be taken as a definite solution for finding an optimal sampling time. For our purposes, since the data set is noisy (refer to the plot of training load or power output), it suffices to pick a slower sampling time, say Ts=tau/5. This ensures that we don't sample a lot of noise into our modeling data.

Solution 2:

2c)

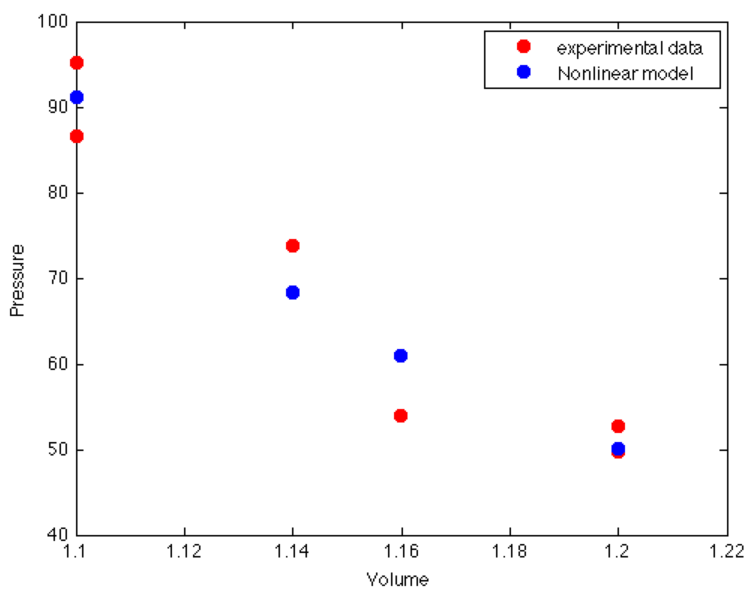

Figure 3: Comparing experimental data with nonlinear model predictions

1e) For, 1st order, no zeros, no dead time:

$SSE = 410.59$
$\sigma_\epsilon^2 = 0.1700$

for 2nd order, no zeros, no dead time:

$$\hat{\theta} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \end{bmatrix} \qquad X = \begin{bmatrix} y_0 & 0 & u_0 \\ y_1 & y_0 & u_1 \\ \vdots & \vdots & \vdots \\ y_{N-1} & y_{N-2} & u_{N-1} \end{bmatrix}$$

$\sigma_\epsilon^2 = 0.1698$
$SSE = 409.85$

for 2nd order, 1 zero, no dead time

$$\hat{\theta} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}, \quad X = \begin{bmatrix} y_0 & 0 & u_0 & 0 \\ y_1 & y_0 & u_1 & u_0 \\ \vdots & & \vdots & \\ y_{N-1} & y_{N-2} & u_{N-1} & u_{N-2} \end{bmatrix}$$

$\sigma_\epsilon^2 = 0.1694$
$SSE = 408.74$

We see that in terms of performance and model complexity, 1st order model gives satisfactory results. We find no significant improvement in the performance by increasing the model order or the number of zeros.

Therefore, it suffices to pick a 1st order model for the January workout session.

2) a) Nonlinear model is given as:

$$P = \frac{a}{V-b} - \frac{c}{V(V+b)}$$

P: output/measurement
V: input
a, b, c: unknown parameters

To estimate the model parameters, we use the MATLAB command "lsqcurvefit", with the initial conditions as $[a_0; b_0; c_0] = [0.5\ 0.5\ 0.5]$

The parameter estimates are: $\hat{a} = 9.9774$
$\hat{b} = 0.9853$
$\hat{c} = -9.4664$

2b) The above nonlinear regression model can also be written as follows:

$$P(V-b)V(V+b) = aV(V+b) - c(V-b)$$
$$P(V^2-b^2)V = aV^2 + aVb - cV + cb$$

$$PV^3 = Pb^2V + aV^2 + aVb - cVb + cb$$
$$PV^3 = \beta PV + \delta V^2 + \epsilon V + \gamma \quad \text{(linear in parameters)}$$

Where; $\beta = b^2$; $\delta = a$; $\epsilon = ab - c$ and $\gamma = cb$

$\Rightarrow$ Thus my colleague is correct about the alternate model form.

2c) Using the linear-in-parameter model and applying LSR, we obtain

$$\underline{\hat{\theta}} = \begin{bmatrix} \beta \\ \delta \\ \epsilon \\ \gamma \end{bmatrix}; \quad \underline{X} = \begin{bmatrix} P_1V_1 & V_1^2 & V_1 & 1 \\ P_2V_2 & V_2^2 & V_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ P_NV_N & V_N^2 & V_N & 1 \end{bmatrix}; \quad \underline{Y} = \begin{bmatrix} P_1V_1^3 \\ P_2V_2^3 \\ \vdots \\ P_NV_N^3 \end{bmatrix}$$

$$\hat{\Theta} = \begin{bmatrix} 1.307 \\ -693.3 \\ 1772.9 \\ -1121.1 \end{bmatrix} \Rightarrow \begin{aligned} \hat{b} &= \pm\sqrt{\beta} = 1.144 \text{ (since, -ve is infeasible)} \\ \hat{a} &= \delta = -693.3 \\ \hat{c} &= \gamma/b = -979.98 \end{aligned}$$

2d) See the plot for comparison between the experimental data and linear and nonlinear model predictions.

	Nonlinear model	linear model
Estimates:	$\hat{a} = 9.9774$ $\hat{b} = 0.9853$ $\hat{c} = -9.4664$	$\hat{a} = -693.3$ $\hat{b} = 1.144$ $\hat{c} = -979.98$

Variance

$$\begin{array}{c} \hat{a} \\ \hat{b} \\ \hat{c} \end{array} \begin{bmatrix} 1.61\times10^{4} & & \\ & 0.0269 & \\ & & 467.7 \end{bmatrix} \qquad \begin{array}{c} \hat{\beta} \\ \hat{\delta} \\ \hat{\varepsilon} \\ \hat{\gamma} \end{array} \begin{bmatrix} 23.60 & & & \\ & 6.634\times10^{8} & & \\ & & 3.64\times10^{9} & \\ & & & 1.26\times10^{9} \end{bmatrix}$$

[off diagonal elements are not shown]

σ_e^2 40.128 1.36×10^{10}

Clearly the nonlinear model outperforms the linear model.

e) Given an option, additional experiments can be performed at molar volume 114 and 116 m^3. This is to ensure that we have enough replicates to estimate the model parameters from the noisy data set.

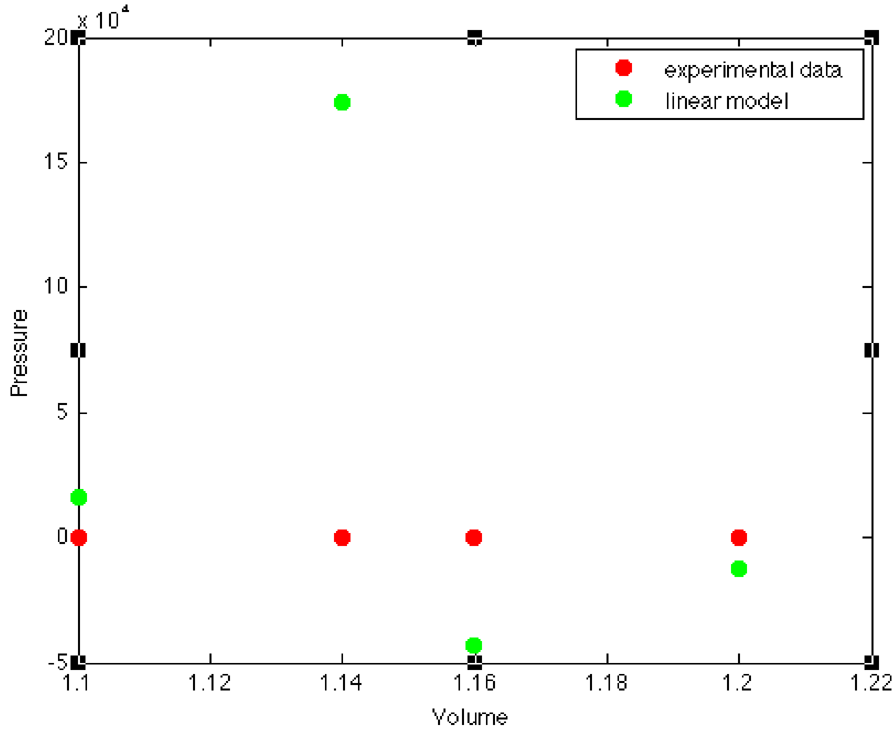

Figure 4 : Comparing the experimental data with the linear model predictions

2f) from the analysis in the previous section, it is clear that the linear model representation of the EOS is inappropriate.