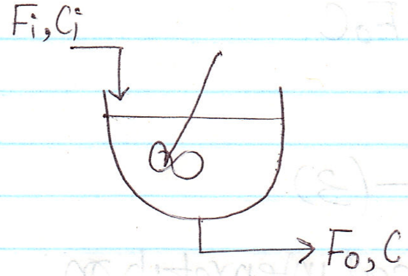
SEMINAR #1.

Consider the mixing process:



Mass Balance

$$\frac{d(\rho A h)}{dt} = \rho F_i - \rho F_o$$

$$\boxed{A\frac{dh}{dt} = F_i - F_o} \quad \text{—— (1)}$$

Component Balance:

$$\frac{d}{dt}(\rho A h c) = F_i C_i - F_o C$$

$$A\frac{dhC}{dt} = F_i C - F_o C$$

$$A\left[h\frac{dC}{dt} + C\frac{dh}{dt}\right] = F_i C - F_o C \quad \text{—(2)}$$

Physical interpretation:

↳ $\left(\frac{dh}{dt}\right)C \equiv$ net instantaneous rate of change of mass of component due to dilution / concentration

↳ $\left(\frac{dC}{dt}\right)h \equiv$ net instantaneous rate of change of mass of component due to transport.

Substituting (1) into (2).

$$(F_i - F_o)C + Ah\frac{dC}{dt} = F_iC_i - F_oC$$

$$Ah\frac{dC}{dt} = \underbrace{F_i(C_i - C)}_{} \quad -(3)$$

↳ physical intepretation:
net influx due to conc. differences.

finally, the full model:

$$\begin{cases} A\frac{dh}{dt} = F_i - F_o & \leftarrow \text{effects of volume change} \\ Ah\frac{dC}{dt} = F_i(C_i - C) & \leftarrow \text{effects of material transport} \end{cases}$$

$$\begin{cases} \frac{dh}{dt} = \frac{1}{A}(F_i - F_o) \\ \frac{dC}{dt} = \frac{1}{Ah}F_i(C_i - C) \end{cases} \quad -(4)$$

Substitute variables:

$$\underline{x} = \begin{bmatrix} h \\ C \end{bmatrix} ; \quad \underline{u} = \begin{bmatrix} F_i \\ F_o \\ C_i \end{bmatrix}$$

The model is:

$$\frac{d\underline{x}}{dt} = \frac{1}{A}\begin{bmatrix} (u_1 - u_2) \\ u_1(u_3 - x_2)/x_1 \end{bmatrix} \begin{matrix} \leftarrow f_1(\underline{x}, \underline{u}) \\ \leftarrow f_2(\underline{x}, \underline{u}) \end{matrix} \quad -(5)$$

linearizing model (5) requires the calculation of the Jacobians.

$$\therefore \cancel{\nabla_{\underline{x}} \underline{f} = \frac{1}{A} \quad 0 \quad 0 \quad -u_1(u_3-u_2)}$$

$$\nabla_{\underline{x}} \underline{f} = \frac{1}{A} \begin{bmatrix} 0 & 0 \\ -\dfrac{u_1(u_3-x_2)}{x_1^2} & -\dfrac{u_1}{x_1} \end{bmatrix}$$

$$\nabla_{\underline{u}} \underline{f} = \frac{1}{A} \begin{bmatrix} 1 & -1 & 0 \\ \dfrac{(u_3-x_2)}{x_1} & 0 & \dfrac{u_1}{x_1} \end{bmatrix}.$$

To evaluate the Jacobians, we need some steady-state values.

$$A = 2\,m^2;\; c_i^* = c^* = 1\,kg/m^3;\; F_i^* = F_0^* = 1000\,m^3/hr;\; h^* = 1\,m.$$

$$\therefore \underline{\underline{A}} = \nabla_{\underline{x}} \underline{f}\Big|_{\underline{x}^*, \underline{u}^*} = \begin{bmatrix} 0 & 0 \\ 0 & -500 \end{bmatrix}$$

$$\underline{\underline{B}} = \nabla_{\underline{u}} \underline{f}\Big|_{\underline{x}^*, \underline{u}^*} = \begin{bmatrix} +0.5 & -0.5 & 0 \\ 0 & 0 & 500 \end{bmatrix}.$$

Then the linearized model is:

$$\frac{d\underline{x}'}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & -500 \end{bmatrix} \underline{x}' + \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 0 & 500 \end{bmatrix} \underline{u}'$$

Now, the output eq^{n}s could be anything (lets output all the variables)

$$\underline{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}' + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{u}'$$

Note that x', u', y' are all in their deviation form

Q → is the system stable?

eigenvalues of $\underline{\underline{A}} \Rightarrow \lambda_1 = 0; \lambda_2 = -500$.

eigenvectors $\underline{\underline{V}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Q → why?

∴ The state-transition matrix is given as

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-500t} \end{bmatrix}$$

↳ Why didn't I have to do all the fancy linear algebra?

recall

$$e^{\underline{\underline{A}}t} = \underline{\underline{V}} e^{\underline{\underline{\Lambda}}t} \underline{\underline{V}}^T = \underline{\underline{V}} e^{\underline{\underline{\Lambda}}t} \underline{\underline{V}}^{-1}$$

finally the solution to the linearized model is:

$$\underline{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-500t} \end{bmatrix} \underline{x}'(0) + \int_0^t \begin{bmatrix} 1 & 0 \\ 0 & e^{-500\tau} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 0 & 500 \end{bmatrix} \underline{u}'(t-\tau)\, d\tau$$

or,

$$\underline{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-500t} \end{bmatrix} \underline{x}'(0) + \int_0^t \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 0 & 500e^{-500\tau} \end{bmatrix} \underline{u}'(t-\tau)\,d\tau .$$

Some Questions.

1) Suppose: $\underline{x}'(0) = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$; $\underline{u}'(t) = 0$

then what would be the long term response of the linearized model?

$$\lim_{t \to \infty} \underline{x}'(t) = \lim_{t \to \infty} \begin{bmatrix} 1 & 0 \\ 0 & e^{-500t} \end{bmatrix} \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}$$

$$= \lim_{t \to \infty} \begin{bmatrix} 0.5 \\ 2e^{-500t} \end{bmatrix}.$$

$$\underbrace{\lim_{t \to \infty} x'(t) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}}_{\text{in terms of deviation.}} \quad \text{or} \quad \underbrace{\lim_{t \to \infty} x(t) = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}}_{\text{in terms of absolute values.}}$$

Q. Why does the initial value for the concentration goes to zero but not the initial height?

Q. Given that this is an integrating system, will step changes in each of the inputs cause the outputs to grow (ramp)?