Solution 1:

The continuous-time nonlinear state space model for the Belousov Zhabotinsky reaction is given as:

$$\frac{dx_1(t)}{dt} = u_1 - x_1 - \frac{4x_1x_2}{1+x_1^2}$$
$$\frac{dx_2(t)}{dt} = u_2x_2\left(1 - \frac{x_2}{1+x_1^2}\right)$$
$$\boldsymbol{y}(t) = \boldsymbol{x}(t)$$

Part1:

1a) Normal operations

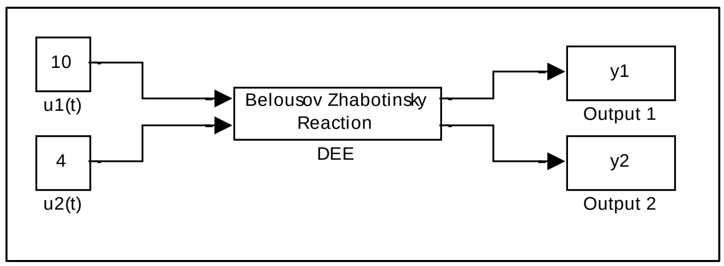

Figure 1: SIMULINK sheet for normal operations

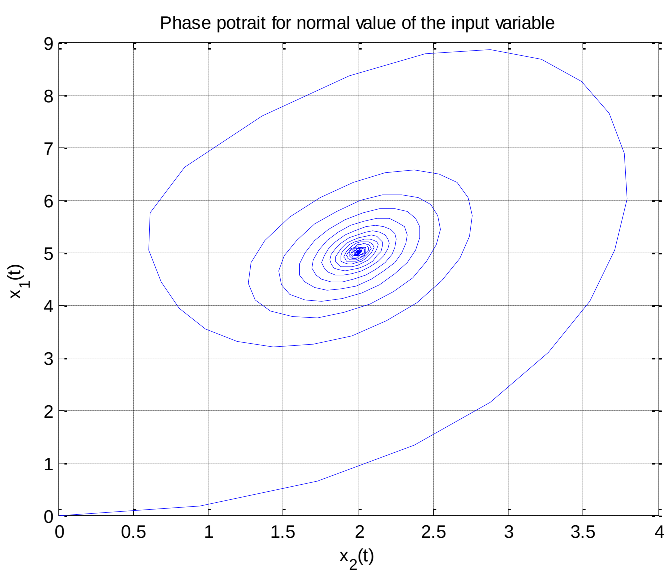

Figure 2: Phase plane portrait for normal operations

1b) Abnormal operations

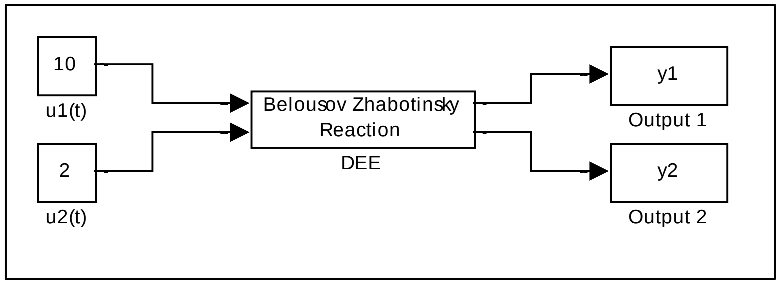

Figure 3: SIMULINK sheet for abnormal operations

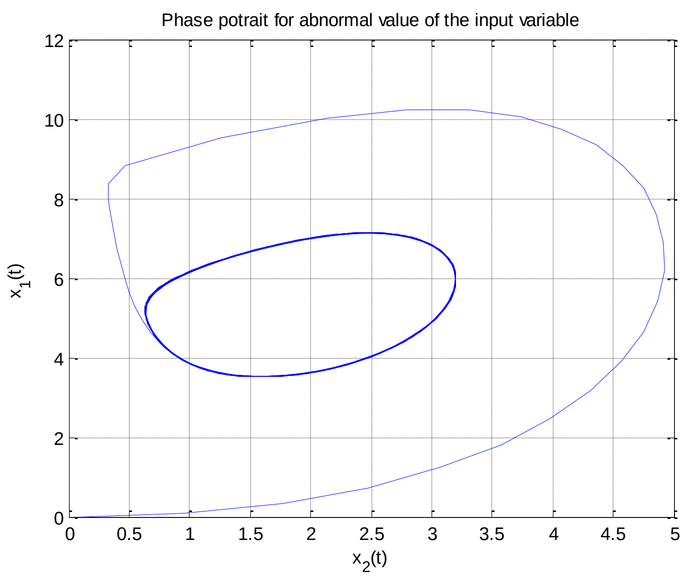

Figure 4: Phase plane portrait for abnormal operations

1c)

- Normal operating conditions: the system is stable (refer to Figure 2) as the outputs (y=x) settles at (2, 5).
- Abnormal operating conditions: the system is marginally stable (refer to Figure 4) with sustained oscillations around (2, 5)

Note: Clearly, choice of an appropriate operating condition is critical for this system/process

Part 2:

2a) Linearizing the nonlinear model around the normal operating mode: (**u₀**= [10 4]^T ; **x₀**= [2 5]^T)

Computing the required Jacobian matrices:

$A = \nabla_x f|_{x_0,u_0} =$

$$\begin{pmatrix} \frac{8\,x1^2\,x2}{(x1^2+1)^2} - \frac{4\,x2}{x1^2+1} - 1 & -\frac{4\,x1}{x1^2+1} \\ \frac{2\,u2\,x1^2\,x2}{(x1^2+1)^2} - u2\left(\frac{x2}{x1^2+1} - 1\right) & -\frac{u2\,x1}{x1^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{5} & -\frac{8}{5} \\ \frac{32}{5} & -\frac{8}{5} \end{pmatrix}$$

B=$\nabla_u f|_{x_0,u_0}$=

$$\begin{pmatrix} 1 & 0 \\ 0 & -x1\left(\frac{x2}{x1^2+1} - 1\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Linearized continuous state space model around the normal operating conditions is given as:

$$\frac{dx'(t)}{dt} = \begin{bmatrix} \frac{7}{5} & -\frac{8}{5} \\ \frac{32}{5} & -\frac{8}{5} \end{bmatrix} x'(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u'(t)$$

$$y'(t) = x'(t)$$

Note that all the variables are in deviation form.

2a) Linearizing the nonlinear model around the abnormal operating mode: ($\mathbf{u_0}$= [10 2]^T ; $\mathbf{x_0}$= [2 5]^T)

Computing the required Jacobian matrices:

A=$\nabla_x f|_{x_0,u_0}$=

$$\begin{pmatrix} \frac{8\,x1^2\,x2}{(x1^2+1)^2} - \frac{4\,x2}{x1^2+1} - 1 & -\frac{4\,x1}{x1^2+1} \\ \frac{2\,u2\,x1^2\,x2}{(x1^2+1)^2} - u2\left(\frac{x2}{x1^2+1} - 1\right) & -\frac{u2\,x1}{x1^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{5} & -\frac{8}{5} \\ \frac{16}{5} & -\frac{4}{5} \end{pmatrix}$$

$B = \nabla_u f|_{x_0,u_0} =$

$$\begin{pmatrix} 1 & 0 \\ 0 & -x1\left(\frac{x2}{x1^2+1} - 1\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Linearized continuous state space model around the <u>abnormal operating conditions</u> is given as:

$$\frac{dx'(t)}{dt} = \begin{bmatrix} \frac{7}{5} & -\frac{8}{5} \\ \frac{16}{5} & -\frac{4}{5} \end{bmatrix} x'(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u'(t)$$

$$y'(t) = x'(t)$$

Again, note that all the variables are in deviation form.

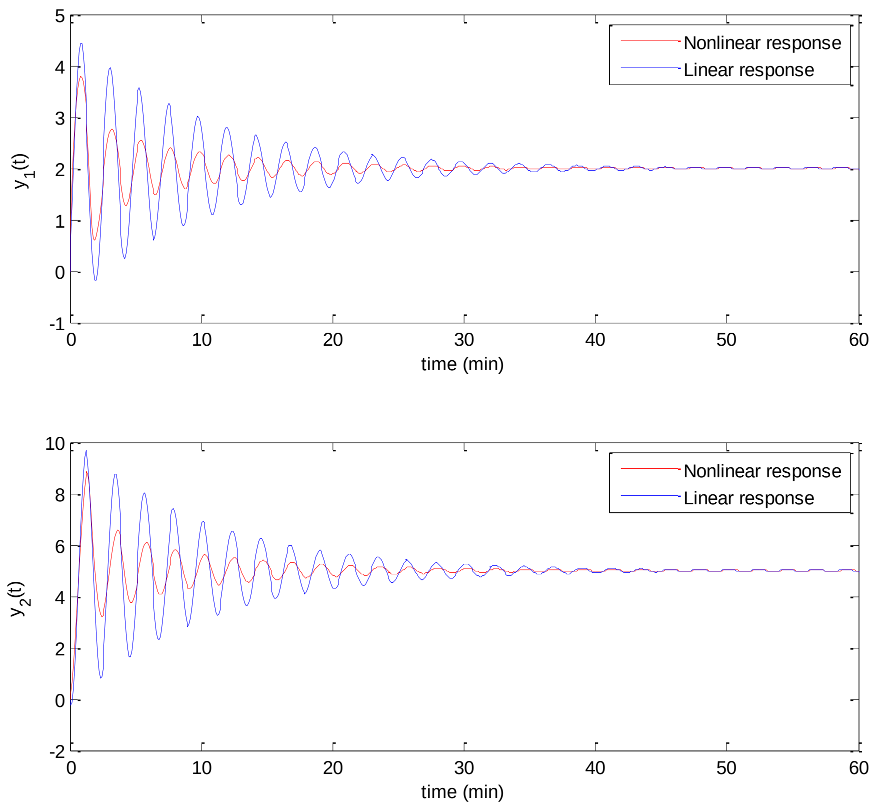

Figure 5: Comparing the nonlinear and linear model response at normal operating conditions

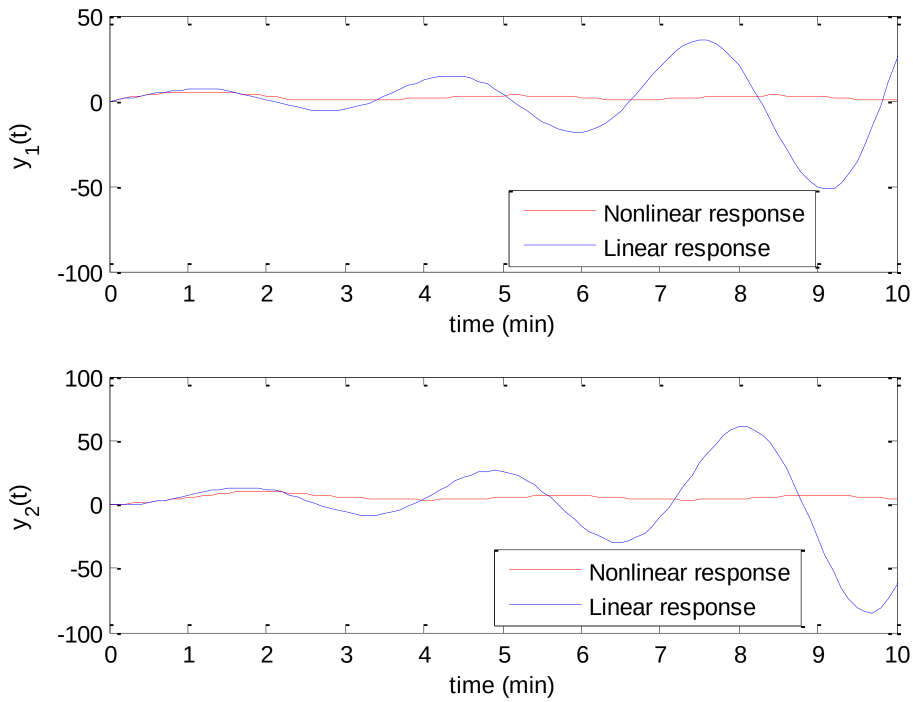

Figure 6: Comparing the nonlinear and linear model response at abnormal operating conditions

Linearization around the normal operation: Even though the linearized model does not approximate the transit behaviour well other than the oscillatory trend, the long term dynamics are well captured by the linear model. See Figure 5 for the comparison.

Linearization around the abnormal operation: Linearizing the nonlinear model around the abnormal operations makes the linearized model unstable. See Figure 6 for the comparison.

Note: This exercise highlights that a model cannot be linearized effectively at all operating conditions. The linearization technique is effective only in those operating conditions where the system nonlinearity is not severe (nonlinearity contour mapping for nonlinear systems is an important and active area of research). Therefore, choice of a good linearization point plays a critical role in deciding the performance or quality of the linearized system. In some extreme situations, a poor choice of linearization point can also make an otherwise stable (or marginally stable) system, unstable (See Figure 6).

Part 6:

See Figure 7 and 8

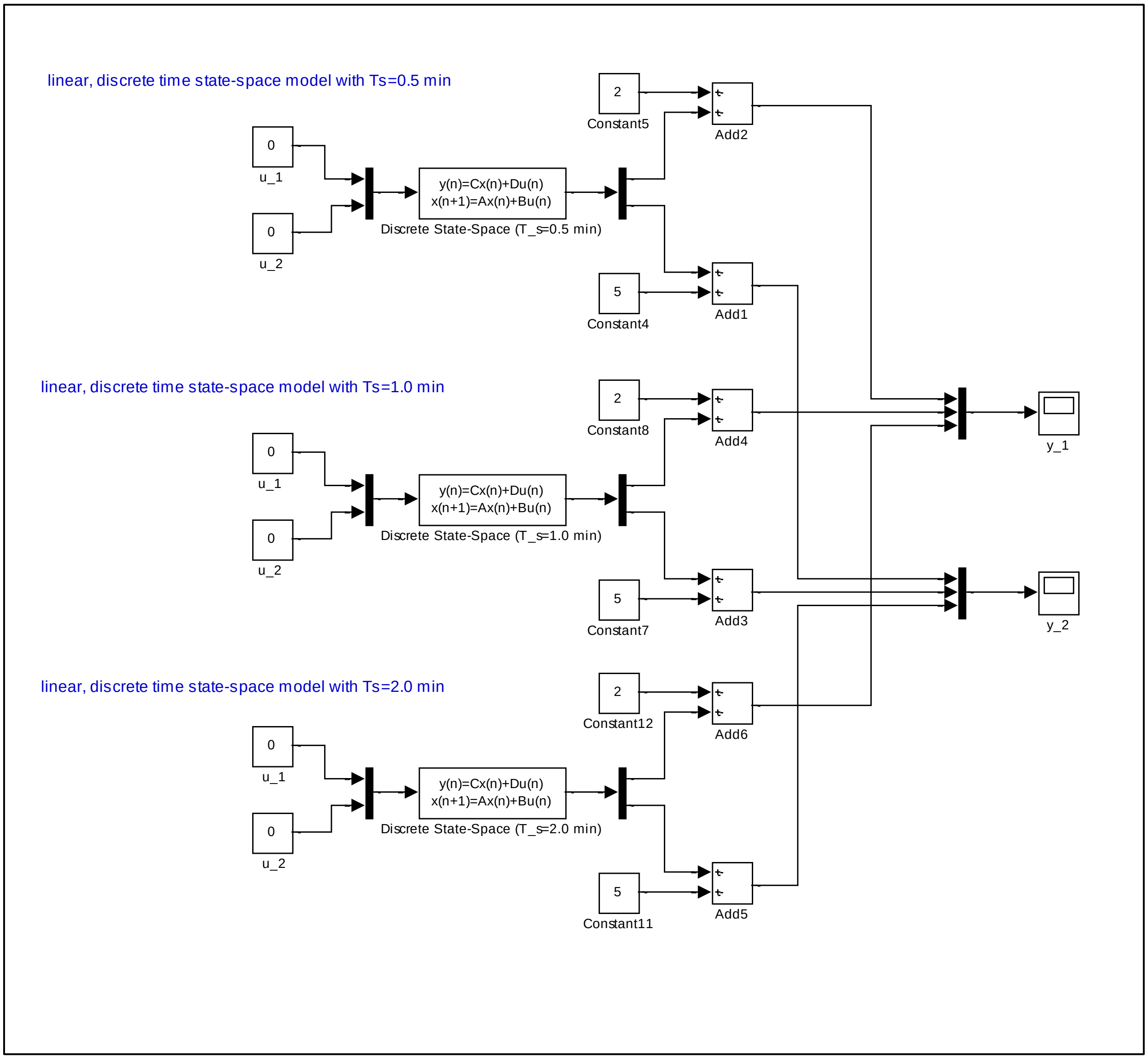

Figure 7: SIMULINK sheet for linear, discrete-time state space models

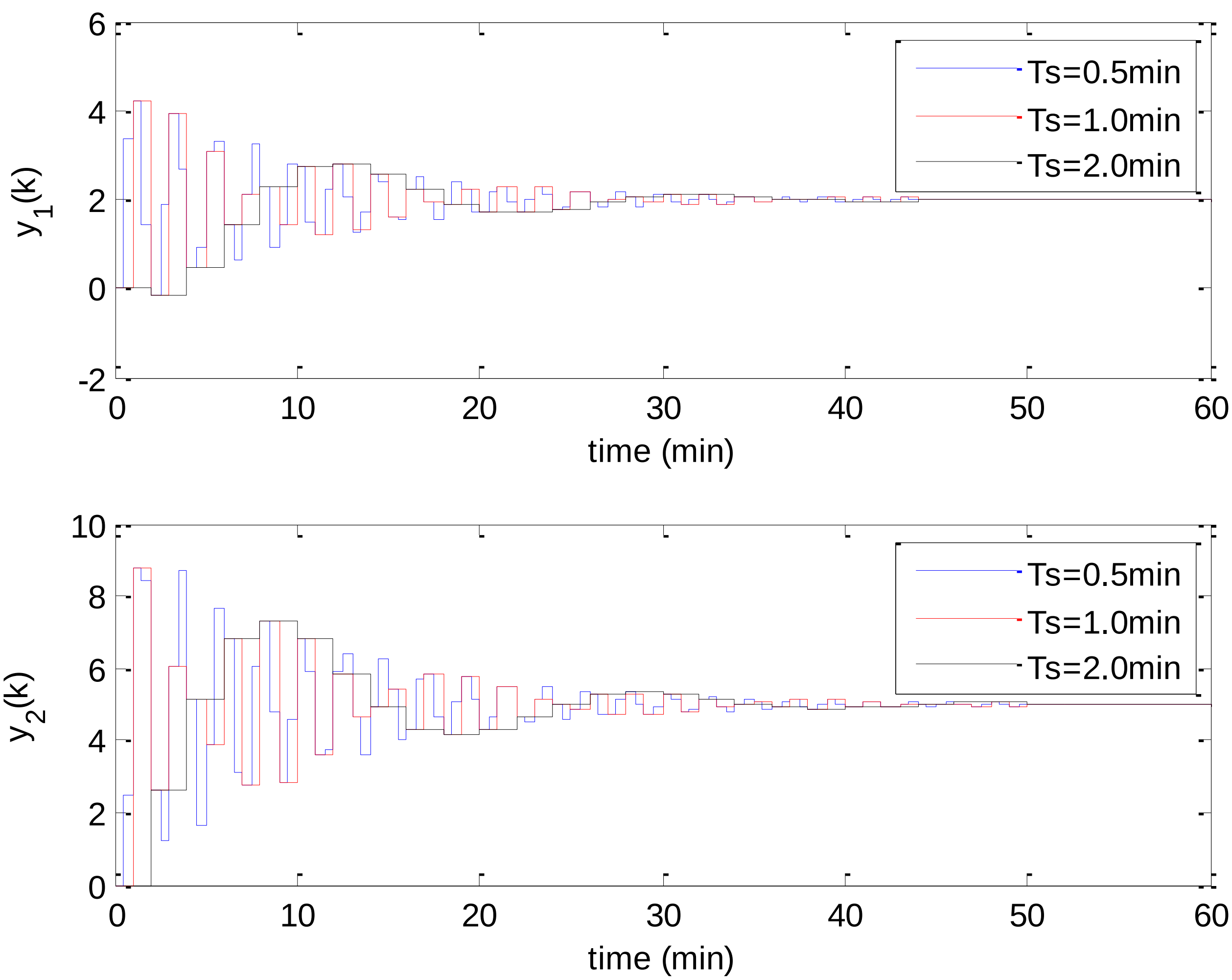

Figure 8: Response for discrete-time state space models with different sampling time Ts

Part 8:

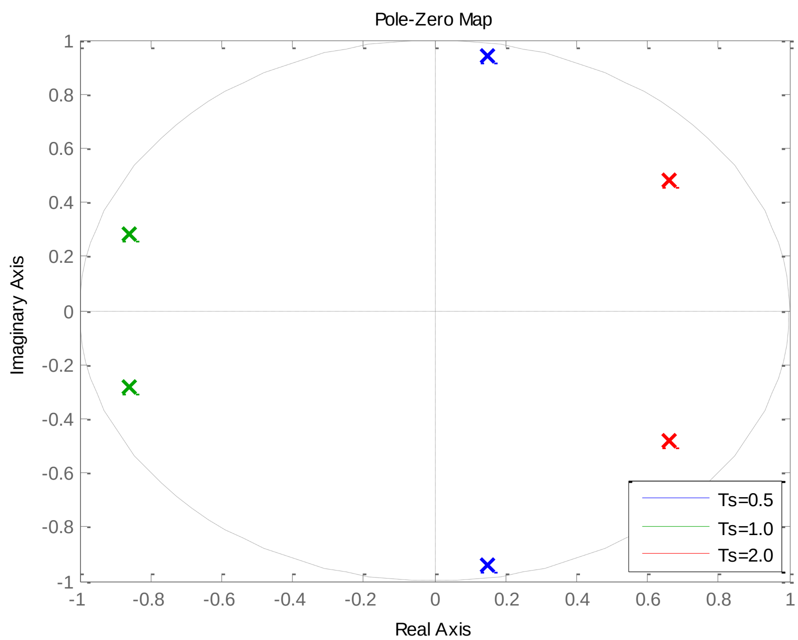

Figure 9: Poles of different discrete-time transfer functions

From Figure 9, it is evident that the poles of the discrete time transfer function for all the chosen sampling time are within the unit circle. This suggests that the discrete-time model is stable. (Note that this is only shown for the model linearized around the normal operating conditions)

Part 9:

Ts=0.5 min seems to be an appropriate choice for the sampling time, since it captures the peaks and troughs of the linear, continuous time response fairly well. In Figure 10, continuous-time response for the linearized model around the normal operating conditions is compared against the discrete-time model with Ts=0.5min. Note that the plotted responses are zoomed so as to get a better view of the approximation.

Clearly for Ts=1min and 2 min, the approximation is poor as both the sampling intervals miss the peaks and troughs of the transient oscillations (see Figure 11 and 12); however, it should be noted that all three discrete-time model captures the long term response fairly well (see Figure 8). Hence, if the final objective is just to know the steady state value, then using a discrete-time model with Ts=2.0 min is probably the best choice, since it keeps the model simple and requires relatively less computation to reconstruct the original continuous time response as compared to the models corresponding to Ts=0.5min and 1.0min.

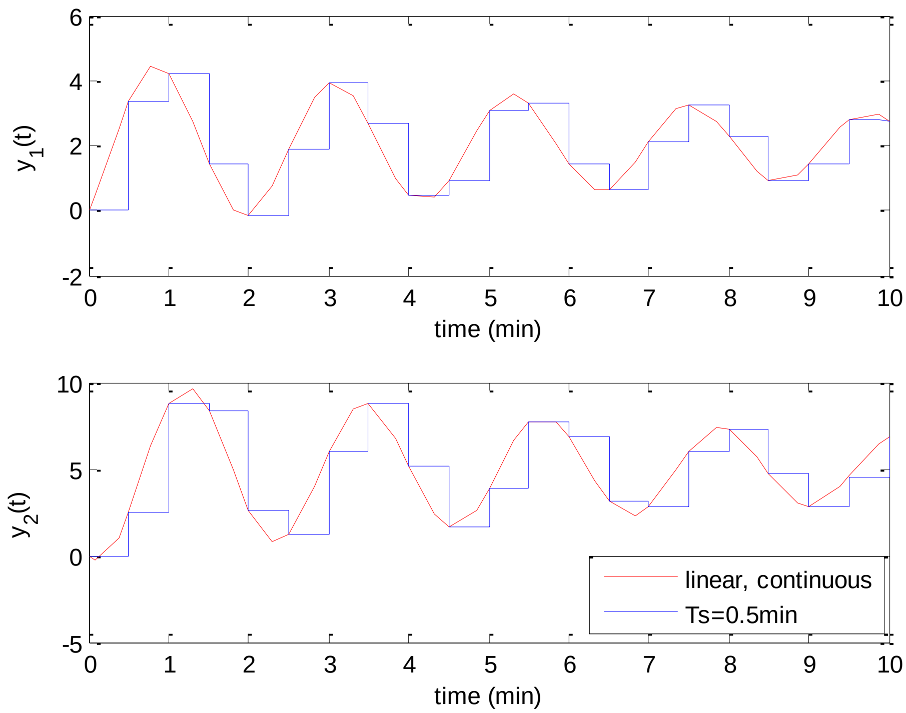

Figure 10: Continuous-time, linear response with discrete-time response

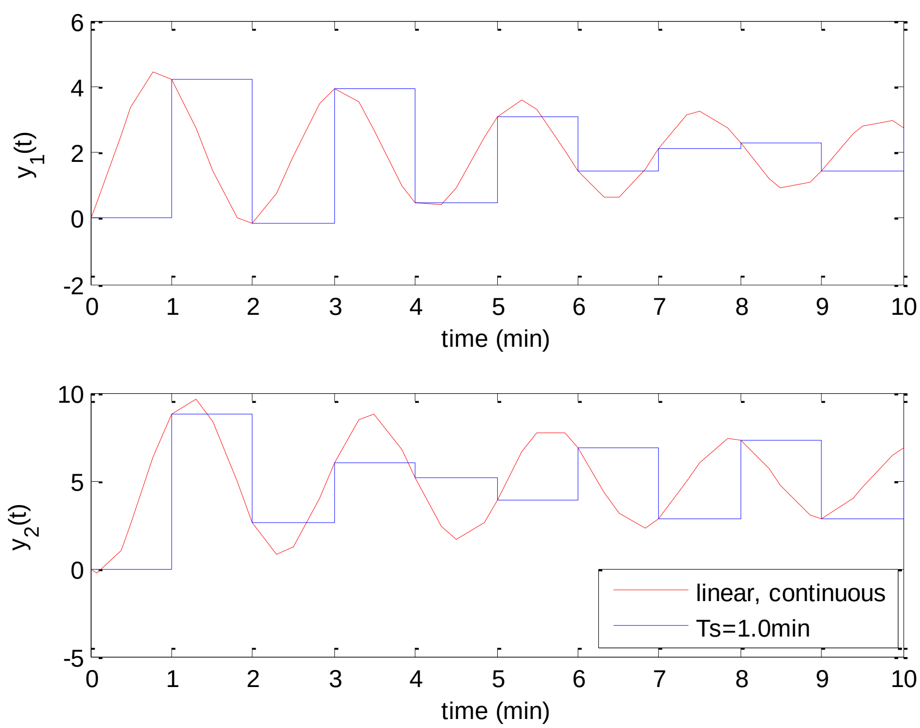

Figure 11: Continuous-time, linear response with discrete-time response

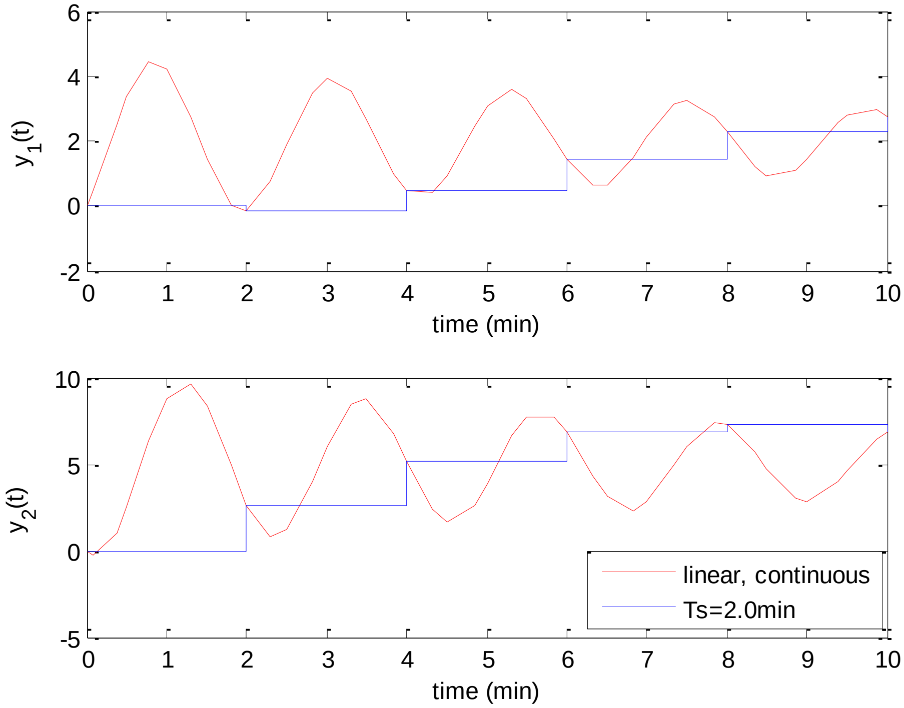

Figure 12: Continuous-time, linear response with discrete-time response