CHE 572.

ASSIGNMENT #2 SOLUTIONS.

m_T : Total mass of the solution inside the tank.
V_T : Total volume of the solution inside the tank.
ρ_T : density of the liquid solution inside the tank.
m_W : mass of water entering the system.
m_c : mass of the cleaning solution.
m_o : Total mass of the solution moving out from the tank.
ρ_W : density of water.
ρ_c : density of the cleaning solution.
ρ_o : output solution density.
h : height of the liquid inside the tank.
R : radius of the tank.
$X_{c,in}$: mass fraction of cleaning solution in F_c
X_o : mass fraction of cleaning solution in F_o
X : mass fraction of cleaning solution in V_T.

Inputs		Outputs			States
manip.	dist.	Meas.	unmeas.	control	
P_c	F_c		F_o		m_T
	m_W		h		~~m_W~~
			V_T		~~m_c~~
			$m_{c,in}$		m_c
			m_o		
			h		
			X_o		
			X		

Assumptions.

1) kettle is perfectly mixed, i.e., $X = X_o$.

2) $X_{c,in} = 1.0$

3) $\rho_T \approx \rho_c \approx \rho_o \approx \rho_W =$ constant for all time 't'

Overall mass balance:

$$\frac{dm_T}{dt} = \dot{m}_w + \dot{m}_c - \dot{m}_o$$

$$\frac{d}{dt}\rho_T V_T = \rho_w F_w + \rho_c F_c - \rho_o F_o$$

$$\frac{dV_T}{dt} = F_w + F_c - F_o \quad (\because \rho_T \approx \rho_w \approx \rho_c \approx \rho_o = \text{constant}) \quad \text{— (1)}$$

from Appendix A, we have

$$V_T = \frac{\pi}{3} h^2 (3R - h) \quad \text{— (2)}$$

differentiating (2) w.r.t 't', we have

$$\frac{dV_T}{dt} = \frac{\pi}{3}\left(6Rh - 3h^2\right)\frac{dh}{dt}$$

$$\frac{dV_T}{dt} = \pi h (2R - h)\frac{dh}{dt} \quad \text{— (3)}$$

Substituting (3) into (1) yields

$$\pi h (2R-h)\frac{dh}{dt} = F_w + k_c P_c^2 - k_o\sqrt{h}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{F_w + k_c P_c^2 - k_o\sqrt{h}}{\pi h (2R-h)}} \quad \text{— (4)}$$

mass balance for cleaning solution:

$$\frac{dm_c}{dt} = \dot{m}_c - m_{c,out}$$

$$\frac{d(m_T x)}{dt} = \dot{m}_c x_{c,in} - \dot{m}_o x_o$$

from Assumption A $\Rightarrow x = x_o$; from Assumption B $\Rightarrow x_{c,in} = 1.0$

$$\therefore \frac{d(m_T x_o)}{dt} = \dot{m}_c - \dot{m}_o x_o$$

$$\Rightarrow \frac{d}{dt}(\rho_T V_T x_o) = \rho_c k_c P_c^2 - \rho_o k_o \sqrt{h} . x_o$$

$$\Rightarrow \frac{d(V_T x_o)}{dt} = k_c P_c^2 - k_o \sqrt{h} x_o \quad \text{(Using Assumption C)}.$$

Using chain rule:

$$V_T \frac{d x_o}{dt} + x_o \frac{d V_T}{dt} = k_c P_c^2 - k_o \sqrt{h} x_o . \quad \text{— (5)}$$

Substituting (2) & (3) into (5)

$$\frac{\pi}{3} h^2 (3R-h) \frac{d x_o}{dt} + x_o \pi h (2R-h) \frac{dh}{dt} = k_c P_c^2 - k_o \sqrt{h} x_o \quad \text{— (6)}$$

Substituting (4) into (6)

$$\frac{\pi}{3} h^2 (3R-h) \frac{d x_o}{dt} + x_o \cancel{\pi h (2R-h)} \frac{\left[F_w + k_c P_c^2 - k_o \sqrt{h}\right]}{\cancel{\pi h (2R-h)}} = k_c P_c^2 - k_o \sqrt{h} x_o$$

$$\Rightarrow \frac{\pi}{3} h^2 (3R-h) \frac{d x_o}{dt} + x_o \left(F_w + k_c P_c^2 - k_o \sqrt{h}\right) = k_c P_c^2 - k_o \sqrt{h} x_o$$

$$\Rightarrow \frac{\pi}{3} h^2 (3R - h) \frac{dX_o}{dt} = k_c P_c^2 (1 - X_o) - F_w X_o$$

$$\boxed{\frac{dX_o}{dt} = \frac{k_c P_c^2 (1 - X_o) - F_w X_o}{\frac{\pi}{3} h^2 (3R - h)}} \quad \text{—} (7)$$

finally the process can be represented as:

→ State Equations:

$$\begin{bmatrix} \frac{dh}{dt} \\ \frac{dX_o}{dt} \end{bmatrix} = \begin{bmatrix} \frac{F_w + k_c P_c^2 - k_o \sqrt{h}}{\pi h (2R - h)} \\ \frac{k_c P_c^2 (1 - X_o) - F_w X_o}{\frac{\pi}{3} h^2 (3R - h)} \end{bmatrix}$$

→ Output equations:

$$\begin{bmatrix} h \\ X_o \\ F_o \\ F_c \end{bmatrix} = \begin{bmatrix} h \\ X_o \\ k_o \sqrt{h} \\ k_c P_c^2 \end{bmatrix}$$

Note: The choice of output eqns are arbitrary and should be decided based on the intended end use e.g., control, optimization.

Substitute variables:

$$\underline{x} = \begin{bmatrix} h \\ X_o \end{bmatrix} ; \quad \underline{u} = \begin{bmatrix} F_w \\ P_c \end{bmatrix} ; \quad \underline{y} = \begin{bmatrix} h \\ X_o \\ F_o \\ F_c \end{bmatrix}$$

then the model is.

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} \left[\dfrac{u_1 + k_c u_2^2 - k_o\sqrt{x_1}}{\pi x_1(2R - x_1)} \right] \\ \\ \left[\dfrac{k_c u_2^2(1 - x_2) - u_1 x_2}{\frac{\pi}{3} x_1^2 (3R - x_1)} \right] \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} x_1 \\ x_2 \\ k_o\sqrt{x_1} \\ k_c u_2^2 \end{bmatrix}$$

3) Model linearization:

$$\nabla_{\underline{x}} \underline{f} = \begin{bmatrix} \dfrac{u_1 + k_c u_2^2 - k_o\sqrt{x_1}}{\pi x_1(2r - x_1)} - \dfrac{u_1 + k_c u_2^2 - k_o\sqrt{x_1}}{\pi x_1^2(2r - x_1)} - \dfrac{k_o}{2\pi x_1^{3/2}(2r - x_1)} , & 0 \\ \\ \dfrac{9(2r - x_1)(u_1 x_2 - k_c u_2^2 + k_c u_2^2 x_2)}{\pi x_1^3 (3r - x_1)^2} , & -\dfrac{3(k_c u_2^2 + u_1)}{\pi x_1^2(3r - x_1)} \end{bmatrix}$$

$$\nabla_{\underline{u}} \underline{f} = \begin{bmatrix} \dfrac{1}{\pi x_1(2r - x_1)} , & \dfrac{2k_c u_2}{\pi x_1(2r - x_1)} \\ \\ \dfrac{-3x_2}{\pi x_1^2(3r - x_1)} , & \dfrac{-6k_c u_2(x_2 - 1)}{\pi x_1^2(3r - x_1)} \end{bmatrix}$$

$$\nabla_{\underline{x}}\, \underline{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ k_0/2\sqrt{x_1} & 0 \\ 0 & 0 \end{bmatrix} \; ; \; \nabla_{\underline{u}}\, \underline{g} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2k_c u_2 \end{bmatrix}.$$

Steady state data:

$x_1 = h = 1.0\ m$; $x_2 = x_0 = x_{c,in} = 50/(150+50) = 0.25$

$u_1 = F_w = 150\ m^3/hr$.

$u_2 = P_c = 0.50$

Also:

- $R = 1\ m/hr$.
- $F_0 = F_w + F_c = k_0\sqrt{h} \Rightarrow 150+50 = k_0\sqrt{1} \Rightarrow k_0 = 200 \frac{m^3}{hr\sqrt{m}}$.
- $F_c = k_c P_c^2 \Rightarrow 50 = k_c \cdot (0.5)^2 \Rightarrow k_c = 200\ m^3/hr$.

Evaluating the Jacobians at steady state (SS)

$$\underline{\underline{A}} \triangleq \nabla_{\underline{x}} \underline{f}\Big|_{SS} = \begin{bmatrix} -31.831 & 0 \\ 0 & -95.493 \end{bmatrix}$$

$$\underline{\underline{B}} \triangleq \nabla_{\underline{u}} \underline{f}\Big|_{SS} = \begin{bmatrix} 1/\pi & 63.6363 \\ -0.11931 & 71.5909 \end{bmatrix}.$$

$$\underline{\underline{C}} \triangleq \nabla_{\underline{x}} \underline{g}\Big|_{SS} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 100 & 0 \\ 0 & 0 \end{bmatrix} ; \underline{\underline{D}} \triangleq \nabla_{\underline{u}} \underline{g}\Big|_{SS} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 200 \end{bmatrix}.$$

finally, the linearized model is given as:

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

Note: The variables are defined in their deviation form about the steady state values.

4) Degrees of freedom:

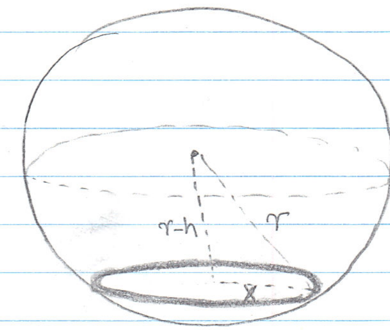
# of variables = # of states + # of inputs + # of outputs = 2+2+4 = 8
# of time derivatives = 2.
# of equations = 6.
# of initial conditions = 2.
# of specified inputs = Z.

$\therefore$ DOF = 8+2−6−2−Z = 2−Z.

$\therefore$ To obtain a unique sol^n; DOF = 0
$\Rightarrow$ 2−Z = 0
Z = 2.

$\therefore$ Both the inputs must be specified to obtain a unique solution (F_w, P_c)

Appendix A.



Volume of a circular disk of radius 'x' and thickness 'dh' is given as

$$dV = (\pi x^2)\, dh$$

Volume of the sphere : $\int_0^V dv = \int_0^{2r} \pi x^2\, dh$

$$V = \int_0^h \pi\left[r^2 - (r-h)^2\right] dh \quad (\because r^2 = x^2 + (r-h)^2)$$

$$V = \int_0^h \pi(2rh - h^2)\, dh$$

$$V = \pi\left(2r\frac{h^2}{2} - \frac{h^3}{3}\right)\Big|_0^h$$

$$V = \frac{\pi}{3}(3rh^2 - h^3)$$

$$\boxed{V = \pi/_3\, h^2(3r - h)}$$

Check your answer

When $h = 0 \Rightarrow V = 0$

When $h = 2r \Rightarrow V = \frac{4\pi h^3}{3}$

In order to simulate the nonlinear system, the dee block of Simulink should be used. In Figure 1 the Simulink model and settings of DEE block have been shown. Note that we assume that the system is initially at steady state.

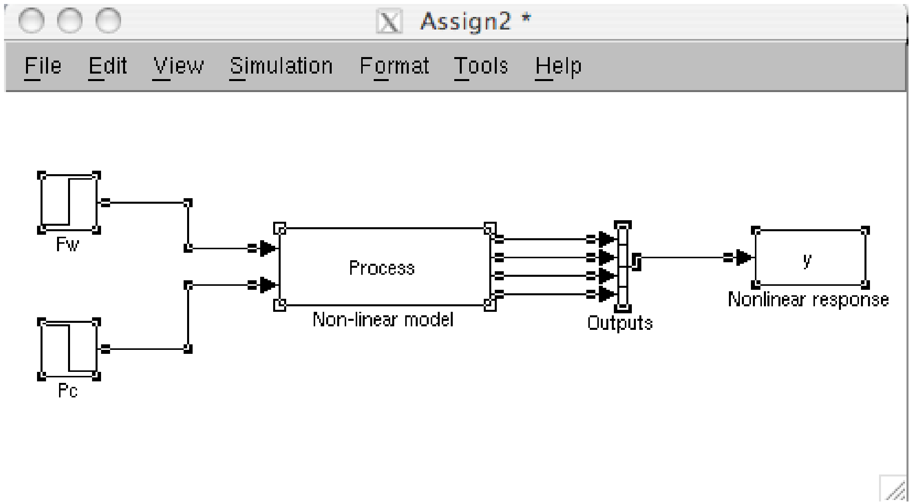


Figure 1: SIMULINK model for the nonlinear system

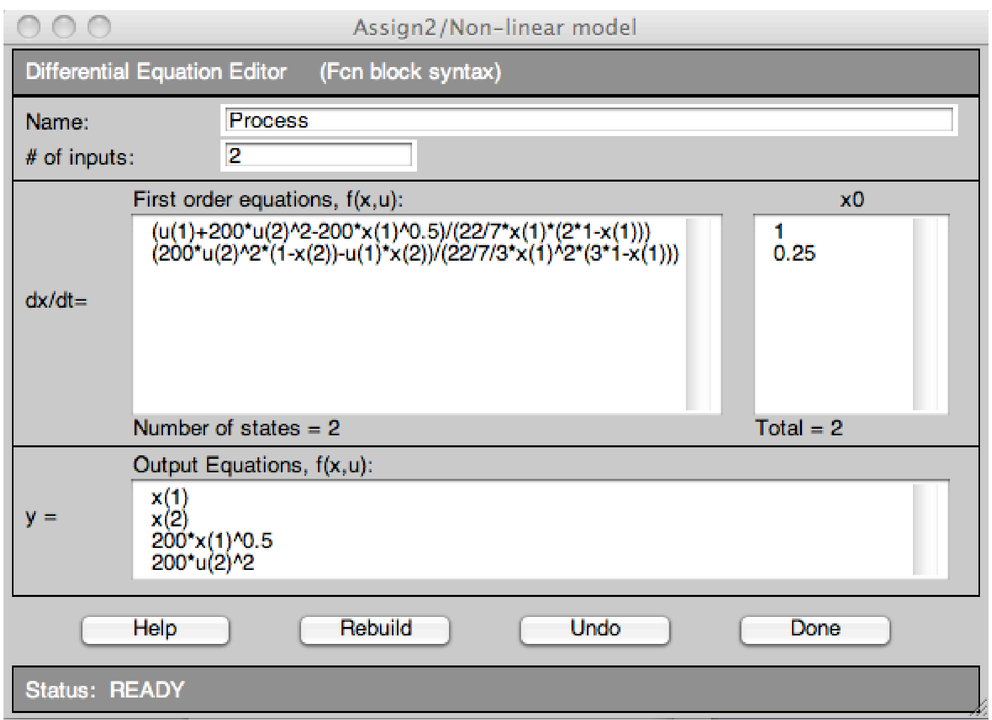


Figure 2: DEE block settings

In order to model the linearized system, one can use the DEE block, but there is block called ”state space” which can be found under the ”continuous” in the library browser. The Simulink model and the settings of the state space block are shown in Figures 3-4. Note that if you choose to use DEE block to simulate the linear system, the initial values and steady state values of input variables should be set as zero (as the linearized equations are in terms of deviation variables)

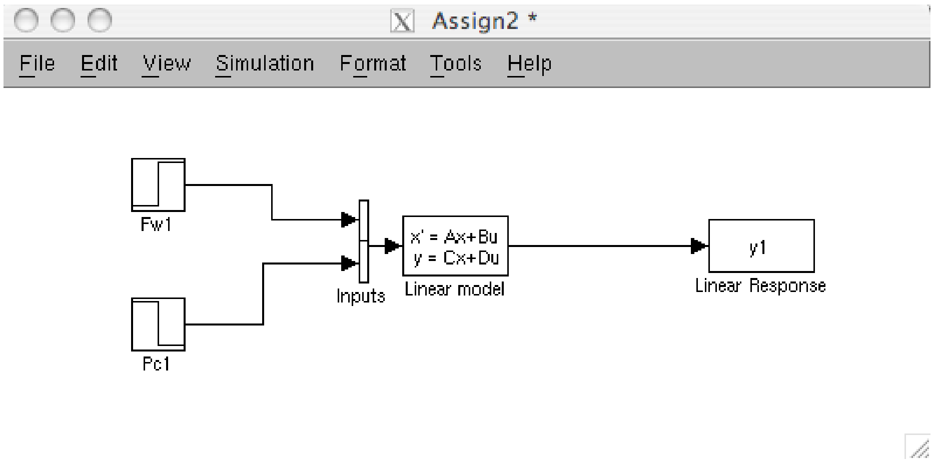


Figure 3: SIMULINK model of the linearized system

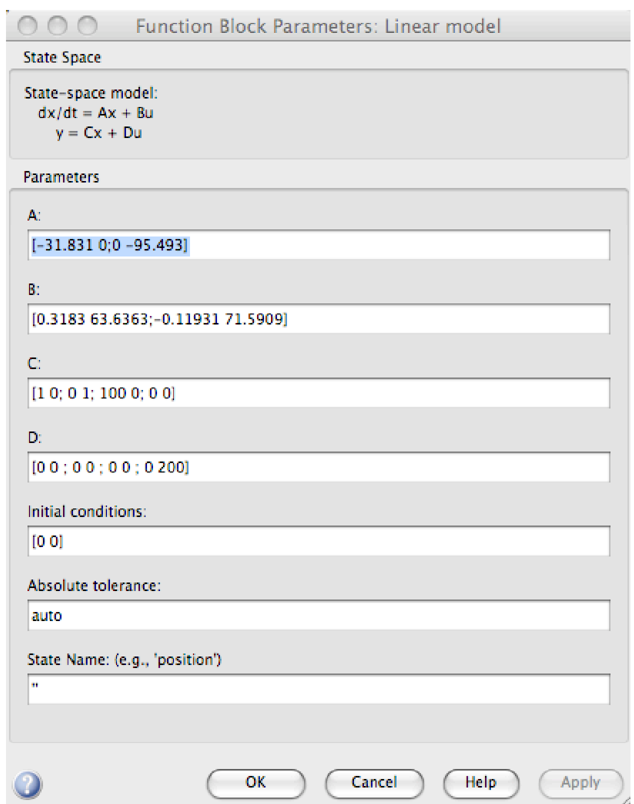


Figure 4: Linear state space model block settings

Figures 5-8 illustrate the simulation results for question 5a-5d

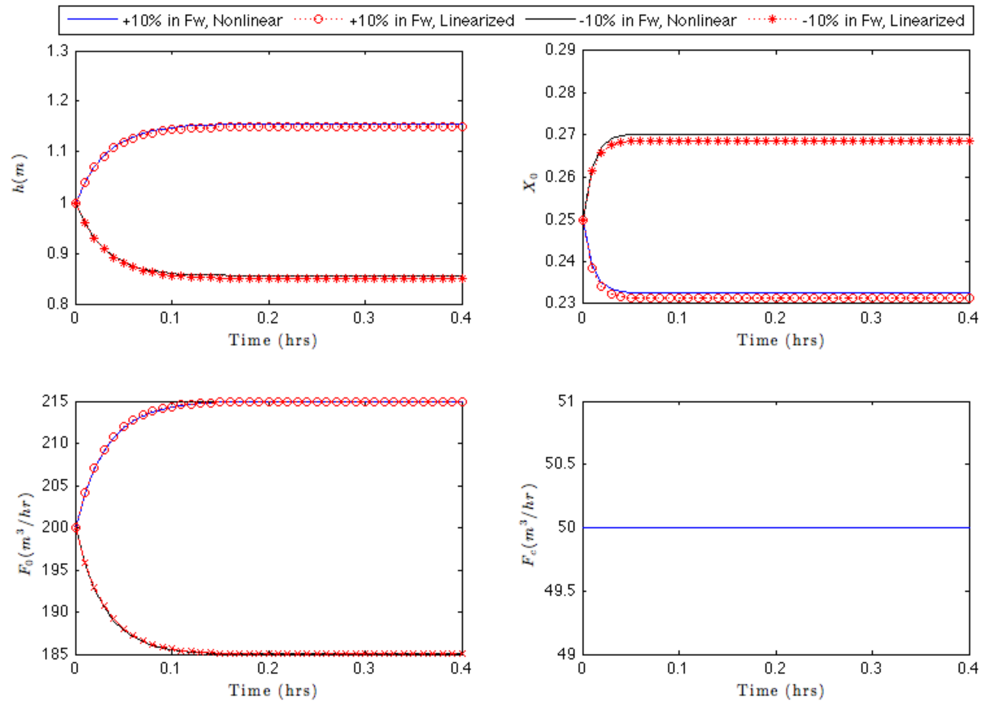


Figure 5: 10% change in Fw

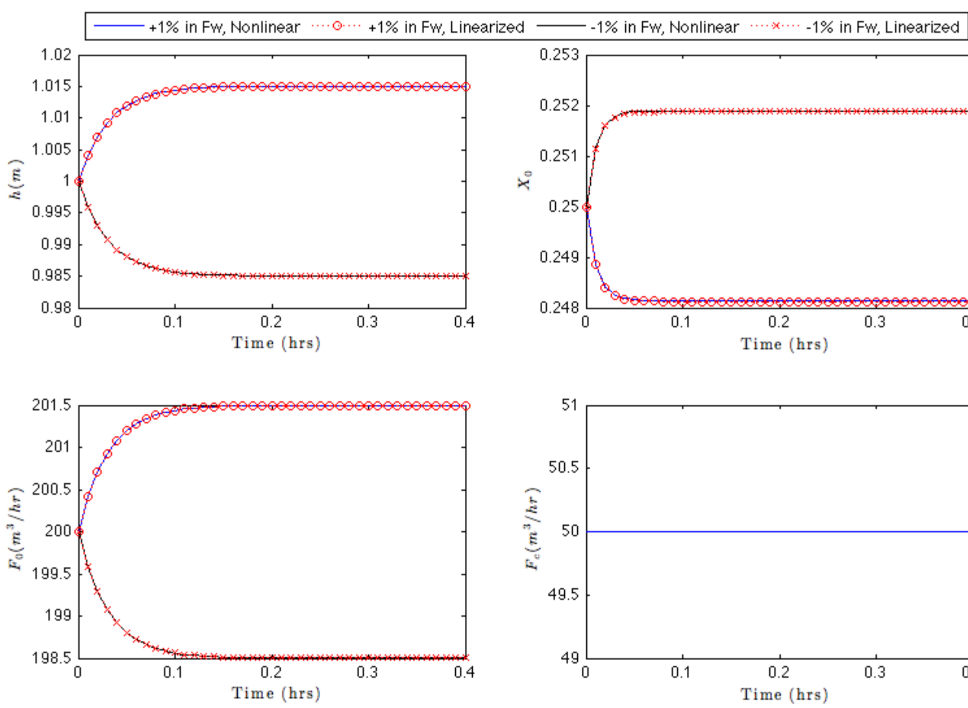


Figure 6: 1% change in Fw

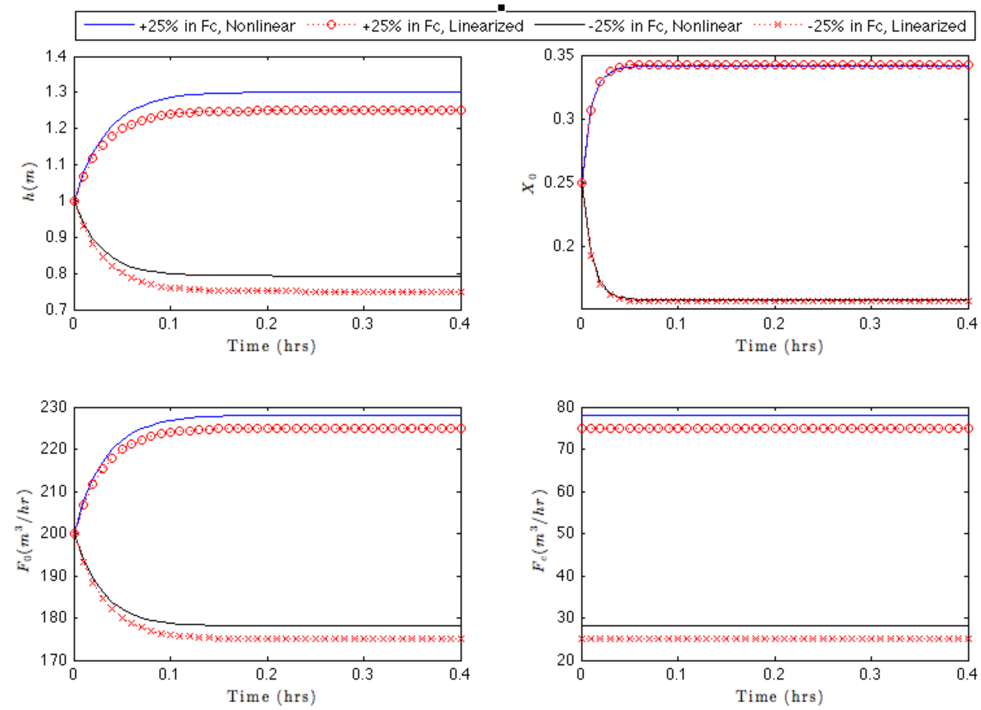


Figure 7: 25% in Fc

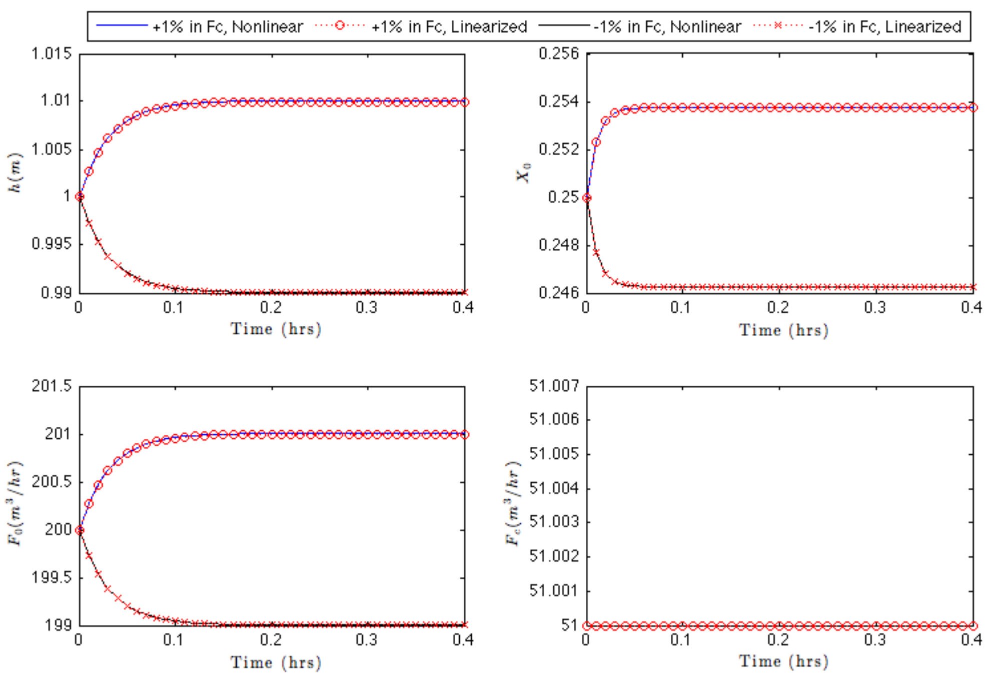


Figure 8:1% change in Fc

Comment on the results:

If you have a close look at the figures, the linearized model does surprisingly well for both the small and larger step changes to the inputs. (This is not always the case). As expected, the deviation between the linear and nonlinear models changes based on the step size and direction. You should expect that the linear model would differ more for larger steps. Also, you can see that the difference between the models is exhibited in both steady-state gains and in the transient responses.