

CH E 572

Assignment #3 (Solution)

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SOLUTION OF QUESTION NO. 1

The given system is:

$$\begin{aligned} \frac{dx_1}{dt} &= 10 - x_1 - \frac{4x_1x_2}{1+x_1^2} \\ \frac{dx_2}{dt} &= ux_1 \left(1 - \frac{x_2}{1+x_1^2} \right) \end{aligned} \quad (1)$$

The initial conditions for the reactor are $\mathbf{x} = [0 \ 2]^T$. The normal operating input value for the reactor is $u = 3.5$ and the process has an equilibrium point at $\mathbf{x} = [2 \ 5]^T$.

a)

You can check if the given point is an equilibrium point by determining whether $\mathbf{x} = [2 \ 5]^T$ is a solution to the steady-state problem (i.e., see if the right-hand side of the equations is zero for the given values of \mathbf{x} and u).

b)

The Simulink model was setup as given below:

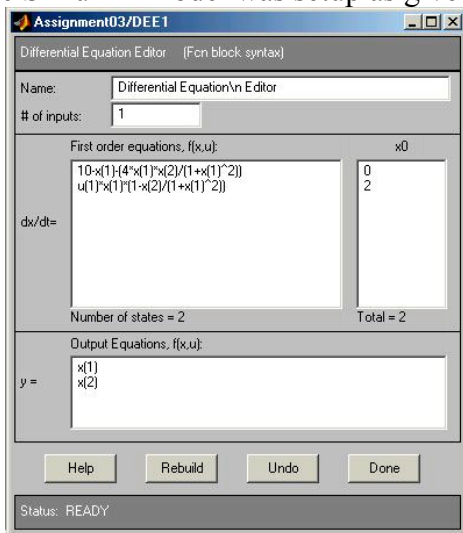


Fig # 1 : Inside the DEE block

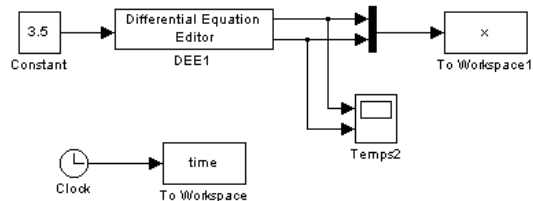


Fig # 2 : The Simulink Block Diagram

In this case the relative tolerance (*In the menu bar go to, Simulation>Simulation parameters>Relative tolerance*) was set at 10^{-9} . It must be set at a small value to get smooth curves like the ones given here. The simulation was run for 200 seconds.

c)

The time trace of the state variables for the three feed rate conditions are given below:

- (i) The normal operation, where $u = 3.5$,

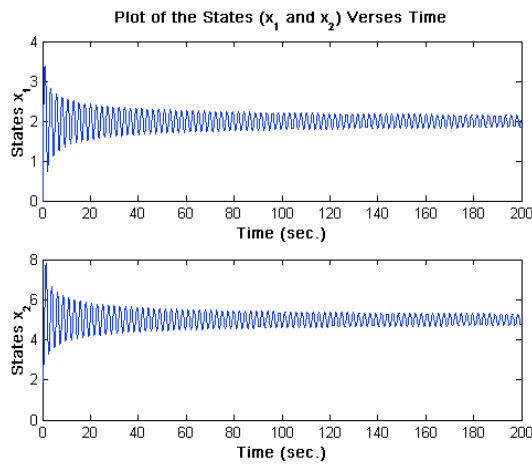


Fig # 3 : Plot of the States for $u = 3.5$

- (ii) A step change to the input $\Delta u = 0.5$ at time $t = 0$,

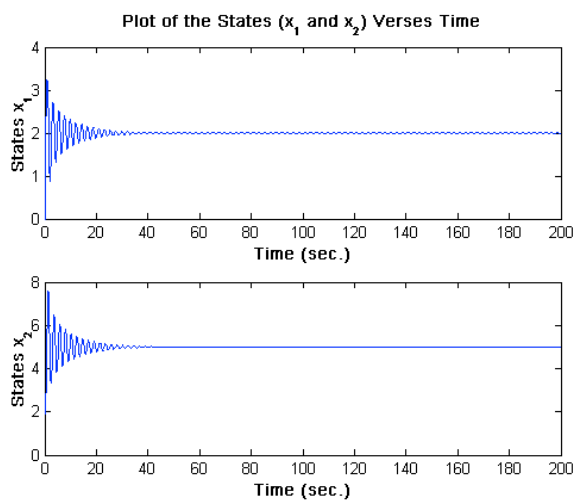


Fig # 4 : Plot of the States for step change to the input $\Delta u = 0.5$ at time $t = 0$

- (iii) A step change to the input $\Delta u = -0.5$ at time $t = 0$,

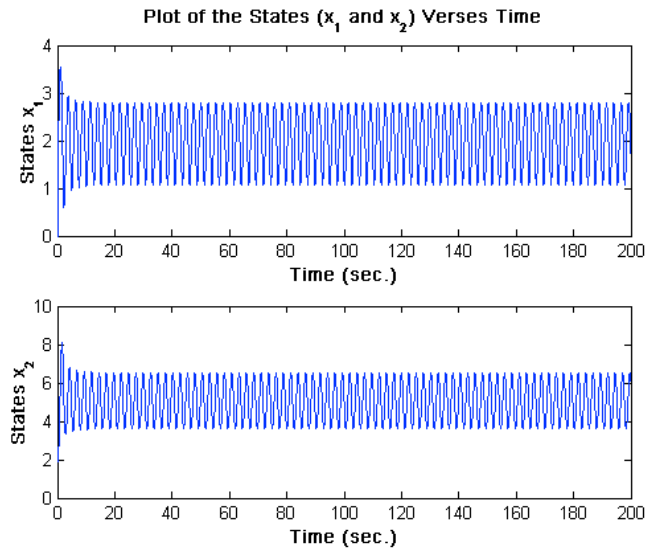


Fig # 5 : Plot of the States for step change to the input $\Delta u = -0.5$ at time $t = 0$

d)

The phase portrait of the state variables for the three feed rate conditions are given below:

- (i) The normal operation, where $u = 3.5$,

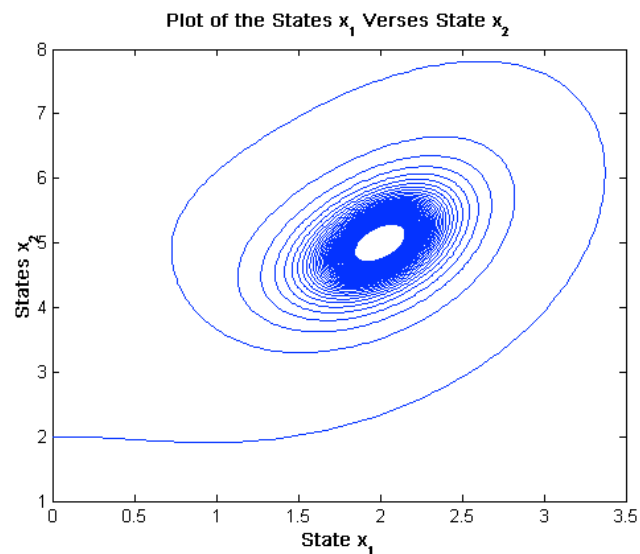


Fig # 6 : Plot of the States for $u = 3.5$

- (ii) A step change to the input $\Delta u = 0.5$ at time $t = 0$,

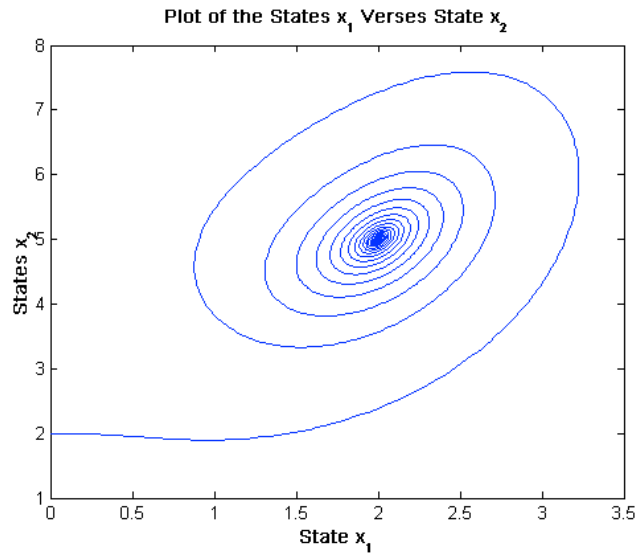


Fig # 7 : Plot of the States for step change to the input $\Delta u = 0.5$ at time $t = 0$

- (iii) A step change to the input $\Delta u = -0.5$ at time $t = 0$,

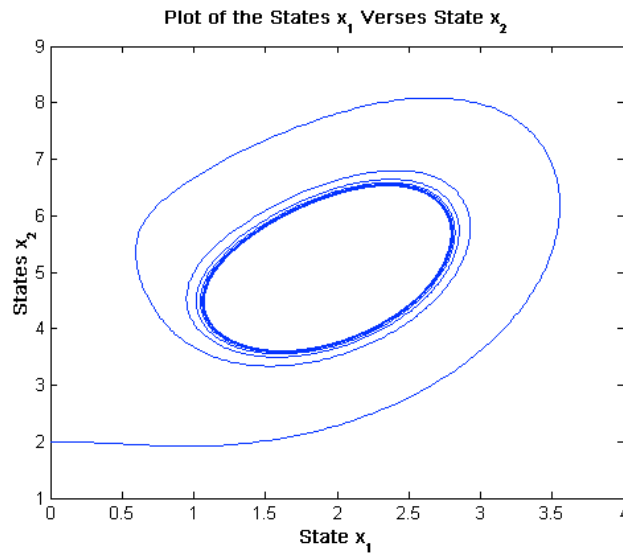


Fig # 8 : Plot of the States for step change to the input $\Delta u = -0.5$ at time $t = 0$

e)

$$A = \nabla_x f|_{x_0, u_0} = \begin{bmatrix} -1 - \frac{4x_2}{1+x_1^2} + \frac{8x_1^2 x_2}{(1+x_1^2)^2} & -\frac{4x_1}{1+x_1^2} \\ u_1 \left(1 - \frac{x_2}{1+x_1^2}\right) + \frac{2u_1 x_1^2 x_2}{(1+x_1^2)^2} & -\frac{u_1 x_1}{1+x_1^2} \end{bmatrix}_{x_0, u_0} = \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix}$$

$$B = \nabla_u f|_{x_0, u_0} = \begin{bmatrix} 0 \\ x_1 \left(1 - \frac{x_2}{1+x_1^2}\right) \end{bmatrix}_{x_0, u_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{dX}{dt} = \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \quad (2)$$

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

It should be noted in the State Space equations (Equation (2)) given above that, the process input has no influence on the output. So, whatever input is given, it does not change the process output. The linearized model fails to represent the process output at any other point except the steady state point.

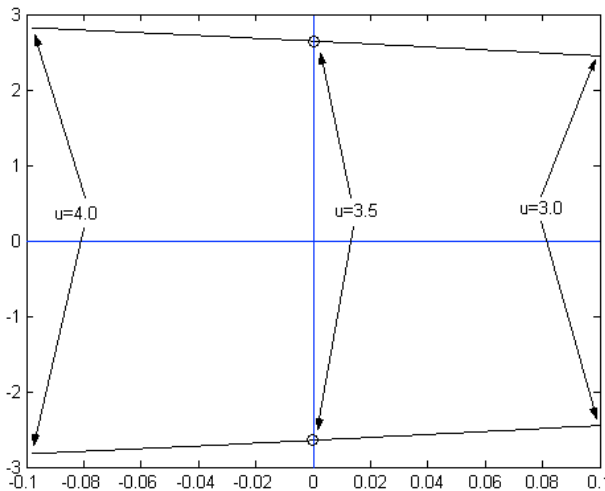


Fig # 10 : The Eigenvalues for Different u 's ($u=3.0$ to 4.0)

From Figure (10) it can be clearly noted that, the eigenvalues corresponding to $u = 3.5$ lie on the imaginary axis. This indicates that the process will be marginally stable at this point. If it moves to the right by a very small amount, that is if u decreases

infinitesimally, it will become unstable and spiral away from the equilibrium point. On the other hand, if u is increased by a small margin; the process will spiral into the equilibrium point at, $\mathbf{x} = [2 \ 5]^T$. So, $u = 3.5$ is a bifurcation point for this process. It will be wrong to linearize around this point, as this will give us erroneous results for minor changes in u (see:

<http://monet.physik.unibas.ch/~elmer/pendulum/bif.htm#attractor>
to learn more about bifurcations).

Can you reconcile the behavior of the eigenvalues shown in Figure 10 and the behavior exhibited by the nonlinear system shown in Figures 6, 7 and 8?

f,g,h,i)

Applying Laplace on Equation (2) gives,

$$s\underline{X}(s) - x(0) = \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s)$$

$$\underline{Y}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s)$$

Since the system is not initially at steady state then $x(0) \neq 0$ and cannot be eliminated.

$$\left(s\underline{I} - \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix} \right) \underline{X}(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s) + x(0)$$

$$\underline{Y}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s)$$

We are interested in the poles of this transfer function to make deductions about the stability of the process. For that purpose, we replace the $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

$$\underline{X}(s) = \left(s\underline{I} - \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix} \right)^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \underline{U}(s) + \left(s\underline{I} - \begin{bmatrix} 1.4 & -1.6 \\ 5.6 & -1.4 \end{bmatrix} \right)^{-1} x(0)$$

$$\underline{Y}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{X}(s)$$

And, now substituting the first equation into the second gives,

$$\underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s-1.4 & 1.6 \\ -5.6 & s+1.4 \end{bmatrix}^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \right\} \underline{U}(s) + \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s-1.4 & 1.6 \\ -5.6 & s+1.4 \end{bmatrix}^{-1} \right\} x(0)$$

$$\text{or, } \underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1.4}{s^2+7} & \frac{-1.6}{s^2+7} \\ \frac{5.6}{s^2+7} & \frac{s-1.4}{s^2+7} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \right\} \underline{U}(s) + \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1.4}{s^2+7} & \frac{-1.6}{s^2+7} \\ \frac{5.6}{s^2+7} & \frac{s-1.4}{s^2+7} \end{bmatrix} \right\} x(0)$$

$$\text{or, } \underline{Y}(s) = \left\{ \begin{bmatrix} \frac{s+1.4}{s^2+7} & \frac{-1.6}{s^2+7} \\ \frac{5.6}{s^2+7} & \frac{s-1.4}{s^2+7} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \right\} \underline{U}(s) + \begin{bmatrix} \frac{s+1.4}{s^2+7} & \frac{-1.6}{s^2+7} \\ \frac{5.6}{s^2+7} & \frac{s-1.4}{s^2+7} \end{bmatrix} x(0) \quad (3)$$

It is clearly visible from Equation (3) that, the denominator of both the equations will be $s^2 + 7$. So, the poles of the transfer functions will be, $s = \pm\sqrt{7}i$. Which indicates that the process will show marginally stable behavior for $u = 3.5$. The states of the process will keep on oscillating indefinitely.

Finally if we replace $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we will have:

$$\underline{Y}(s) = \begin{bmatrix} \frac{s+1.4}{s^2+7} & \frac{-1.6}{s^2+7} \\ \frac{5.6}{s^2+7} & \frac{s-1.4}{s^2+7} \end{bmatrix} x(0)$$

Since matrix B in the state space form of the system is zero, any change in the input will not affect Y and input-output transfer function is zero. Therefore if the initial condition was zero (steady state) the output would stay at zero (steady state). In this problem the initial conditions are not zero that will affect the output. Since the linear system is based on deviation variables, the initial condition should be converted to deviation variable as

well. $x(0) = \begin{bmatrix} 0 - 2 \\ 2 - 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$. Then

$$\underline{Y}(s) = \begin{bmatrix} \frac{-2s+2}{s^2+7} \\ \frac{-3s-7}{s^2+7} \end{bmatrix}$$

The above equation is the trajectory of output variable in Laplace domain. By taking the inverse Laplace of it the trajectory of output can be calculated.

As it is shown in this problem, it is difficult to include the effects of initial conditions in transfer function models. This would be much more difficult if we had $B \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus,

transfer function models are not really appropriate for the type of simulation we would like to run. State-space models, however, incorporate initial conditions very naturally and since LTI state-space and the transfer functions that are derived from them are “equivalent” representations, then we can use our linearized state-space models to do the simulation.

In order to set up the state-space block the initial condition should be set to $\begin{bmatrix} -2 & -3 \end{bmatrix}$.

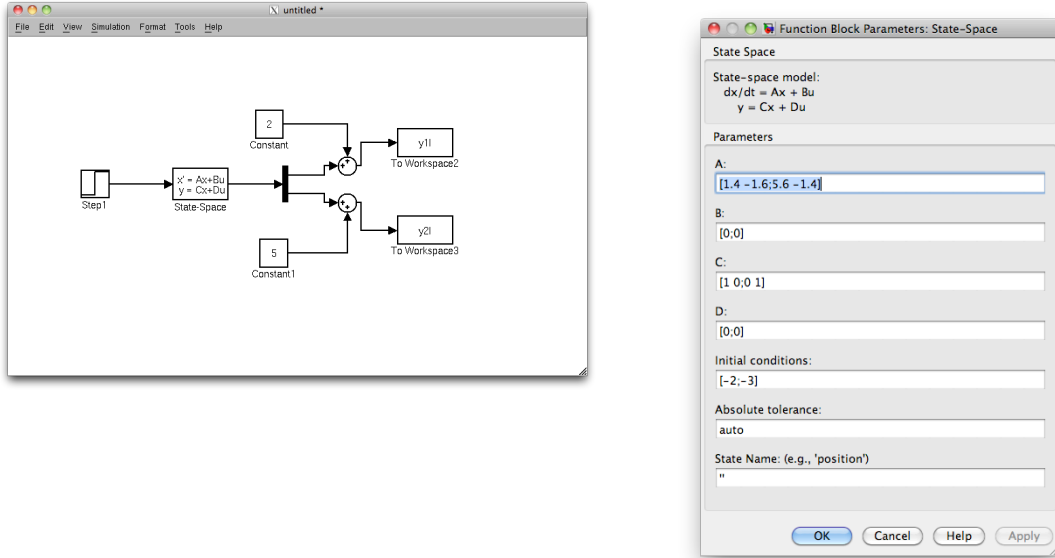


Fig # 11 : The Simulink Block Diagram Used to Compare the TwoTypes of Models

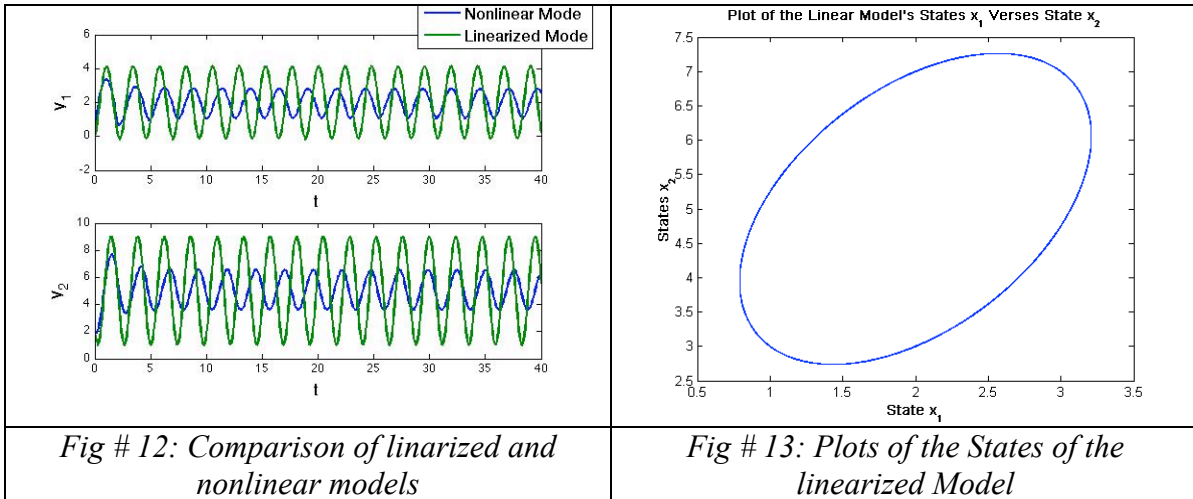


Fig # 12: Comparison of linearized and nonlinear models

Fig # 13: Plots of the States of the linearized Model

The original nonlinear model and the linear state-space model are compared in Figure 12 for the negative change in input. You should note that the sine waves have different amplitudes for the nominal value of u . Since in the linearized model the input u has no effect on the states or the outputs, changing u will not result in any different behavior than is shown in Figure 12. Thus the linearized model will not adequately represent the behavior of the nonlinear model, even locally.

This is an example where linearization results in a poor model.

SOLUTION OF QUESTION NO. 2

The given system is:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 - 2x_2 \\ \frac{dx_2}{dt} &= 2x_1 - x_2 + u \\ y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2\end{aligned}\tag{1}$$

The process is at steady state at $x(0) = [0 \ 0]^T$.

a)

The Jacobian of the set of ODE given in Equations (1) is:

$$Jac = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

Evaluating the eigenvalues of the Jacobian gives, $0 \pm i\sqrt{3}$. Since the real part of the eigenvalues is zero, the system is marginally stable.

b)

The eigenvalues of the system are at $0 \pm i\sqrt{3}$. This indicates that the process will become oscillatory after any kind of excitation and the oscillation will persist indefinitely. So, for both step input and impulse input, the process will start to oscillate. The only difference will be that the center of oscillation will be different for the two inputs. For the impulse response the oscillations will be about $\mathbf{x}=[0 \ 0]^T$; whereas, for the step response, the oscillations will be about some nonzero steady-state value.

c)

The system is linear and can be represented as a linear state-space model as following

$$A = \nabla_{\mathbf{x}} f \Big|_{x_0, u_0} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

$$B = \nabla_u f \Big|_{x_0, u_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \nabla_x g|_{x_0, u_0} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$D = \nabla_u g|_{x_0, u_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The State Space model will be:

$$\frac{dx}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

(2)

The simulink model is given in Figure 2.

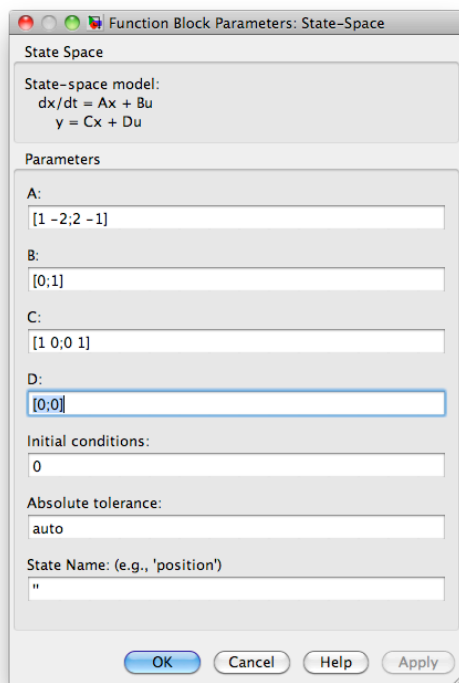


Fig # 1 : Inside the State-Space block

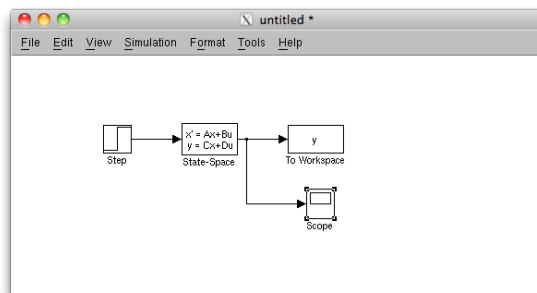


Fig # 2 : The Simulink Block Diagram (for Step response)

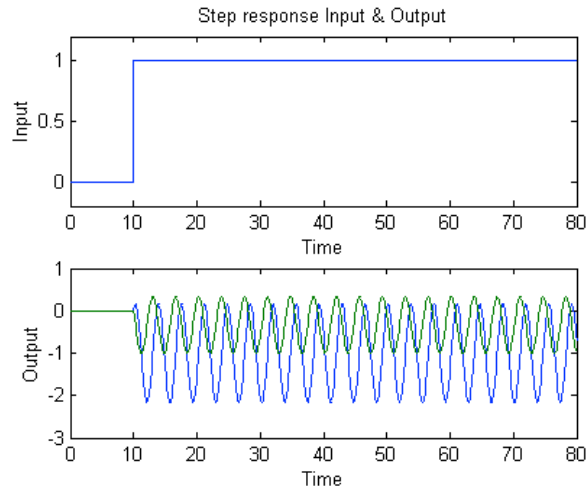


Fig # 3 : Step Response Input and Output

```
subplot (2,1,1); plot (t,u); xlabel ('Time'); ylabel ('Input');
subplot (2,1,2); plot (t,y); xlabel ('Time'); ylabel ('Output');
title ('Step response Input & Output');
```

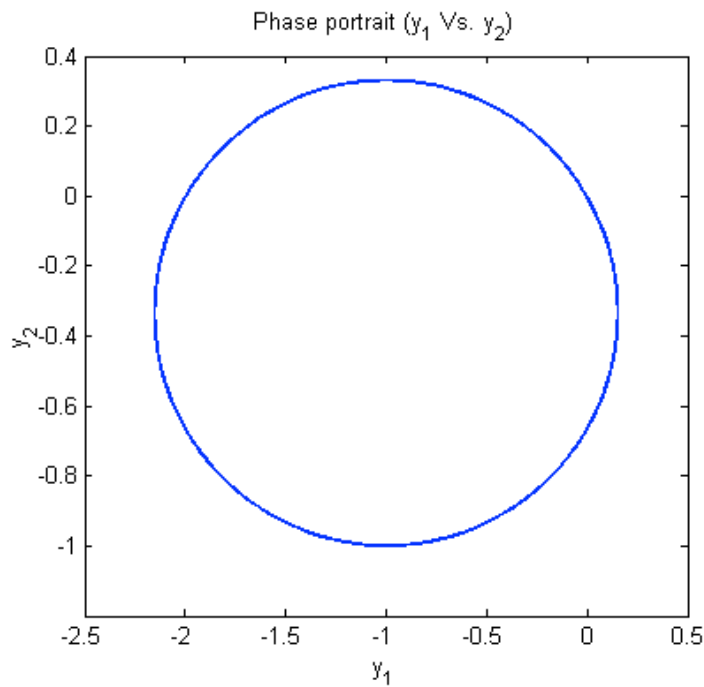


Fig # 4 : Phase Portrait

```
figure; plot (y(:,1),y(:,2));
title ('Phase portrait (y_1 Vs. y_2)'); xlabel ('y_1'); ylabel ('y_2');
```

d)

Applying Laplace on Equation (2) gives,

$$s\underline{X}(s) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{U}(s)$$

$$\underline{Y}(s) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s)$$

or,
$$\left(s\underline{I} - \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \right) \underline{X}(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{U}(s)$$

$$\underline{Y}(s) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{X}(s) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \underline{U}(s)$$

Since, $\underline{Y}(s) = \left[\underline{C}(s\underline{I} - \underline{A})^{-1} \underline{B} + \underline{D} \right] \underline{U}(s)$, so,

$$\underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(s\underline{I} - \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \underline{U}(s)$$

or,
$$\underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} s-1 & 2 \\ -2 & s+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \underline{U}(s)$$

or,
$$\underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} s-1 & 2 \\ -2 & s+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \underline{U}(s)$$

$$\text{Since, } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{so, } \begin{bmatrix} s-1 & 2 \\ -2 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s-1)(s+1) - 2 \times (-2)} \begin{bmatrix} s+1 & -2 \\ 2 & s-1 \end{bmatrix}$$

$$\underline{Y}(s) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{s^2 - 1 + 4} \begin{bmatrix} s+1 & -2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \underline{U}(s)$$

or,
$$\underline{Y}(s) = \frac{1}{(s^2 + 3)} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ s-1 \end{bmatrix} \right\} \underline{U}(s)$$

or,
$$\underline{Y}(s) = \frac{1}{(s^2 + 3)} \begin{bmatrix} s-1-2 \\ -2-s+1 \end{bmatrix} \underline{U}(s)$$

or,
$$\underline{Y}(s) = \begin{bmatrix} \frac{s-3}{(s^2 + 3)} \\ -\frac{s+1}{(s^2 + 3)} \end{bmatrix} \underline{U}(s)$$

e)

From Equation (2) we know that,

$$\frac{dx}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

In order to discretize the system we need to calculate $\Phi(T_s) = e^{AT_s}$. To calculate Φ , one can use 'expm' command of MATLAB. Note that $e^{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \neq \begin{bmatrix} e^a & e^b \\ e^c & e^d \end{bmatrix}$.

For $T_s = \frac{2\pi}{\sqrt{3}}$ seconds,

$$\Phi\left(T_s = \frac{2\pi}{\sqrt{3}}\right) = e^{A\frac{2\pi}{\sqrt{3}}} = e^{\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\left(\frac{2\pi}{\sqrt{3}}\right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma\left(T_s = \frac{2\pi}{\sqrt{3}}\right) = (\Phi - I)A^{-1}B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Gamma\left(T_s = \frac{2\pi}{\sqrt{3}}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the first matrix is a zero matrix, all the terms in the resultant matrix will be zero too.

So, the discrete state space model will be:

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(k)$$

For $T_s = \frac{\pi}{\sqrt{3}}$ seconds,

$$\Phi\left(T_s = \frac{\pi}{\sqrt{3}}\right) = e^{A\frac{\pi}{\sqrt{3}}} = e^{\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\left(\frac{\pi}{\sqrt{3}}\right)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Gamma\left(T_s = \frac{\pi}{\sqrt{3}}\right) = (\Phi - I)A^{-1}B = \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Gamma\left(T_s = \frac{2\pi}{\sqrt{3}}\right) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

so, $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{1 \times (-1) - 2 \times (-2)} \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{-1+4} \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Again,

$$\Gamma\left(T_s = \frac{2\pi}{\sqrt{3}}\right) = (\Phi - I)A^{-1}B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \end{bmatrix}$$

So, the discrete state space model will be:

$$x(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(k) + \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(k)$$

For $T_s = \frac{\pi}{5\sqrt{3}}$ seconds,

$$\Phi\left(T_s = \frac{\pi}{5\sqrt{3}}\right) = e^{A \frac{\pi}{5\sqrt{3}}} = e^{\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \left(\frac{\pi}{5\sqrt{3}}\right)} = \begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix}$$

$$\Gamma\left(T_s = \frac{\pi}{5\sqrt{3}}\right) = (\Phi - I)A^{-1}B = \left(\begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or, $\Gamma\left(T_s = \frac{\pi}{5\sqrt{3}}\right) = \begin{bmatrix} 0.1484 & -0.6787 \\ 0.6787 & -0.5303 \end{bmatrix} \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

or, $\Gamma\left(T_s = \frac{\pi}{5\sqrt{3}}\right) = \begin{bmatrix} 0.1484 & -0.6787 \\ 0.6787 & -0.5303 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0.1484 \times \frac{2}{3} - 0.6787 \times \frac{1}{3} \\ 0.6787 \times \frac{2}{3} - 0.5303 \times \frac{1}{3} \end{bmatrix}$

or, $\Gamma\left(T_s = \frac{\pi}{5\sqrt{3}}\right) = \begin{bmatrix} -0.1273 \\ 0.2757 \end{bmatrix}$

So, the discrete state space model will be:

$$x(k+1) = \begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix} x(k) + \begin{bmatrix} -0.1273 \\ 0.2757 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} y(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(k)$$

f)

$$\text{For } T_s = \frac{2\pi}{\sqrt{3}}, \Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\Phi - \lambda I| = 0$$

$$\text{or, } \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{or, } \left| \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \right| = 0$$

$$\text{or, } (1-\lambda)(1-\lambda) = 0$$

$$\text{or, } \lambda = 1, 1$$

This point is on the unit circle.

$$\text{For, } T_s = \frac{\pi}{\sqrt{3}}, \Phi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|\Phi - \lambda I| = 0$$

$$\text{or, } \left| \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{or, } \left| \begin{bmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \right| = 0$$

$$\text{or, } (-1-\lambda)(-1-\lambda) = 0$$

$$\text{or, } \lambda = -1, -1$$

This point is on the unit circle.

$$\text{For, } T_s = \frac{\pi}{5\sqrt{3}}, \Phi = \begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix}$$

$$\begin{aligned} |\Phi - \lambda I| &= 0 \\ \text{or, } \begin{vmatrix} 1.1484 - \lambda & -0.6787 \\ 0.6787 & 0.4697 - \lambda \end{vmatrix} &= 0 \\ \text{or, } \begin{vmatrix} 1.1484 - \lambda & -0.6787 \\ 0.6787 & 0.4697 - \lambda \end{vmatrix} &= 0 \\ \text{or, } (1.1484 - \lambda)(0.4697 - \lambda) - (0.6787)(-0.6787) &= 0 \\ \text{or, } \lambda^2 - 1.6181\lambda + 0.5394 + 0.4606 &= 0 \\ \text{or, } \lambda^2 - 1.6181\lambda + 1 &= 0 \\ \text{or, } \lambda &= 0.809 \pm 0.5877i \end{aligned}$$

Since, $\sqrt{(0.809)^2 + (0.5877)^2} = 1$, so this point is also on the unit circle.

So, for all three cases, we will get *marginally stable systems*.

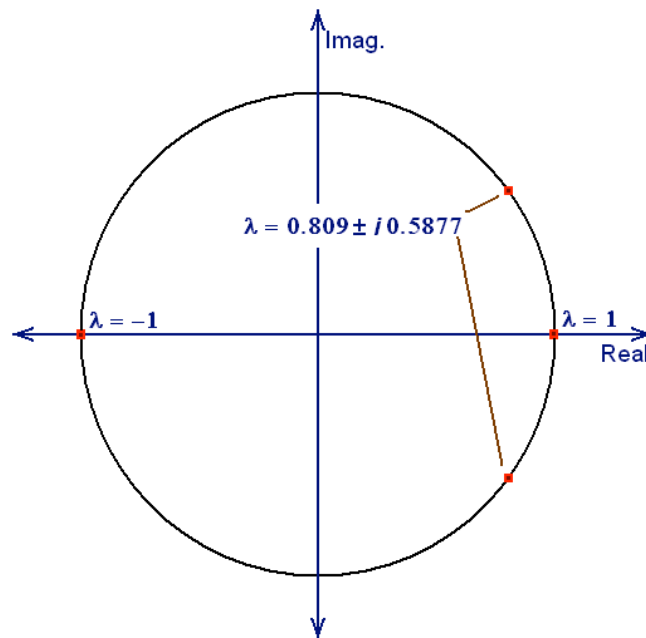


Fig # 5 : Plot of Eigen Values on a Unit Circle

g)

All three models are marginally stable. The first one ($T_s = \frac{2\pi}{\sqrt{3}} \text{sec.}$) has eigenvalues at $\lambda = 1$, the integrator mode. The second model ($T_s = \frac{\pi}{\sqrt{3}} \text{sec}$) has eigenvalues at $\lambda = -1$, the ringing mode. The last one ($T_s = \frac{\pi}{5\sqrt{3}} \text{sec}$) is also on the unit circle. Thus all three models are marginally stable.

.....

h)

We know, $\overline{GH}(z) = C((zI - \Phi)^{-1} \Gamma + D$

For $T_s = \frac{2\pi}{\sqrt{3}}$, $\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So,

$$\begin{aligned} G(z) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left((zI - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For $T_s = \frac{\pi}{\sqrt{3}}$, $\Phi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So,

$$\begin{aligned} \overline{GH}(z) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} z+1 & 0 \\ 0 & z+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now,

$$\left(\begin{bmatrix} z+1 & 0 \\ 0 & z+1 \end{bmatrix} \right)^{-1} = \frac{1}{(z+1)(z+1)} \begin{bmatrix} z+1 & 0 \\ 0 & z+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(z+1)} & 0 \\ 0 & \frac{1}{(z+1)} \end{bmatrix}$$

$$\begin{aligned}
GH(z) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{(z+1)} & 0 \\ 0 & \frac{1}{(z+1)} \end{bmatrix} \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{4}{3(z+1)} \\ -\frac{2}{3(z+1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-4-2}{3(z+1)} \\ \frac{-4+2}{3(z+1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-2}{(z+1)} \\ \frac{-0.6667}{(z+1)} \end{bmatrix}
\end{aligned}$$

For $T_s = \frac{\pi}{5\sqrt{3}}$, $\Phi = \begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix}$, $\Gamma = \begin{bmatrix} -0.1273 \\ 0.2757 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
So,

$$\begin{aligned}
GH(z) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1.1484 & -0.6787 \\ 0.6787 & 0.4697 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.1273 \\ 0.2757 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} z-1.1484 & 0.6787 \\ -0.6787 & z-0.4697 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.1273 \\ 0.2757 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
\left(\begin{bmatrix} z-1.1484 & 0.6787 \\ -0.6787 & z-0.4697 \end{bmatrix} \right)^{-1} &= \frac{1}{(z-1.1484)(z-0.4697)+0.4606} \begin{bmatrix} z-0.4697 & -0.6787 \\ 0.6787 & z-1.1484 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(z-0.4697)}{(z^2-1.6181z+1)} & -\frac{0.6787}{(z^2-1.6181z+1)} \\ \frac{0.6787}{(z^2-1.6181z+1)} & \frac{(z-1.1484)}{(z^2-1.6181z+1)} \end{bmatrix}
\end{aligned}$$

i)

The simulation results are given below:

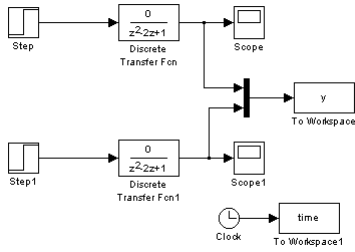


Fig # 6 : Discrete Transfer Function Model Based on Sampling time of $T_s = \frac{2\pi}{\sqrt{3}}$ sec.

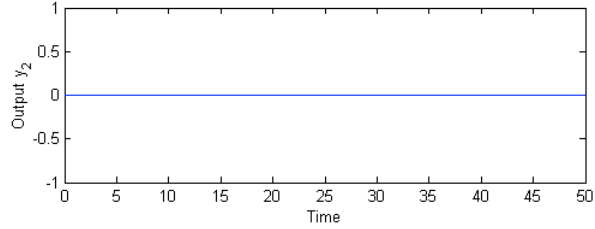
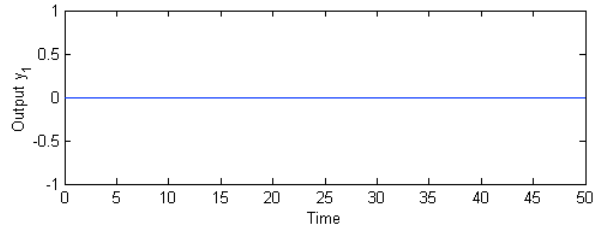


Fig # 7 : Output of Model Based on Sampling Time of $T_s = \frac{2\pi}{\sqrt{3}}$ sec.

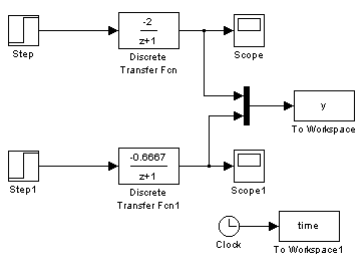


Fig # 8 : Discrete Transfer Function Model Based on Sampling time of $T_s = \frac{\pi}{\sqrt{3}}$ sec.

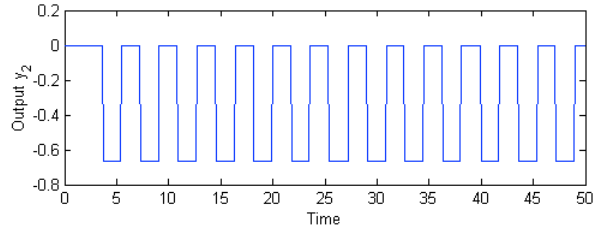
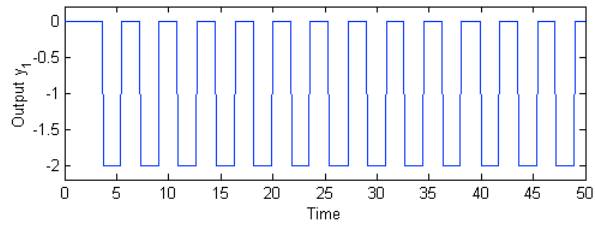


Fig # 9 : Output of Model Based on Sampling Time of $T_s = \frac{\pi}{\sqrt{3}}$ sec.

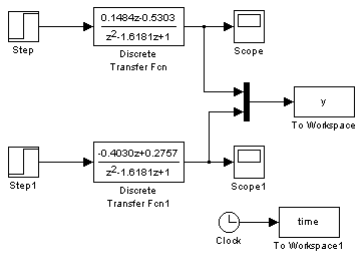


Fig # 10 : Discrete Transfer Function Model Based on Sampling time of $T_s = \frac{\pi}{5\sqrt{3}}$ sec.

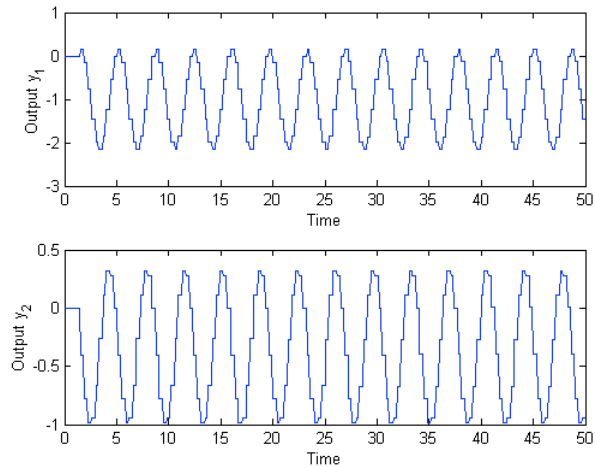


Fig # 11 : Output of Model Based on Sampling Time of $T_s = \frac{\pi}{5\sqrt{3}}$ sec.

j)

The thing to be noted here that, we must sample data at a rate that is more than double the frequency natural of the process. The results we have here are classic examples of what can happen if 1) the sample and process oscillation periods are identical ($T_s = \frac{2\pi}{\sqrt{3}}$ sec), 2) we sample twice per process oscillation (the Shannon frequency) ($T_s = \frac{\pi}{\sqrt{3}}$ sec), and 3) we sample at a faster rate than the process oscillates ($T_s = \frac{\pi}{5\sqrt{3}}$ sec).

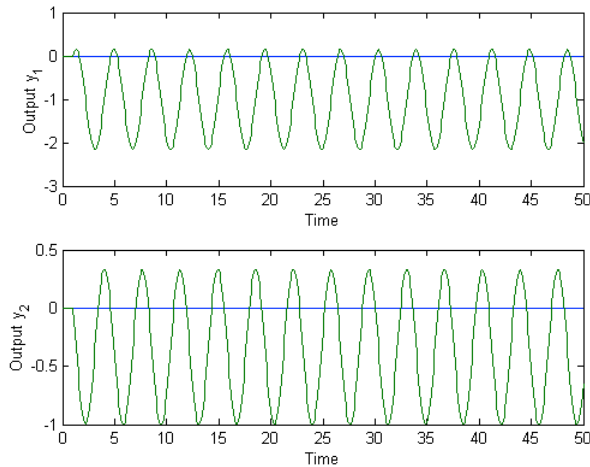


Fig # 12 : Actual Output and the Discrete Transfer Function
 Model Based on Sampling time of $T_s = \frac{2\pi}{\sqrt{3}} \text{ sec}$

In Figure 12, the sampling and oscillation frequencies are identical. So, if it is assumed that the first sample was taken at the crest of a sine wave, the next one will be taken at the next crest. So, it will give a straight line.

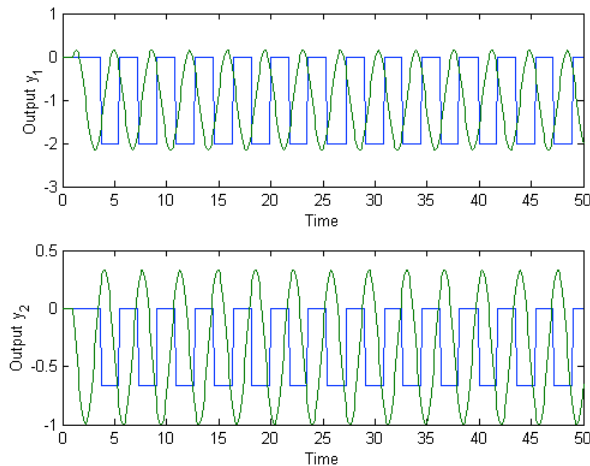


Fig # 13 : Actual Output and the Discrete Transfer Function
 Model Based on Sampling time of $T_s = \frac{\pi}{\sqrt{3}} \text{ sec}$

In Fig. 13, we are sampling at Shannon frequency for this process (i.e. sampling rate is twice that of the oscillation frequency). Here we do capture the dynamics of the process, but in a wrong way (observe, we have a peak where we are supposed to have a trough). Thus the simulation seems “out of phase” with the process. Since, we are sampling only two points from every cycle of the sine wave, another sine wave of different amplitude

and phase can fit the data. That is exactly what has happened here. So, Shannon sampling is not enough for most of the cases.

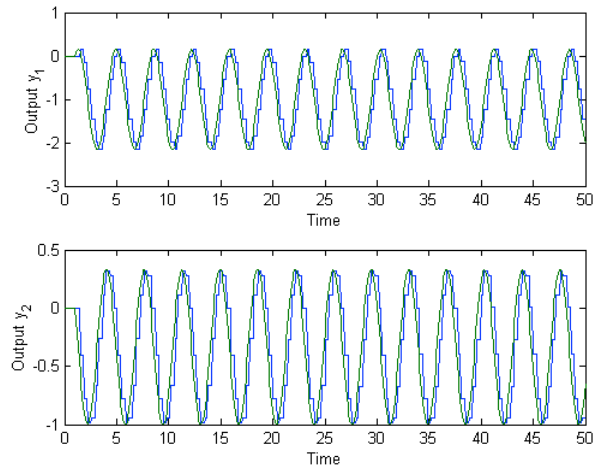


Fig # 14 : Actual Output and the Discrete Transfer Function

$$\text{Model Based on Sampling time of } T_s = \frac{\pi}{5\sqrt{3}} \text{ sec}$$

In Fig 14, we are sampling at a much faster rate than the previous two cases. So, we are getting a better representation of the true scenario. But, we must also keep in mind that sampling at a faster rate does make the model more complex.