Variation of Geometric Invariant Theory and Derived Categories

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Based on joint work with M. Ballard (U. Wisconsin) and Ludmil Katzarkov (U. Miami and U. Vienna).
1 Motivating Example
2 Background on GIT
3 General results
4 Landau-Ginzburg models and factorizations
5 RG-flow and a theorem of Orlov
Outline

1 Motivating Example

2 Background on GIT

3 General results

4 Landau-Ginzburg models and factorizations

5 RG-flow and a theorem of Orlov
Consider $k[x_0, x_1, x_2]$ with the $\mathbb{G}_m$-action with weights $(1, 1, n)$. We define $\mathbb{P}(1 : 1 : n)$ as the smooth global quotient Deligne-Mumford stack, $[(\text{Spec } k[x_0, x_1, x_2] \setminus 0)/\mathbb{G}_m]$. Characters of $\mathbb{G}_m$, $\lambda \mapsto \lambda^i$, give line bundles, $\mathcal{O}(i)$, and a tilting object, $T$, is given by,

$$T := \bigoplus_{i=0}^{n+2} \mathcal{O}(i)$$

**Quiver for $\mathbb{P}(1 : 1 : 4)$:**
Hirzebruch Surfaces

Consider the total space of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ with the $\mathbb{G}_m$-action given by dilating the fibers. The Hirzebruch surface, $\mathbb{F}_n$, is defined as the projective bundle, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, which represents the smooth global quotient stack, $[\text{tot}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))/\text{zero section}/\mathbb{G}_m]$. A tilting object, $T$, is given by,

$$T := \mathcal{O} \oplus \pi^* \mathcal{O}(1) \oplus \mathcal{O}_\pi(1) \oplus \pi^* \mathcal{O}(1) \otimes \mathcal{O}_\pi(1).$$
Motivating Example

Comparing

Quiver for $\mathbb{P}(1 : 1 : 4)$:

$\begin{align*}
&x_2 \\
&x_0 \\
&x_1
\end{align*}$

Quiver for $\mathbb{P}_n$:

$\begin{align*}
&x_0 \\
&x_1 \\
&x_2
\end{align*}$
A semi-orthogonal decomposition of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^\perp$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & T_{m-1} & \rightarrow & \cdots & \rightarrow & T_2 & \rightarrow & T_1 & \rightarrow & T \\
& & \downarrow & & & \downarrow & & \downarrow & & \downarrow \\
& & A_m & & & & A_2 & & A_1 \\
\end{array}
$$

where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$. 

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VGIT and Derived Categories
Hirzebruch surfaces

- If $n < 2$, there is a semi-orthogonal decomposition

$$D^b(\text{coh } \mathbb{F}_n) = \langle E_1, \ldots, E_{2-n}, D^b(\text{coh } \mathbb{P}(1, 1, n)) \rangle$$

with $E_i$ exceptional objects.

- If $n = 2$, we have an equivalence

$$D^b(\text{coh } \mathbb{F}_n) = D^b(\text{coh } \mathbb{P}(1, 1, 2)).$$

- If $n > 2$, there is a semi-orthogonal decomposition

$$D^b(\text{coh } \mathbb{P}(1, 1, n)) = \langle E_1, \ldots, E_{n-2}, D^b(\text{coh } \mathbb{F}_n) \rangle$$

with $E_i$ exceptional objects.
Outline

1 Motivating Example
2 Background on GIT
3 General results
4 Landau-Ginzburg models and factorizations
5 RG-flow and a theorem of Orlov
• $X$ is a smooth quasi-projective variety over an algebraically closed field, $k$ of characteristic zero,

• $G$ is a linearly reductive algebraic group acting on $X$,

• $L$ is a $G$-equivariant ample line bundle on $X$,

The semi-stable locus is an open subset,

$$X^{ss}(L) := \{ x \in X \mid \exists f \in H^0(X, L^n)^G \text{ with } n \geq 0, f(x) \neq 0, \text{ and } X_f \text{ affine} \}$$

For us, the GIT quotient corresponding to this data is the global quotient stack $[X^{ss}(L)/G]$. We can vary the $G$-equivariant structure on $L$ by choosing characters, $\chi$, in the dual group, $\hat{G} := \text{Hom}(G, \mathbb{G}_m)$. We denote the GIT quotient corresponding to this linearization by $X//L(\chi)$. 
The unstable locus, $A_\chi$, is the complement of the semistable locus in $X$. Let $X$ be proper or affine. There exists a fan in $\hat{G}_R$ with support the set of characters in $\hat{G}$ with $X^{ss} \neq \emptyset$. For each $\chi \in \hat{G}$, we have a cone

$$C_\chi = \{ \mu \in \hat{G}_R : A_\mu \subset A_\chi \}.$$ 

These are the cones of the fan. The characters on the relative interiors of the cones have equal unstable loci.

The maximal cones in the GIT fan are called the chambers. The codimension one cones are called walls.

If $G$ is Abelian, the GIT fan is the GKZ fan.
We can realize \( F_n \) as a GIT quotient of \( \mathbb{A}^4 \) by the subgroup
\[
\{ (r, r^{-n}s, r, s) : r, s \in \mathbb{G}_m \} \subset \mathbb{G}_m^4.
\]
Write \( k[x, y, u, v] \) for the ring of regular functions on \( \mathbb{A}^4 \).
The GIT fan for this quotient is

\[
\begin{align*}
\mathbb{P}(1, 1, n)
\end{align*}
\]

We have labeled rays by the variables with that associated character and labeled the chambers according to their toric stacks.
Let $\lambda : \mathbb{G}_m \rightarrow G$ be a one parameter subgroup of $G$. Let $L(\lambda)$ be the centralizer of $\lambda$ in $G$. Let $Z$ denote the $\lambda$-fixed locus. For simplicity, assume $Z$ is connected. The $G$-invariant subvariety, $Z$, inherits an $L(\lambda)/\lambda(\mathbb{G}_m)$-action and an induced linearization (pulling back $L$). Let $Z_\lambda$ denote the semi-stable locus and let $S^\pm_\lambda$ be

$$S^+_\lambda := \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x \in Z_\lambda\}$$

$$S^-_\lambda := \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \in Z_\lambda\}.$$

Denote by $G \cdot S^\pm_\lambda$ the orbit of $S^\pm_\lambda$ under the $G$-action.
Let $X$ be a smooth projective variety equipped with the action of reductive algebraic group, $G$. Choose an ample line bundle, $L$, with an equivariant structure.

**Theorem (Kempf, Hesselink, Kirwan, Ness)**

There exist finitely many one-parameter subgroups, $\lambda_i : \mathbb{G}_m \to G$, with

$$X^{\text{us}}(L) = G \cdot S_{\lambda_1}^+ \cup \cdots \cup G \cdot S_{\lambda_p}^+. $$
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Suppose that we have a one-parameter family of linearizations, $L_t$, such that

- $X^{ss}(L_0) = G \cdot S_{\lambda_1}^+ \cup \cdots \cup G \cdot S_{\lambda_p}^+ \cup X^{ss}(L_t)$ for $t > 0$
- $X^{ss}(L_0) = G \cdot S_{\lambda_1}^- \cup \cdots \cup G \cdot S_{\lambda_p}^- \cup X^{ss}(L_t)$ for $t < 0$.

For example, $X$ is proper or $X$ is affine space and $G$ is Abelian. Denote the quotient by $t > 0$ as $X/\!/+\!+$ and denote the quotient by $t < 0$ as $X/\!/-\!-\!$. If $p = 1$, let $Y$ be the GIT quotient associated to $Z_{\lambda_1}$. Choose a fixed point $x \in Z_{\lambda_i}$. Let $\mu_i$ be the sum of the weights of $\mathbb{G}_m$-action on the normal bundles to $G \cdot S_{\lambda_i}^+$ and $G \cdot S_{\lambda_i}^-$ restricted to $x$. 

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VGIT and Derived Categories
Main theorem

Theorem (Ballard-F-Katzarkov, Halpern-Leinster)

Fix $d_1, \ldots, d_p \in \mathbb{Z}$.

- If $\mu_i > 0$ for all $1 \leq i \leq p$, then there exists a left-admissible fully-faithful functor,

$$
\Phi_{d_1, \ldots, d_p} : D^b(\text{coh } X/\sim) \rightarrow D^b(\text{coh } X/\triangleright).
$$

If $p = 1$, then there also exists fully-faithful functors,

$$
\Upsilon_j^- : D^b(\text{coh } Y) \rightarrow D^b(\text{coh } X/\triangleright),
$$

and a semi-orthogonal decomposition,

$$
D^b(\text{coh } X/\triangleright) = \langle \Upsilon_{d}^- D^b(\text{coh } Y), \ldots, \\
\Upsilon_{\mu-d-1}^- D^b(\text{coh } Y), \Phi_d D^b(\text{coh } X/\sim) \rangle.
$$
Theorem (Ballard-F-Katzarkov, Halpern-Leinster)

Fix $d_1, \ldots, d_p \in \mathbb{Z}$.

- If $\mu_i = 0$ for all $1 \leq i \leq p$, then there exist an equivalence,

$$\Phi_{d_1, \ldots, d_p} : D^b(\text{coh } X \sslash -) \to D^b(\text{coh } X \sslash +).$$
Main theorem

Theorem (Ballard-F-Katzarkov, Halpern-Leinster)

Fix $d_1, \ldots, d_p \in \mathbb{Z}$.

- If $\mu < 0$ for all $1 \leq i \leq p$, then there exists a left-admissible fully-faithful functor,

$$\Psi_{d_1, \ldots, d_p} : D^b(\text{coh } X/\langle + \rangle) \to D^b(\text{coh } X/\langle - \rangle)$$

If $p = 1$, then there also exists fully-faithful functors,

$$\Upsilon_j^+ : D^b(\text{coh } Y) \to D^b(\text{coh } X/\langle + \rangle),$$

and a semi-orthogonal decomposition,

$$D^b(\text{coh } X/\langle - \rangle) = \langle \Upsilon_{-d}^+ D^b(\text{coh } Y), \ldots, \Upsilon_{\mu-d+1}^+ D^b(\text{coh } Y), \Psi_d D^b(\text{coh } X/\langle + \rangle) \rangle.$$
If the stabilizer of $x$ is $\mathbb{G}_m$, then $\mu_i > 0$ if and only if the canonical linearization lies on the positive plane for the separating hyperplane corresponding to the wall (normalized so that the $+$ chamber is positive).

In the toric case i.e. if $X$ is the Cox ring of a toric variety, $X$, and $G = \hat{\text{Pic}}(X)$, then a separating hyperplane is given by pairing with a one parameter subgroup, $\lambda$, explicitly, $\langle \lambda, - \rangle$ is an element of $\text{Pic}(X)^*_R$ and we choose $\lambda$ to be primitive in $\text{Hom}(\text{Pic}(X), \mathbb{Z})$. In this case, $p = 1$ corresponding to this 1-parameter subgroup and we get the strongest possible result. Furthermore, $\mu = \langle \lambda, -K_X \rangle$. 

The GIT fan for this quotient is

We have labeled rays by the variables with that associated character and labeled the chambers according to their toric stacks.
Hirzebruch surfaces

- If $n < 2$, there is a semi-orthogonal decomposition
  \[ D^b(\text{coh } \mathbb{F}_n) = \langle E_1, \ldots, E_{2-n}, D^b(\text{coh } \mathbb{P}(1, 1, n)) \rangle \]
  with $E_i$ exceptional objects.
- If $n = 2$, we have an equivalence
  \[ D^b(\text{coh } \mathbb{F}_n) = D^b(\text{coh } \mathbb{P}(1, 1, 2)). \]
- If $n > 2$, there is a semi-orthogonal decomposition
  \[ D^b(\text{coh } \mathbb{P}(1, 1, n)) = \langle E_1, \ldots, E_{n-2}, D^b(\text{coh } \mathbb{F}_n) \rangle \]
  with $E_i$ exceptional objects.
Let $B$ be a quasi-projective algebraic variety, and $\mathcal{E}$ be a vector bundle over $B$. Consider the $\mathbb{G}_m$-action on the total space of $\mathcal{E}$ given by dilating the fibers. There are two linearizations of trivial bundle corresponding to the identity character and inverse. The corresponding GIT-quotients are $\mathbb{P}(\mathcal{E})$ and the empty set. In this case $Y \cong B$ and we get:

$$D^b(\text{coh } \mathbb{P}(\mathcal{E})) = \langle \pi^* D^b(\text{coh } B), \ldots, \pi^* D^b(\text{coh } B)(\mathcal{O}_\pi(n - 1)) \rangle.$$ 

This explains why the Hirzebruch surface and weighted projective stacks of the previous example decompose further.
A theorem of Kawamata

Theorem

Let $X$ be a smooth projective toric DM stack. Then $\text{D}^b(\text{coh } X)$ admits a full exceptional collection.

Idea of Proof:
A theorem of Kawamata

**Theorem**

Let $X$ be a smooth projective toric DM stack. Then $\mathcal{D}^b(\text{coh } X)$ admits a full exceptional collection.

**Idea of Proof:** Let $\Sigma_0$ denote the chamber corresponding to the nef cone of $X$. Choose a sufficiently straight line path in the secondary fan starting at the anti-canonical divisor, passing through $\Sigma_0$ and ending in the complement of the pseudo-effective cone which doesn’t passing through any cones of codimension 2.
General results

A theorem of Kawamata

Theorem

Let $X$ be a smooth projective toric DM stack. Then $\mathbf{D}^b(\text{coh } X)$ admits a full exceptional collection.

Idea of Proof:
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Theorem

Let $X$ be a smooth projective toric DM stack. Then $D^b(\text{coh } X)$ admits a full exceptional collection.

Idea of Proof:
The path, $l$, passes through a finite number of walls, $\tau_1, \ldots, \tau_s$ on its way through the “chambers”, $\Sigma_0, \ldots, \Sigma_{s-1}$ and finally to the complement. The value of $\mu$ is a pairing of the wall with the anti-canonical class, which is always positive because the anti-canonical class is always behind you along this path. where $\Sigma_s$ denotes the complement of the secondary fan as opposed to a chamber, thus the quotation marks. Let $X_i$ be the GIT quotient corresponding to the chamber $\Sigma_i$ with $X := X_0$ and $X_s = \emptyset$ so that $D^b(\text{coh } X_s) := 0$. Let $Y_i$ denote the GIT quotients coming from the walls as appearing the theorem.
A theorem of Kawamata

Theorem

Let $X$ be a smooth projective toric DM stack. Then $\mathcal{D}^b(\text{coh } X)$ admits a full exceptional collection.

Idea of Proof:
We obtain a SOD decomposition:

$$\mathcal{D}^b(\text{coh } X_i) = \langle \mathcal{D}^b(\text{coh } X_{i-1}), \mathcal{D}^b(\text{coh } Y_i), \ldots, \mathcal{D}^b(\text{coh } Y_i)(\mu_i - 1) \rangle.$$ 

Hence, combining these SODs, we obtain:

$$\mathcal{D}^b(\text{coh } X) = \langle \mathcal{D}^b(\text{coh } Y_1), \ldots, \mathcal{D}^b(\text{coh } Y_1), \ldots, \mathcal{D}^b(\text{coh } Y_s), \ldots, \mathcal{D}^b(\text{coh } Y_s) \rangle.$$
A theorem of Kawamata

**Theorem**

Let $X$ be a smooth projective toric DM stack. Then $D^b(\text{coh } X)$ admits a full exceptional collection.

**Idea of Proof:**

$$D^b(\text{coh } X) = \langle D^b(\text{coh } Y_1), \ldots, D^b(\text{coh } Y_1), \ldots, D^b(\text{coh } Y_s), \ldots, D^b(\text{coh } Y_s) \rangle.$$ 

Now, $\dim(Y_i) < \dim(X)$ for all $i$. Arguing by induction on the dimension, we may assume that $D^b(\text{coh } Y_i)$ admits a full exceptional collection for all $i$. The base case is a point. Following the inductive process and the functors involved in the theorem, everything can be done with sheaves.
Theorem

Various compactifications of the moduli space of $\mathbb{P}^m$ with $n$ distinct marked points admit full exceptional collections.

Idea of Proof:
General results

Moduli Spaces

GIT fan, \( n = 3, m = 1 \), intersection with the hyperplane, \( w_1 + w_2 + w_3 = 1 \) inside \( \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \) by \( \text{PSL}(2) \).
Various compactifications of the moduli space of $\mathbb{P}^m$ with $n$ distinct marked points admit full exceptional collections.

Idea of Proof: Let $\Sigma_0$ denote the chamber corresponding to a given compactification of the moduli space of $n$-points on $\mathbb{P}^m$. Choose a sufficiently straight line path in the secondary fan starting at the anti-canonical divisor, passing through $\Sigma_0$ and ending in a chamber for which the GIT quotient is empty which doesn’t passing through any cones of codimension 2.
Theorem

Various compactifications of the moduli space of $\mathbb{P}^m$ with $n$ distinct marked points admit full exceptional collections.

Idea of Proof:
The path, $l$, passes through a finite number of walls, $\tau_1, \ldots, \tau_s$ on its way through the chambers, $\Sigma_0, \ldots, \Sigma_s$. The value of $\mu$ is always positive because the anti-canonical class is always behind you along this path (by one of our simplifications). Let $X_i$ be the GIT quotient corresponding to the chamber $\Sigma_i$ with $X := X_0$ and $X_s = \emptyset$ so that $D^b(\text{coh} \, X_s) := 0$. Let $Y_i$ denote the GIT quotients coming from the walls as appearing the theorem.
Theorem

Various compactifications of the moduli space of $\mathbb{P}^m$ with $n$ distinct marked points admit full exceptional collections.

Idea of Proof:

We obtain a SOD decomposition:

$$D^b(\text{coh } X_i) = \langle D^b(\text{coh } X_{i-1}), D^b(\text{coh } Y_i), \ldots, D^b(\text{coh } Y_i)(\mu_i - 1) \rangle.$$ 

Hence, combining these SODs, we obtain:

$$D^b(\text{coh } X) = \langle D^b(\text{coh } Y_1), \ldots D^b(\text{coh } Y_1), \ldots, D^b(\text{coh } Y_s), \ldots, D^b(\text{coh } Y_s) \rangle.$$
Various compactifications of the moduli space of $\mathbb{P}^m$ with $n$ distinct marked points admit full exceptional collections.

Idea of Proof:

$$D^b(\text{coh } X) = \langle D^b(\text{coh } Y_1), \ldots D^b(\text{coh } Y_1), \ldots, D^b(\text{coh } Y_s), \ldots, D^b(\text{coh } Y_s) \rangle.$$  

All the $Y_i$ are points, so we get an exceptional collection.
Outline

1. Motivating Example
2. Background on GIT
3. General results
4. Landau-Ginzburg models and factorizations
5. RG-flow and a theorem of Orlov
A gauged Landau-Ginzburg model (gauged LG-model) is the quadruple, \( (X, G, L, w) \), with \( X, G, L, \) and \( w \) as above.
“Coherent sheaves” on a gauged LG-model, $(X, G, L, w)$ are called factorizations.
Factorizations

“Coherent sheaves” on a gauged LG-model, \((X, G, L, w)\) are called factorizations.

**Definition**

A **factorization** of a gauged LG-model, \((X, G, L, w)\), consists of a pair of coherent \(G\)-equivariant sheaves, \(\mathcal{E}^{-1}\) and \(\mathcal{E}^{0}\), and a pair of \(G\)-equivariant \(\mathcal{O}_X\)-module homomorphisms,

\[
\phi_{\mathcal{E}}^{-1} : \mathcal{E}^{0} \otimes L^{-1} \rightarrow \mathcal{E}^{-1} \\
\phi_{\mathcal{E}}^{0} : \mathcal{E}^{-1} \rightarrow \mathcal{E}^{0}
\]

such that the compositions, \(\phi_{\mathcal{E}}^{0} \circ \phi_{\mathcal{E}}^{-1} : \mathcal{E}_0 \otimes L^{-1} \rightarrow \mathcal{E}^{0}\) and \(\phi_{\mathcal{E}}^{-1} \otimes L \circ \phi_{\mathcal{E}}^{0} : \mathcal{E}^{-1} \rightarrow \mathcal{E}^{-1} \otimes L\), are isomorphic to multiplication by \(w\).
Factorizations

Definition

A factorization of a gauged LG-model, \((X, G, L, w)\), consists of a pair of coherent \(G\)-equivariant sheaves, \(\mathcal{E}^{-1}\) and \(\mathcal{E}^{0}\), and a pair of \(G\)-equivariant \(\mathcal{O}_X\)-module homomorphisms,

\[
\phi_{\mathcal{E}}^{-1} : \mathcal{E}^{0} \otimes L^{-1} \to \mathcal{E}^{-1} \\
\phi_{\mathcal{E}}^{0} : \mathcal{E}^{-1} \to \mathcal{E}^{0}
\]

such that the compositions, \(\phi_{\mathcal{E}}^{0} \circ \phi_{\mathcal{E}}^{-1} : \mathcal{E}^{0} \otimes L^{-1} \to \mathcal{E}^{0}\), and \(\phi_{\mathcal{E}}^{-1} \otimes L \circ \phi_{\mathcal{E}}^{0} : \mathcal{E}^{-1} \to \mathcal{E}^{-1} \otimes L\), are isomorphic to multiplication by \(w\).

The category of factorizations is an Abelian category akin to the category of complexes of coherent sheaves. An appropriate (dg) localization of this category by “acyclic factorizations” yields the derived category of matrix factorizations, \(MF([X/G], w)\).
We can also add in a potential, i.e. a $G$-invariant regular function, $w \in \Gamma(X)$, and an auxiliary group action, $H$ for which $w$ is $H$-invariant. Assume, in addition to the previous conditions on the linearization, that $X^{ss}(L_0)$ admits an $G \times H$-invariant affine cover. We let $X//\pm$ be the quotient by $G \times H$, using on the semi-stable locus of $G$. Similarly, $Y$ is now the further quotient by $H$. $w$ induces sections of line bundles on $X//\pm$ and $Y$ which we will denote $w_{\pm}$ and $w_Y$. 
Main theorem

Theorem (Ballard-F-Katzarkov)

Fix $d_1, \ldots, d_p \in \mathbb{Z}$.

- If $\mu_i > 0$ for $1 \leq i \leq p$, then there exists a left-admissible fully-faithful functor,

$$\Phi_{d_1, \ldots, d_p} : \text{MF}(X/\!, w^-) \to \text{MF}(X/\!+, w_+)$$

If $p = 1$, then there exists fully-faithful functors,

$$\Upsilon^-_j : \text{MF}(Y, w_Y) \to \text{MF}(X/\!+, w_+),$$

and a semi-orthogonal decomposition,

$$\text{MF}(X/\!+, w_+) = \langle \Upsilon^-_{-\mu-d+1} \text{MF}(Y, w_Y), \ldots,$$

$$\Upsilon^-_{-d} \text{MF}(Y, w_Y), \Phi_d \text{D(coh } X/\!, w_-) \rangle.$$
Main theorem

Theorem (Ballard-F-Katzarkov)

Fix $d_1, \ldots, d_p \in \mathbb{Z}$.

- If $\mu_i = 0$ for $1 \leq i \leq p$, then there exist an equivalence,

$$
\Phi_{d_1, \ldots, d_p} : \text{MF}(X // -, w_-) \to \text{MF}(X // +, w_+).
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Main theorem

Theorem (Ballard-F-Katzarkov)

Fix $d \in \mathbb{Z}$.

- If $\mu_i < 0$ for $1 \leq i \leq p$, then there exist fully-faithful functors,
  \[ \Psi_{d_1, \ldots, d_p} : \text{MF}(X / / +, w_+) \to \text{MF}(X / / -, w_-) \]

  If $p = 1$, then there exists fully-faithful functors,
  \[ \Upsilon_j^+ : \text{MF}(Y, w_Y) \to \text{MF}(X / / +, w_+), \]

and a semi-orthogonal decomposition,

\[ \text{MF}(X / / -, w_-) = \langle \Upsilon_{d_p}^+ \text{MF}(Y, w_Y), \ldots, \Upsilon_{\mu-d+1}^+ \text{MF}(Y, w_Y), \Psi_d \text{MF}(X / / +, w_+) \rangle. \]
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Theorem (Isik, Shipman)

Let $X$ be a variety and let $\sigma : \mathcal{O}_X \to E$ be a section of a vector bundle, $E$. Let $Z$ denote the zero locus of $\sigma$ and assume that all components of $Z$ have codimension equal to the rank of $E$. There is an equivalence

$$D^b(\text{coh } Z) \cong \text{MF}(\text{tot } E^\vee, w, \mathbb{G}_m)$$

where $w$ is the regular function induced by $\sigma$ and the $\mathbb{G}_m$ is the dilation action on the fibers of $\text{tot } E^\vee$.
Orlov’s Theorem

The following concept comes from work of Herbst, Hori, and Page, and was described mathematically in the Calabi-Yau case by Segal and Shipman. Consider a hyper surface $Z$, in $\mathbb{P}(V)$ defined by $f \in H^0(\mathcal{O}(d))$. By Isik/Shipman’s theorem we have:

$$D^b(\text{coh } Z) \cong \text{MF}(\text{tot } \mathcal{O}(-d), f, \mathbb{G}_m).$$

Now we want to do VGIT to reproduce a theorem of Orlov. Consider the $\mathbb{G}_m$-action on $\mathbb{C} \times V$ with weights, $-d$ and 1. There are two GIT quotients: tot $\mathcal{O}(-d)$ and $V$. Applying our main result we obtain:
Let $R = \text{Sym}(V)$.

**Theorem (Orlov, hypersurface/commutative case)**

1. If $n + 1 - d > 0$, there is a semi-orthogonal decomposition,
   \[ D^b(\text{coh} Z) = \langle \mathcal{O}_Z(d - n), \ldots, \mathcal{O}_Z, \text{MF}(R, f, \mathbb{Z}) \rangle. \]

2. If $n + 1 - d = 0$, there is an equivalence of triangulated categories,
   \[ D^b(\text{coh} Z) = \langle \text{MF}(R, f, \mathbb{Z}) \rangle. \]

3. If $n + 1 - d < 0$, there is a semi-orthogonal decomposition,
   \[ \text{MF}(R, f, \mathbb{Z}) \cong \langle k, \ldots, k(n + 2 - d), D^b(\text{coh} Z) \rangle. \]
A generalization of Herbst and Walcher

Theorem (Isik, Shipman)

Let $X$ be a variety and let $\sigma : \mathcal{O}_X \rightarrow E$ be a section of a vector bundle, $E$. Let $Z$ denote the zero locus of $\sigma$ and assume that all components of $Z$ have codimension equal to the rank of $E$. There is an equivalence

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where $w$ is the regular function induced by $\sigma$ and the $\mathbb{G}_m$ is the dilation action on the fibers of $\text{tot } E^\vee$. 
A generalization of Herbst and Walcher

**Theorem (Isik, Shipman)**

Let $X$ be a variety and let $\sigma : \mathcal{O}_X \rightarrow E$ be a section of a vector bundle, $E$. Let $Z$ denote the zero locus of $\sigma$ and assume that all components of $Z$ have codimension equal to the rank of $E$. There is an equivalence

$$D^b(\text{coh } Z) \cong \text{MF}(\text{tot } E^\vee, w, \mathbb{G}_m)$$

where $w$ is the regular function induced by $\sigma$ and the $\mathbb{G}_m$ is the dilation action on the fibers of tot $E^\vee$.

Let $X$ be a smooth toric variety, $D_1, ..., D_s$ be divisors, and $E := \bigoplus_{i=1}^{s} \mathcal{O}(D_i)$. Suppose $f_i$ are section of $\mathcal{O}(D_i)$ forming a complete intersection, $Z$, and let $w$ be the corresponding map on $E^\vee$. We have,

$$D^b(\text{coh } Z) \cong \text{MF}(\text{tot } E^\vee, w, \mathbb{G}_m).$$
A generalization of Herbst and Walcher

Let $X$ be a smooth toric variety, $D_1, ..., D_s$ be divisors, and $E := \bigoplus_{i=1}^{s} O(D_i)$. Suppose $f_i$ are sections of $O(D_i)$ forming a complete intersection, $Z$, and let $w$ be the corresponding map on $E^\vee$. We have,

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Now suppose the classes of the $D_i$ are all nef and lie on a wall separating the nef cone, $\Sigma$, from some other chamber of the GKZ fan, $\Sigma'$. Let $X'$ be the GIT quotient corresponding to the chamber $\Sigma'$. Looking instead at the secondary fan of $E^\vee$, it turns out that the pullback of $\Sigma$ and $\Sigma'$ are also chambers in this secondary fan, and the GIT quotients are both $E^\vee$, once as a bundle over $X$ and the other as a bundle over $X'$. 

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$$D^b(\text{coh } Z) \cong \text{MF}(\text{tot } E^\vee_X, w, \mathbb{G}_m),$$

and

$$D^b(\text{coh } Z') \cong \text{MF}(\text{tot } E^\vee_{X'}, w, \mathbb{G}_m).$$

Let $Y$ be the GIT quotient obtained by taking a generic character in the wall, and thinking of the wall itself as a secondary fan of a toric variety.
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Theorem (Herbst-Walcher, Ballard-F-Katzarkov)

1. If $\mu > 0$, we have a SOD of $\mathcal{D}^b(\text{coh } Z)$,

$$\langle \text{MF}(Y, w_Y)(-\mu - d + 1), \ldots, \text{MF}(Y, w_Y)(-d), \mathcal{D}^b(\text{coh } Z') \rangle,$$

2. If $\mu = 0$,

$$\mathcal{D}^b(\text{coh } Z) = \mathcal{D}^b(\text{coh } Z').$$

3. If $\mu < 0$, we have a SOD of $\mathcal{D}^b(\text{coh } Z')$

$$\langle \text{MF}(Y, w_Y)(-d), \ldots, \text{MF}(Y, w_Y)(\mu - d + 1), \mathcal{D}^b(\text{coh } Z) \rangle.$$
Homological Projective Duality

\[ \text{D}(\text{coh } \mathcal{Y}, w) \xleftrightarrow{\text{VGIT}} \text{D}(\text{coh } \mathcal{Y}', w) \]

\[ \text{RG-flow} \quad \text{Orlov} \quad \text{RG-flow} \]

\[ \text{D}^b(\text{coh } \mathcal{X}) \leftarrow \text{HPD} \rightarrow \text{D}^b(\text{coh } \mathcal{X}') \]
Let $\mathcal{L}_i$ be a collection of ample line bundles on $B$ for $1 \leq i \leq r$, and consider the vector bundle, $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{L}_i$. Let $X$ be the projective bundle, $\pi : \mathbb{P}(\mathcal{E}) \to B$ and $V := \text{H}^0(\mathcal{E})$. The relative bundle, $\mathcal{O}_{\pi}(1)$, provides an embedding, $j : X \to \mathbb{P}(V^*)$. The universal hyperplane section, $\mathcal{X} \subset X \times \mathbb{P}(V)$, is the zero section of a section, $s \in \mathcal{O}_{\pi}(1) \boxtimes \mathcal{O}(1)$ (which we will explicitly describe later on). Let $\mathcal{Y}$ be the total space of $\mathcal{O}_{\pi}(-1) \boxtimes \mathcal{O}(-1)$ quotiented by fiberwise dilation by $\mathbb{G}_m$. Employing Isik/Shipman’s Theorem we obtain an equivalence $\mathcal{D}^b(\text{coh } \mathcal{X}) \cong \mathcal{D}(\text{coh } \mathcal{Y}, \omega)$. We now describe a variation of linearization on $\mathcal{E} \times \mathbb{P}(V) \times \mathbb{A}^1$ which yields, $\mathcal{Y}$, on the one hand, and $\mathcal{E} \times \mathbb{P}(V)$ on the other.
Consider the $\mathbb{G}_m$-action on $\mathcal{E}^* \times \mathbb{P}(V) \times \mathbb{A}^1$, given by dilating the fibers of $\mathcal{E}^*$, inverted dilation on $\mathbb{A}^1$, and acting trivially on $\mathbb{P}(V)$. Add an auxiliary $\mathbb{G}_m$-action given by the usual action of $\mathbb{G}_m$ on $\mathbb{A}^1$ and acting trivially on the other components. Consider the trivial line bundle, so that the GIT quotients are determined by a character of $\mathbb{G}_m$ i.e. an integer.
Consider the $\mathbb{G}_m$-action on $E^* \times \mathbb{P}(V) \times \mathbb{A}^1$, given by dilating the fibers of $E^*$, inverted dilation on $\mathbb{A}^1$, and acting trivially on $\mathbb{P}(V)$. Add an auxiliary $\mathbb{G}_m$-action given by the usual action of $\mathbb{G}_m$ on $\mathbb{A}^1$ and acting trivially on the other components. Consider the trivial line bundle, so that the GIT quotients are determined by a character of $\mathbb{G}_m$ i.e. an integer. Let $\mathcal{Y}'$ denote the global quotient stack of $E^* \times \mathbb{P}(V)$ by the fiberwise dilation by $\mathbb{G}_m$. From this description, it follows that,

- $(E^* \times \mathbb{P}(V) \times \mathbb{A}^1)/\!/+ = \mathcal{Y},$
- $(E^* \times \mathbb{P}(V) \times \mathbb{A}^1)/\!- = \mathcal{Y'},$
- $Y = Z \times \mathbb{P}(V) \times 0 \cong B \times \mathbb{P}(V),$

where in the GIT quotient we mod out by the auxiliary action as well.
Let $p : \mathcal{E}^* \times \mathbb{P}(V) \times \mathbb{A}^1 \rightarrow \mathbb{P}(V)$ be the projection to $\mathbb{P}(V)$ and $q : \mathcal{E}^* \times \mathbb{P}(V) \times \mathbb{A}^1 \rightarrow \mathcal{E}^*$ be the projection to $\mathcal{E}$. We can define a potential, $w \in H^0(p^*\mathcal{O}_{\mathbb{P}(V)}(1))$ by the pairing of $V$ with $V^*$. Let $u$ be a global coordinate on $\mathbb{A}^1$ so that, restricting to this component of $\mathcal{E}^* \times \mathbb{P}(V) \times \mathbb{A}^1$, $g \cdot x = g^{-1}x$ for $g \in \mathbb{G}_m$. Identifying $H^0(\mathcal{O}_{\mathbb{P}(V)}(1))$ with $V^*$ and thinking of $V$ as global functions on $\mathcal{E}^*$, let $e_{i,j}, \ldots, e_{i,j}$ be a basis for $H^0(\mathcal{L}_j)$ so that ranging over all $i,j$ we obtain a basis of $V$, then the potential is explicitly,

$$w := u \sum_{i,j} p^* e_{i,j}^* q^* e_{i,j}.$$
HPD example

Let $\pi_1 : \mathcal{Y} \to \mathcal{E}^*$ and $\pi_2 : \mathcal{Y} \to \mathbb{P}(V^*)$ be the two projections. One checks that,

- $w_+ = \langle s, - \rangle$,
- $w_- = \sum_{i,j} \pi_1^* e_{i,j}^* \pi_2^* e_{i,j}$,
- $w_Y = 0$. 
HPD example

Let $\pi_1 : Y' \to \mathcal{E}^*$ and $\pi_2 : Y' \to \mathbb{P}(V^*)$ be the two projections. One checks that,

- $w_+ = \langle s, - \rangle$,
- $w_- = \sum_{i,j} \pi_1^* e_i^* \pi_2^* e_{i,j}$,
- $w_Y = 0$.

With $\mathcal{L} = \mathcal{O}_\pi(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)$, by our main theorem, we obtain a semi-orthogonal decomposition,

$$D^b(\text{coh } Y, w_+) = \langle D^b(\text{coh } Y', w_-), D^b(\text{coh } B \times \mathbb{P}(V)), \ldots, D^b(\text{coh } B \times \mathbb{P}(V)) \rangle$$
With $\mathcal{L} = \mathcal{O}_\pi(1) \boxtimes \mathcal{O}_\mathbb{P}(V)(1)$, by our main theorem, we obtain a semi-orthogonal decomposition,

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Now, examining the explicit description of the potential, $w_-$, we see that it is the same as pairing with the section,

$$t = \bigoplus_{i=1}^r \sum_{j} e^*_i e_{i,j} \in \bigoplus_{i=1}^r \mathcal{L}_i \boxtimes \mathcal{O}_\mathbb{P}(V)(1).$$

The zero set of this section is the complete intersection of zero loci of $\sum_j e^*_i e_{i,j}$,

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Employing RG-flow here yields, $D(\text{coh} \mathcal{Y}', w_-) \cong D^b(\text{coh} \mathcal{X}')$. Piecing this all together gives $D^b(\text{coh} \mathcal{X}) =$

$$\langle D^b(\text{coh} \mathcal{X}'), D^b(\text{coh} B \times \mathbb{P}(V)), \ldots, D^b(\text{coh} B \times \mathbb{P}(V))((r-2)\mathcal{O}_{\pi}(1)\boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)) \rangle.$$

The variety, $\mathcal{X}'$, is a homological projective dual of $X$. 
Employing RG-flow here yields, $D(\text{coh } \mathcal{U}', w_-) \cong D^b(\text{coh } \mathcal{X}')$. Piecing this all together gives $D^b(\text{coh } \mathcal{X}) = \langle D^b(\text{coh } \mathcal{X}'), D^b(\text{coh } B \times \mathbb{P}(V)), \ldots, D^b(\text{coh } B \times \mathbb{P}(V))((r-2)\mathcal{O}_\pi(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)) \rangle$.

The variety, $\mathcal{X}'$, is a homological projective dual of $X$. Consider the composition, $f : \mathcal{X}' \hookrightarrow B \times \mathbb{P}(V) \to \mathbb{P}(V)$. A point $p \in \mathbb{P}(V)$, corresponds to a collection of sections, $s_i \in H^0(B, \mathcal{L}_i)$ for $i = 1, \ldots, r$ up to rescaling. The fiber of $f$ over $p$, is the intersection, $\bigcap_{i=1}^r Z(s_i) \subseteq B$. When $r = s + 1$, the image of $f$ is precisely the so called resultant variety, consisting of points corresponding to degenerate collections of sections $s_i \in H^0(B, \mathcal{L}_i)$ i.e. collections of sections with nonempty intersection. The resultant variety is the “classical projective dual” i.e. the set of singular hyperplane sections of $X$ in $\mathbb{P}(V)$.