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Stockpiling and Shortages (the “Toilet Paper Paper”)

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Stockpiling and Shortages*

(the “Toilet Paper Paper”)

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June 6, 2021

Abstract

Widespread stockpiling of everyday household items (e.g., toilet paper) occurred in the wake of the Covid-19 pandemic, resulting in shortages of these items in stores. Both phenomena reinforce each other: The expectation of shortages causes stockpiling behavior, which in turn amplifies the shortages, which in turn encourages more stockpiling. In this paper, I examine this feedback loop. In the model, households want to consume one unit of a good per period, but can store more than one unit at a cost. Aggregate supply of the good may be insufficient to meet aggregate demand, but prices cannot adjust to equate supply and demand. I characterize stationary equilibria in which households maintain target inventories of z units. Thus, stockpiling arises if $z > 1$, and I demonstrate that this can be an equilibrium even for very small aggregate supply-demand imbalances. In particular, in many equilibria the incentive to stockpile is driven mostly by the stockpiling behavior of other households, and not by the fundamental supply shortage. Transitional dynamics are examined as well.

Keywords: Storage; consumer inventories; stockpiling; multiple equilibria; rationing; mean field games; search models.

JEL codes: C61, C62, C73, D15.

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There is enough in the whole country for the coming ten years. We can all poop for ten years.

Dutch prime minister Mark Rutte on the toilet paper supply, March 2020

1 Introduction

This paper examines a dynamic environment with the following properties: (i) Consumers want to consume a constant amount of a storable good in every period; (ii) the aggregate per-period supply may fall below aggregate demand due to an exogenous shock; (iii) prices are fixed and cannot adjust to balance supply and demand. In this environment, individual consumers may attempt to smooth consumption by accumulating inventories. However, stockpiling behavior has an externality, as it exacerbates the supply shortages experienced by other consumers. This, in turn, increases their incentive to stockpile, which reduces the availability of the good even further. The objective of this paper is to examine this feedback loop—that is, how small underlying supply shortages can be magnified via consumers' rational inventory responses.

The aforementioned market conditions were present, for example, during the early days of the Covid-19 pandemic (see, e.g., Wang *et al.* 2020; Micalizzi *et al.* 2021). As supply chains were being disrupted due to travel restrictions as well as hygiene measures at factories and logistics facilities, a stable supply of everyday grocery items was no longer guaranteed. Sensing that they may not be able to acquire these items in the future, consumers started buying up available quantities of storable products such as dried pasta, flour, canned goods, and toilet paper, resulting in rows of empty supermarket shelves.¹ Stores were reluctant to implement price increases for these items—likely because of a combination of legal constraints and reputational concerns—which meant that the usual price mechanism was unavailable to balance supply and demand. Similarly, in May 2021 a cyberattack on Colonial Pipeline Corporation unexpectedly reduced the supply of transportation fuels in the Eastern United States and caused long queues at gas stations.² The emergence of these queues indicated that drivers were filling up their vehicles before their tanks were near empty (as most drivers normally would). While gas stations did

¹In addition to supply interruptions, there were also several demand increasing effects. Some consumers stored additional supplies to last through a potential quarantine period. Furthermore, consumption of certain goods shifted from workplaces to homes, increasing the demand for home-use varieties of these goods. For example, the toilet paper used in many workplaces comes in large rolls that fit high-capacity dispensers, and the coffee consumed in offices is often sold in packs suitable for commercial coffee makers. Moreover, many offices procure these items from specialized vendors that do not sell to consumers. Thus, separate markets exist for workplace and home-use varieties of these goods, and the supply of workplace varieties cannot easily be redirected to meet increased demand for home-use varieties. The model developed here is expressed in terms of supply reductions instead of such demand increases, but can be recast to apply to the latter case as well (see Section 6).

²See, e.g., <https://www.reuters.com/technology/colonial-pipeline-halts-all-pipeline-operations-after-cybersecurity-attack-2021-05-08/> (retrieved June 3, 2021).

raise prices, price gauging laws prohibited them from implementing increases sufficient to fully balance supply and demand.

In both situations discussed above, underlying supply disruption were likely magnified by consumers' stockpiling responses: The supply shortages consumers actually experienced may have been much less severe, had consumers refrained from stockpiling. To examine this two-way relationship between supply shortages and stockpiling behavior, I develop a dynamic model with a continuum of households, each wanting to consume one unit of a certain good (e.g., toilet paper) per period, the price of which is fixed. Households can store, at a cost, up to a certain maximum quantity of the good, but cannot resell it to other households. The aggregate per-period supply of the good is also fixed, and if it is less than the aggregate consumption requirement, some households will be rationed. A decision rule describes a household's optimal inventory behavior in each period. A symmetric equilibrium is a decision rules that is optimal for a household if all other households adopt the same decision rule. Stockpiling arises if, in equilibrium households buy and store more than the single unit consumed per period.

I show that, if there is no underlying supply shortage—that is, if the aggregate supply in each period is sufficient to meet the aggregate consumption requirement of one unit per household—the unique equilibrium is for each household to obtain exactly one unit of the good per period. Thus, stockpiling cannot merely be a “self-fulfilling prophecy”—it requires an underlying, fundamental supply-demand imbalance. If such a fundamental imbalance exists, however, equilibria may emerge in which households store more than one unit, and may, in fact, store up to their capacity limit. These outcomes can arise even if the fundamental supply shortage is negligible. In particular, I show that the overall degree of stockpiling in any equilibrium can be decomposed into a fundamental component driven by the underlying supply shortage and an excess component caused by the reinforcing effect, and the magnitude of the second component can be multiple times that of the first component. In such stockpiling equilibria, households experience shortages with a much higher likelihood than what is indicated by fundamentals, and will buy large quantities whenever they can find the good.

I give a full characterization of all symmetric, stationary equilibria in what I call z -storage rules. A z -storage rule is a decision rule under which the household tries to maintain a target inventory level of z units, and if the actual inventory falls below this threshold the household attempts to restock to an inventory of z units. Any z -storage rule with $z > 1$ involves stockpiling of units not immediately needed for consumption. Stationary equilibria are generally not unique, and the entire range of possible z -storage rules can be equilibria for generic parameter values. Therefore, the aforementioned positive reinforcing effect, whereby stockpiling behavior creates additional storage incentives, results in equilibrium indeterminacy. I also provide a limited characterization of non-stationary equilibria. This analysis explicitly takes into account the transitional dynamics

that arise when the economy starts out in a state of balanced supply and demand, but then experiences an unexpected supply shock—e.g., because of a pandemic. In both stationary and non-stationary equilibria, welfare decreases in the degree of equilibrium stockpiling.

A classic literature in economics and finance examines price formation in competitive forward markets for storable commodities; see Telser (1958), Turnovsky (1983), Scheinkman and Schechtman (1983), Kawai (1983), Sarris (1984), and Hirshleifer (1989), among others. This line of inquiry was later extended to imperfectly competitive markets; see, e.g., Allaz (1991) and Thille (2003). The models in this literature are meant to characterize the strategies of professional traders in, e.g., agricultural markets. They are, therefore, much differently motivated than the model I examine. For example, in my model, inventories are maintained at the end consumer level, and no sales from inventories can occur (which would be necessary for a professional trader to profit from storage).

Stockpiling by consumers, on the other hand, has been studied in the marketing and industrial organization literatures. Meyer and J. Assunção (1990), Mela *et al.* (1998), Hong *et al.* (2002), Hendel and Nevo (2006a), Hendel and Nevo (2006b), and Ching and Osborne (2020) examine consumer's propensity to stockpile in response to temporary promotional discounts, both theoretically and empirically. An implication from these studies is that, for certain storable consumer goods, price decreases can lead to large increases in units sold even though the underlying consumption demand is relatively price inelastic. That is, demand responses in response to price changes often merely reflect shifts in the timing of purchases. This, in turn, has implications for firms' pricing and promotion strategies; see, e.g., Bell *et al.* (2002), Guo and Villas-Boas (2007), Su (2010), and Gangwar *et al.* (2013). My model of consumer stockpiling differs from this literature in that not prices and aggregate supply quantities are exogenously fixed. Thus, neither can households respond to price changes nor do suppliers make decisions. This choice allows me to focus on consumer stockpiling as an optimal response to other consumers' stockpiling behavior, and not as an optimal response to, say, the expectation of increasing prices.

Finally, my model shares certain similarities with monetary search models. In these models, households cannot directly exchange goods or services due to trade being decentralized without double coincidence of wants; thus, the role of money as a medium of exchange can be studied explicitly. In the seminal monetary search model of Kiyotaki and Wright (1989), agents hold either one or zero units of money; however, Shi (1997), Green and Zhou (1998), Zhou (1999), Camera and Corbae (1999), Berentsen (2000), and others have relaxed this assumption. Berentsen (2000), for example, shows that in a simple extension of the Kiyotaki-Wright model, multiple stationary equilibria exist that have the same money stock and in which agents are willing to accumulate either one or two units of money. These equilibria resemble, in some ways, the 1-storage and 2-storage equilibria

in my model. However, there are several important differences. First, money cannot be consumed, whereas the accumulated good in my model is a consumption good.³ Second, the supply of consumption goods in monetary search models is endogenously determined by agents' production decisions and the supply of money is (explicitly or implicitly) a policy variable. In my model, there is no money and the supply of the consumption good is exogenously fixed and cannot be increased by fiat. Third, money inventories cannot arbitrarily be incremented—for example, the seller's money stock can only increase by the price of the unit sold. In contrast, households in my model will generally increase their inventories in a lumpy fashion. Fourth, while in both types of models inventories provide insurance against non-consumption events, the consequences of this activity are not the same. In my model, one household's inventory causes a negative externality as it increases the likelihood of non-consumption events faced by other households (thereby amplifying their stockpiling incentives). Money inventories, on the other hand, can have a positive externality, as they increase trading possibilities for other households. For example, in the second equilibrium in Berentsen (2000), the equilibrium with more storage exhibits a higher velocity of money, which results in higher welfare compared to the equilibrium with less storage.⁴

The remainder of the paper is organized as follows. In Section 2 I set up the theoretical model and define equilibrium. In Section 3 I define z -storage rules and derive properties of the dynamic system generated by these rules. Section 4 contains the main results, characterizing stationary equilibria in z -storage rules. The same Section also examines non-stationary equilibria. Section 5 contains a welfare analysis of the various equilibria of the model, along with a discussion of some policy implications. Section 6 concludes. Most proofs are in the Appendix.

2 Model

2.1 The economy

The economy is populated by a continuum of households of measure 1. Time is divided into periods indexed by $t = 0, 1, 2, \dots$. There is one consumption good, which can be bought and consumed in integer quantities only. In every period, a household requires one unit of the good. Each household can store up to $K \geq 2$ units of the item from one period to the next (where K is an integer).

³There are some money search models that allow for consumption good inventories (e.g., Li 1994; Molico 2006); however, none studies the questions we are interested in here.

⁴In the higher-inventory equilibrium, all households are willing to produce twice in a row without wanting to consume first. In addition, households with an inventory of two units of money will be able to consume twice in a row without having to produce first. Thus, given the same matching technology, the overall frequency of transactions is increased in the higher-inventory equilibrium.

The economy-wide supply of the good per period is a continuum of measure $m \leq 1$. If $m = 1$, the economy produces exactly as much of the item as is required to meet every household's underlying consumption need. If $m < 1$, the economy experiences an *aggregate supply shortage*.⁵ There is a single store in the economy at which households can obtain the item.

Period t unfolds as follows.

1. At the beginning of the period, the store puts the entire economy-wide supply, m , on its shelves.
2. Household i enters the period with some inventory $s_i^t \in \{0, 1, \dots, K\}$ of the good. If $s_i^t > 0$, the household consumes one unit, reducing its inventory by one. If $s_i^t = 0$, the household cannot consume the item in this period.
3. The household then makes a trip to the store. When it arrives at the store, the household is placed randomly in a queue, so that the measure of households $j \neq i$ that are ahead of i in the queue is a uniformly distributed random variable.
4. The household decides how many units of the item it wants to obtain. If the desired quantity is in stock, the household obtains it. If the desired quantity is not in stock, the household obtains the remaining stock, if positive.

If the store runs out of supply before the final shopper arrives, then some households will leave the store empty-handed. We say that these households experience an *in-store shortage*. Note that this does not mean that these households will not consume in the next period, as they may still have positive inventories in storage.

Household i receives a flow utility of 1 if it consumes the item in a given period (i.e., if $s_i > 0$), and a flow utility of 0 if it does not consume the item (i.e., if $s_i = 0$). In each period, the household also pays a storage cost proportional to its beginning-of-period inventory, λs_i , where $\lambda \in (0, 1)$. Finally, all households discount the future using a common discount factor $\beta \in (0, 1)$. To prevent trivial outcomes where not consuming is optimal, we also assume that $\lambda < \beta$.

We make the following additional assumptions. First, the price of the item is fixed and normalized to zero. Therefore, in our model, prices play no role in equating supply and demand, or in allocating the good to households.⁶ If prices could adjust freely and

⁵In Section 6, we discuss how the model can be adjusted in order to apply to the case of an aggregate demand increase, instead of a supply shortage.

⁶Since each household is quantitatively negligible, behavior in our model will be competitive in the sense that households (i) choose actions that are optimal given economic aggregates and (ii) ignore the impact of their actions on these aggregates. However, because there is no price system to intermediate households' behavior, the solution concept is not competitive equilibrium. Technically, our model is a mean field game, and the solution concept, formally introduced in Section 4, is (subgame perfect) Nash equilibrium.

demand or supply were not entirely inelastic, shortages could not arise. Households are nevertheless prevented from demanding an infinite amount, as they are constrained by a finite storage capacity K . In addition, the storage cost $\lambda > 0$ may reduce demand to below K . Second, a household cannot “resell” or otherwise transfer the good to another household. Third, households cannot dispose of the good they have in storage, and the only way for a household to reduce its inventory is through consumption of one unit per period. Finally, if the store has a positive amount of the good remaining after all households have shopped in a given period, the “unsold” amount is disposed of (this will not happen in any of the equilibria we examine).

2.2 State variables and decision rules

From the perspective of an individual household, it will be convenient to focus on the household’s beginning-of-period- t inventory,

$$s^t \in \{0, \dots, K\},$$

as the relevant state variable. A *decision rule* for the household in period t is a mapping that assigns to each value of s^t the household’s desired beginning-of-period- $(t + 1)$ inventory,

$$\sigma^t(s^t) \in \{\max\{0, s^t - 1\}, \dots, K\}.$$

In period t , the household purchases $\sigma^t(s^t) - \max\{0, s^t - 1\}$ units of the good if the store has this many units in stock. If the store has at least one unit, but fewer than $\sigma^t(s^t) - \max\{0, s^t - 1\}$ units in stock, the household purchases the remaining stock (this will be a probability zero event for each household). If the store has zero units in stock, the household leaves empty-handed.⁷

From the perspective of the aggregate economy, the relevant state in each period is the distribution of inventories across households. This state is given by the vector

$$x^t = (x_0^t, x_1^t, \dots, x_{K-1}^t, x_K^t),$$

where x_k^t is the fraction of households that enter period t with k units in storage. The space of all such states is the K -dimensional unit simplex

$$\Delta_K \equiv \left\{ x \in \mathbb{R}_+^{K+1} : \sum_{k=0}^K x_k = 1 \right\}.$$

⁷The household also observes economy-wide state variables, or has expectations about the values of such variables. These variables will be taken into account when we solve for the equilibrium, as they are obviously important in determining the optimal decision for a household in a given period. However, at this point they do not need to be explicitly listed as arguments of σ^t .

For $x, y \in \Delta_K$, we write $x \succsim y$ (or $y \precsim x$) if x weakly dominates y , in the sense of first-order stochastic dominance.⁸

Suppose all households use the same decision rule σ^t in period t . The aggregate measure of the good that would be purchased in period t if there was no supply constraint (i.e., if $m = \infty$) is given by

$$\theta^t = \sum_{k=0}^K x_k^t [\sigma^t(k) - \max\{0, k-1\}]. \quad (1)$$

Because only a measure $m \leq 1$ is available, the probability that a household arrives at the store and is able to purchase the good is

$$p^t = \min \left\{ \frac{m}{\theta^t}, 1 \right\}. \quad (2)$$

Therefore, the probability that a household experiences an in-store shortage is $1 - p^t$.

Households that find the good are able to execute their desired purchases and will enter period $t+1$ with $\sigma^t(s_i^t)$ units in storage. Households that experience an in-store shortage are unable to execute their desired purchases (if positive) and will enter period $t+1$ with $\max\{0, s_i^t - 1\}$ units in storage. Therefore, given aggregate state x^t and common decision rule σ^t , we can compute next period's state as follows:⁹

$$x_k^{t+1} = \begin{cases} p^t \sum_{k':\sigma^t(k')=K} x_{k'}^t & \text{if } k = K, \\ p^t \sum_{k':\sigma^t(k')=k} x_{k'}^t + (1 - p^t)x_{k+1}^t & \text{if } 0 < k < K, \\ p^t \sum_{k':\sigma^t(k')=0} x_{k'}^t + (1 - p^t)[x_1^t + x_0^t] & \text{if } k = 0. \end{cases} \quad (3)$$

The economy is in a *stationary state* if the distribution of inventories across households is time-invariant, that is

$$x^{t+1} = x^t = x^* \quad \forall t.$$

A special case of a stationary state is a steady state, in which no individual household's inventory changes from period to period.

2.3 Equilibrium definition

Our notion of equilibrium is, essentially, that of Jovanovic and Rosenthal (1988) for anonymous sequential games. In every period, every agent chooses an action that maximizes the agent's discounted continuation payoff in that period, given state variables,

⁸That is, $x \succsim y$ if $\sum_{s=0}^k x_s \leq \sum_{s=0}^k y_s \quad \forall k = 0, \dots, K$.

⁹Note that there can be at most one household that enters the store and finds more than zero, but fewer than $\sigma^t(s_i^t) - \max\{0, s_i^t - 1\}$, units of the item available. This household is of measure zero; hence its purchases have no effect on the evolution of the aggregate state.

and state variables in each period are determined from the previous period's state variables and the distribution of actions across individuals in the previous period. These conditions imply behavior that is the same as that in a subgame perfect Nash equilibrium.¹⁰

We can, however, apply two simplifications to the Jovanovic/Rosenthal definition. First, while Jovanovic and Rosenthal (1988) allow for continuous individual state and action spaces, ours are finite, and the equilibrium definition below is stated accordingly. Second, Jovanovic and Rosenthal (1988) permit mixed strategy equilibria, or—which is equivalent with a continuum of identical agents—asymmetric pure strategy equilibria. We restrict attention to symmetric pure strategy equilibria, that is, equilibria in which all agents adopt the same decision rule in a given period.

With this in mind, to define equilibrium formally we denote by V_s^t the continuation value of being in individual state $s = 0, \dots, K$ in period t . This continuation value can be expressed recursively as follows

$$V_s^t = \begin{cases} \max_{\sigma \in \{s-1, \dots, K\}} \left\{ 1 - s\lambda + \beta \left[p^t V_{\sigma}^{t+1} + (1 - p^t) V_{s-1}^{t+1} \right] \right\} & \text{if } s \geq 1, \\ \max_{\sigma \in \{0, \dots, K\}} \left\{ \beta \left[p^t V_{\sigma}^{t+1} + (1 - p^t) V_0^{t+1} \right] \right\} & \text{if } s = 0. \end{cases} \quad (4)$$

Note that an individual is affected by the aggregate state x^t through the probability p^t . We then make the following definition:

Definition 1. Given $x^0 \in \Delta_K$, an equilibrium is a sequence of probabilities, states, continuation values, and decision rules

$$\left((p^t, x^{t+1}, V^t, \sigma^t) \in [0, 1] \times \Delta_K \times \mathbb{R}^{K+1} \times \{0, \dots, K\}^{K+1} \right)_{t=0,1,2,\dots}$$

such that for each $t \geq 0$ the following holds:

- (i) p^t and x^{t+1} are determined from x^t and σ^t via (1)–(3),
- (ii) $V^t(\cdot)$ and $\sigma^t(\cdot)$ are the value functions and policy functions that solve the Bellman equations (4), for $s = 0, \dots, K$.

In the analysis below, we will further be looking for equilibria that are symmetric not only across households, but also across time. That is, all households apply the same decision rule in every period.

¹⁰Note that the equilibrium conditions themselves are those of Nash equilibrium only, as they do not impose optimality at aggregate states that are not reached in equilibrium. However, no behavior that would be ruled out by the more stringent requirement of subgame perfection can emerge: With a continuum of agents, any aggregate state that is not reached in equilibrium could only be reached through a coordinated deviation by positive measure of agents. Therefore, the Nash equilibrium requirement implies subgame perfect play in the sequential game studied here.

3 z -Storage Rules

In this Section, we examine a specific class of decision rules, called z -storage rules, defined as follows.

Definition 2. For given $z \in \{1, \dots, K\}$, the function $\sigma^t(s^t) = \max\{s^t - 1, z\}$ is called the z -storage rule.

A household that uses a z -storage rule in period t tries to achieve a desired inventory of z units at the beginning of period $t + 1$. If the household has more than z units in storage in period t , it uses one unit in period t and enters period $t + 1$ with one less unit in storage. If the household has z or fewer units in period t , it tries to purchase enough so as to enter period $t + 1$ with z units in storage. We call the z -storage rule the *maximum storage rule* if $z = K$, and the *minimum storage rule* if $z = 1$.¹¹ If a household uses a z -storage rule with $z > 1$, we say that the household *stockpiles*.

We now examine the properties of the dynamical system relating z -storage rules to the evolution of the state variable x^t . The analysis is “mechanical” insofar as it does not yet involve any optimizing on part of households, or any analysis of equilibrium. The optimality of z -storage rules will be examined later in Section 4.1.

If all households use the same z -storage rule in period t , the aggregate quantity of the good households attempt to purchase (which was defined generally in (1)) becomes

$$\theta(x^t|z) = x_z^t + 2x_{z-1}^t + 3x_{z-2}^t + \dots + zx_1^t + zx_0^t, \quad (5)$$

and the probability that a household finds the item in period t (which was defined generally in (2)) becomes

$$p(x^t|z) = \min \left\{ \frac{m}{\theta(x^t|z)}, 1 \right\}. \quad (6)$$

Households that find the good enter period $t + 1$ with z units in storage. The remaining households enter period $t + 1$ with one less unit than they had in period t ; if the household already had zero units in period t they will enter period $t + 1$ with zero units as well. Thus, the transition rule (3) becomes

$$x_k^{t+1} = T_k(x^t|z) \equiv \begin{cases} 0 & \text{if } k = K > z, \\ p(x^t|z) & \text{if } k = z = K, \\ (1 - p(x^t|z))x_{k+1}^t + p(x^t|z) & \text{if } k = z < K, \\ (1 - p(x^t|z))(x_1^t + x_0^t) & \text{if } k = 0, \\ (1 - p(x^t|z))x_{k+1}^t & \text{otherwise.} \end{cases} \quad (7)$$

¹¹Technically, a household could also follow a 0-storage rule, that is, a policy of not buying the good even if it has nothing stored. Because we assume that $\lambda < \beta$, this rule is never optimal, and we can ignore it.

A stationary state of the economy is a fixed point of $T(\cdot|z) : \Delta_K \rightarrow \Delta_K$. Since T is continuous, and Δ_K is compact and convex, a fixed point x^* exists by Brouwer's Fixed Point Theorem. To characterize x^* , fix z and let $p^* \equiv p(x^*|z)$ and $\theta^* \equiv \theta(x^*|z)$. Then (7) implies that $x_k^* = 0$ for all $k > z$ and

$$\begin{aligned} x_z^* &= p^*, \\ x_{z-1}^* &= p^*(1-p^*), \\ x_{z-2}^* &= p^*(1-p^*)^2, \\ &\vdots \\ x_1^* &= p^*(1-p^*)^{K-1}, \\ x_0^* &= 1 - \sum_{k=1}^z x_k^* = 1 - p^* \sum_{k=0}^z (1-p^*)^k = (1-p^*)^z. \end{aligned}$$

Thus, in the stationary state, the attempted purchase quantity, (5), becomes

$$\theta^* = \sum_{k=1}^z (z-k+1)p^*(1-p^*)^{z-k} + z(1-p^*)^z = \frac{1-(1-p^*)^z}{p^*} \geq 1, \quad (8)$$

and the probability that a household finds the good in the store, (6), becomes

$$p^* = \frac{m}{\theta^*} = \frac{mp^*}{1-(1-p^*)^z}. \quad (9)$$

(9) can be solved uniquely for $p^* = 1 - (1-m)^{1/z}$. Therefore, we have shown:

Proposition 1. *Suppose every household uses the same z -storage rule in every period, for $z \in \{1, \dots, K\}$. There is a unique stationary state,*

$$x^* = \underbrace{((1-p^*)^z, p^*(1-p^*)^{z-1}, p^*(1-p^*)^{z-2}, \dots, p^*(1-p^*), p^*, 0, \dots, 0)}_{x_0^*, \dots, x_z^*} \underbrace{(0, \dots, 0)}_{x_{z+1}^*, \dots, x_K^*},$$

where

$$p^* = 1 - (1-m)^{1/z}$$

is the probability that, in any given period, a household finds the good in the store.

Since p^* is the probability that a household can purchase the good, the probability that a household experiences an in-store shortage when all households use the same z -storage rule is $1-p^* = (1-m)^{1/z}$. If $m=1$, this probability is zero for all z ; yet, if $z > 1$ every household stockpiles. This is not inconsistent: For $m=1$, the stationary

(in fact, steady) state associated with the z -storage rule is for every household to have z units in storage, consume one unit per period, and buy one unit per period to keep the household inventory constant at z units. What Proposition 1 implies, then, is that stockpiling cannot cause permanent in-store shortages if the aggregate supply is sufficient to meet aggregate consumption needs. But without in-store shortages there is clearly no need for costly storage. In Section 4, we use this observation to show that there is a unique equilibrium when $m = 1$, which is for all households to use the minimum storage rule.

If $m < 1$, there must necessarily be in-store shortages (it is not possible to have an aggregate shortage but no individual ones). The formula for p^* in Proposition 1 allows us to compute the frequency of these in-store shortages. Suppose there is a one percent aggregate shortfall in the toilet paper supply ($m = 0.99$) and households store five units ($z = 5$). In this case $p^* = 0.6019$, meaning that four out of every ten trips to the store will be unsuccessful. If $m = 0.9$ and $z = 10$, then eight out of every ten trips will be unsuccessful. Therefore, to the household, the experienced in-store shortages appear much more severe than the underlying aggregate supply shortage. However, in a stationary state, any household's consumption probability must be independent of z . Note that in every period, the fraction of households that do not consume is

$$x_0^* = (1 - p^*)^z = 1 - m.$$

Since households are symmetric, this means that the long-run probability that a household consumes the good in any given period is m , which does not depend on the z -storage rule used.¹²

Finally, we turn to the question whether the economy converges to the stationary state. Answering this question is complicated by the fact that the mapping T in (7) fails to be a contraction under common metrics on Δ_K . For example, suppose $m = 1$, $z = K = 5$, and consider the states

$$x = (0, 0, 0, 0, 0, 1) \text{ and } y = \left(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}\right).$$

Under the standard Euclidean norm (i.e., ℓ_2), the distance between x and y is $d(x - y) = \sqrt{1/2}$. Applying (5)–(7), we obtain

$$x' = T(x|5) = (0, 0, 0, 0, 0, 1) \text{ and } y' = T(y|5) = \left(\frac{1}{3}, 0, 0, 0, \frac{1}{3}, \frac{1}{3}\right),$$

and thus $d(x' - y') = \sqrt{2/3}$. Thus, T expands the distance between some points in Δ_K , and this will be the case for alternative metrics as well (including ℓ_1 and ℓ_∞). Therefore,

¹²This is a purely mechanical consequence of inventories being limited to at most K . If the long-run probability of consumption was smaller than m , inventories would eventually exceed the maximum K . Likewise, it cannot be larger than m given that supply is fixed at m .

we cannot simply apply the Contraction Mapping Theorem to establish convergence. Nevertheless, in the Appendix we prove:

Proposition 2. *Suppose every household uses the same z -storage policy in every period. The economy converges to the stationary state x^* from any initial state x^0 . Furthermore, if $x^0 \succsim (\precsim) x^*$ then $p(x^t)$ converges to p^* from above (below).*

4 Equilibrium Analysis

As discussed above, we will be looking for equilibria that are symmetric across households and also across time. That is, all households apply the same decision rule in every period. If the common decision rule used in every period is a z -storage rule, we call such an equilibrium a *z -storage equilibrium*, and these are the equilibria we characterize. We emphasize that we do not restrict households to use a z -storage rule and only characterize the equilibrium value of z under that restriction. Instead, households can use any decision rule, and we identify conditions under which the use of a z -storage rule is optimal for a household if all other households use this rule.

For the case where there is no aggregate supply shortage, we have a clear result:

Proposition 3. *Suppose $m = 1$ and let $x^0 \in \Delta_K$ be any initial state. The unique equilibrium is for every household to use the minimum storage rule in every period.*

In other words, if $m = 1$, the unique equilibrium behavior is for every household to buy and consume one unit of the good in every period. This is true regardless of the initial state x^0 . The result implies that stockpiling cannot merely arise because of a “self-fulfilling prophecy.” One may imagine a situation where there is no aggregate shortage but where there are nonetheless in-store shortages because consumers stockpile, and this behavior is optimal because of the in-store shortages it causes. As Proposition 3 shows, however, this situation cannot occur in our model. For stockpiling to arise in equilibrium, an underlying aggregate shortage must exist. For the remainder of this Section, therefore, we assume that $m < 1$.

4.1 General conditions for the optimality of z -storage rules

To characterize z -storage equilibria, consider the value function in (4) and note that $V_0^t - V_1^t = \lambda - 1 < 0 \forall t$. Furthermore, for $k \geq 2$,

$$V_{k-1}^t - V_k^t = \lambda + \beta \underbrace{\left(p^t \left[\max_{s \geq k-2} V_s^{t+1} - \max_{s \geq k-1} V_s^{t+1} \right] + (1-p^t) \left[V_{k-2}^{t+1} - V_{k-1}^{t+1} \right] \right)}_{\geq 0}.$$

Proceeding recursively, we can write

$$\begin{aligned}
V_{k-1}^t - V_k^t &\geq \lambda + \beta(1-p^t) [V_{k-2}^{t+1} - V_{k-1}^{t+1}] \\
&\geq \lambda + \beta(1-p^t) [\lambda + \beta(1-p^{t+1}) [V_{k-3}^{t+2} - V_{k-2}^{t+2}]] \\
&\quad \vdots \\
&\geq \lambda \left[1 + \sum_{k'=1}^{k-2} \beta^{k'} \prod_{s=0}^{k'-1} (1-p^{t+s}) \right] \\
&\quad + \underbrace{\beta^{k-1} \prod_{s=0}^{k-2} (1-p^{t+s}) [V_0^{t+k-1} - V_1^{t+k-1}]}_{= \lambda - 1} \\
&= \lambda \left[1 + \sum_{k'=1}^{k-1} \beta^{k'} \prod_{s=0}^{k'-1} (1-p^{t+s}) \right] - \beta^{k-1} \prod_{s=0}^{k-2} (1-p^{t+s}) \\
&\equiv D_k(\mathbf{p}^t),
\end{aligned} \tag{10}$$

where \mathbf{p}^t denotes the sequence $(p^t, p^{t+1}, p^{t+2}, \dots)$.

We now make three observations. First, from (10) it is apparent that

$$D_K(\mathbf{p}^t) \geq D_{K-1}(\mathbf{p}^t) \geq \dots \geq D_2(\mathbf{p}^t) \geq D_1(\mathbf{p}^t) = \lambda - 1. \tag{11}$$

Second, suppose that all households use the same z -storage rule ($z \in \{1, \dots, K\}$) from period t onward, and that this rule is optimal from period t onward. Then (10) can be strengthened as follows:

$$V_{k-1}^t - V_k^t = D_k(\mathbf{p}^t) \quad \text{for all } k \leq z+1. \tag{12}$$

Third, let z be an integer such that

$$D_z(\mathbf{p}^t) \leq 0 \quad \text{and} \quad D_{z+1}(\mathbf{p}^t) \geq 0. \tag{13}$$

Then (11), (12), and (13) together imply that

$$V_0^t \leq V_1^t \leq \dots \leq V_z^t \quad \text{and} \quad V_z^t \geq V_{z+1}^t \geq \dots \geq V_K^t.$$

This means that the z -storage rule is also optimal in period $t-1$.

Therefore, given $z \in \{1, \dots, K\}$ we can verify if a z -storage equilibrium exists in the following way: Fix x^0 and set $x^{t+1} = T(x^t|z)$ for all $t \geq 0$. Let (p^0, p^1, p^2, \dots) be the

associated sequence of probabilities of finding the item in the store. If (13) holds for all $t \geq 1$, then using the z -storage rule in every period is optimal for a household if all other households do the same.

Condition (13) can be expressed in terms of the storage cost λ . If we define

$$\bar{\lambda}_z(\mathbf{p}^t) \equiv \frac{\beta^{z-1} \prod_{s=0}^{z-2} (1 - p^{t+s})}{1 + \sum_{k=1}^{z-1} \beta^k \prod_{s=0}^{k-1} (1 - p^{t+s})}, \quad (14)$$

then (13) is equivalent to

$$\lambda \leq \bar{\lambda}_z(\mathbf{p}^t) \quad \text{and} \quad \lambda \geq \bar{\lambda}_{z+1}(\mathbf{p}^t). \quad (15)$$

The first condition in (15) states that the storage cost is low enough for households to want to store at least z units, and the second condition states that the storage cost is high enough for households to not want to store more than z units. For the two “bookend” cases given by the minimum and maximum storage rules, only one of the two conditions is relevant: If $z = 1$, the first condition is automatically satisfied ($D_1(\mathbf{p}^t) = \lambda - 1 < 0$); and if $z = K$, the second condition is unnecessary (as no household can store more than K units).

In (15), the thresholds $\bar{\lambda}_z$ and $\bar{\lambda}_{z+1}$ depend on the probabilities with which households are able to obtain the item in the future. This is not surprising: Optimal inventories are determined not only by the cost of storage but also by its benefit, which is higher if shortages are more likely. Intuitively, if shortages are less likely, the cost of storage needs to decrease to support a given z -storage equilibrium. The following result confirms this:

Lemma 4. *For $z = 2, \dots, K$, $\bar{\lambda}_z(\mathbf{p}^t)$ is decreasing in \mathbf{p}^t .*

4.2 Stationary equilibria and excess stockpiling

We now consider *stationary equilibria* in z -storage rules. In a stationary equilibrium, $x^t = x^* \forall t$. Such an equilibrium is best thought of as describing the long-run, permanent outcome to which the economy has converged to aggregate state x^* and at which it remains. In other words, if the initial state was x^* , the economy would remain at this state permanently in equilibrium (absent any changes in fundamentals, i.e., changes in m , K , β , or λ).¹³

¹³A non-stationary equilibria, on the other hand, is one in which the aggregate state changes over time. That is, a non-stationary equilibrium is one in which the transition from some initial state x^0 to a long-run, stationary state $x^* \neq x^0$ is made an explicit part of equilibrium dynamics. We consider such equilibria in Section 4.3 below.

Fix z and suppose $x^t = x^* \forall t$, where x^* is the stationary state described in Proposition 1. The associated probability of finding the item is $p^t = p^* = 1 - (1 - m)^{1/z} \forall t$. After substituting this value for all probability terms in (14) and simplifying, (15) can be expressed as follows:

$$\frac{\gamma(z)^z(1 - \gamma(z))}{1 - \gamma(z)^{z+1}} \leq \lambda \leq \frac{\gamma(z)^{z-1}(1 - \gamma(z))}{1 - \gamma(z)^z}, \quad (16)$$

where $\gamma(z) \equiv \beta(1 - m)^{1/z}$. For $z < 1 < K$, a stationary z -storage equilibrium exists if condition (16) is satisfied. Likewise, a minimum storage equilibrium exists if the first inequality in (16) holds when $z = 1$, and a maximum storage equilibrium exists if the second inequality in (16) holds when $z = K$.

Proposition 5. *A stationary z -storage equilibrium exists for some $z = 1, \dots, K$.*

Figure 1 characterizes the stationary z -storage equilibria for the case where $\beta = 0.999$ and $K = 6$ are fixed, varying only m and λ . If $m = 1$, the only equilibrium is for every household to employ the minimum storage rule, as predicted by Proposition 3. On the other hand, when $m < 1$, stationary equilibria with stockpiling can emerge in which households stockpile. Moreover, the stationary z -storage equilibria are generally not unique, that is, there may be multiple z -storage equilibria with different values of z for the same model parameters. Figure 1, for values of m close to one (i.e., the supply shortage is very small) and valued of λ close to zero (i.e., storage is relatively cheap), a large number of different z -storage equilibria simultaneously exist.

In particular, in the small red-colored region in Figure 1, all z -storage strategies are equilibria, including the minimum and maximum storage strategies. The following result shows that this is a generic possibility for arbitrarily small but positive aggregate shortages. In other words, a region similar to the small red-colored region in Figure 2 will always exist.

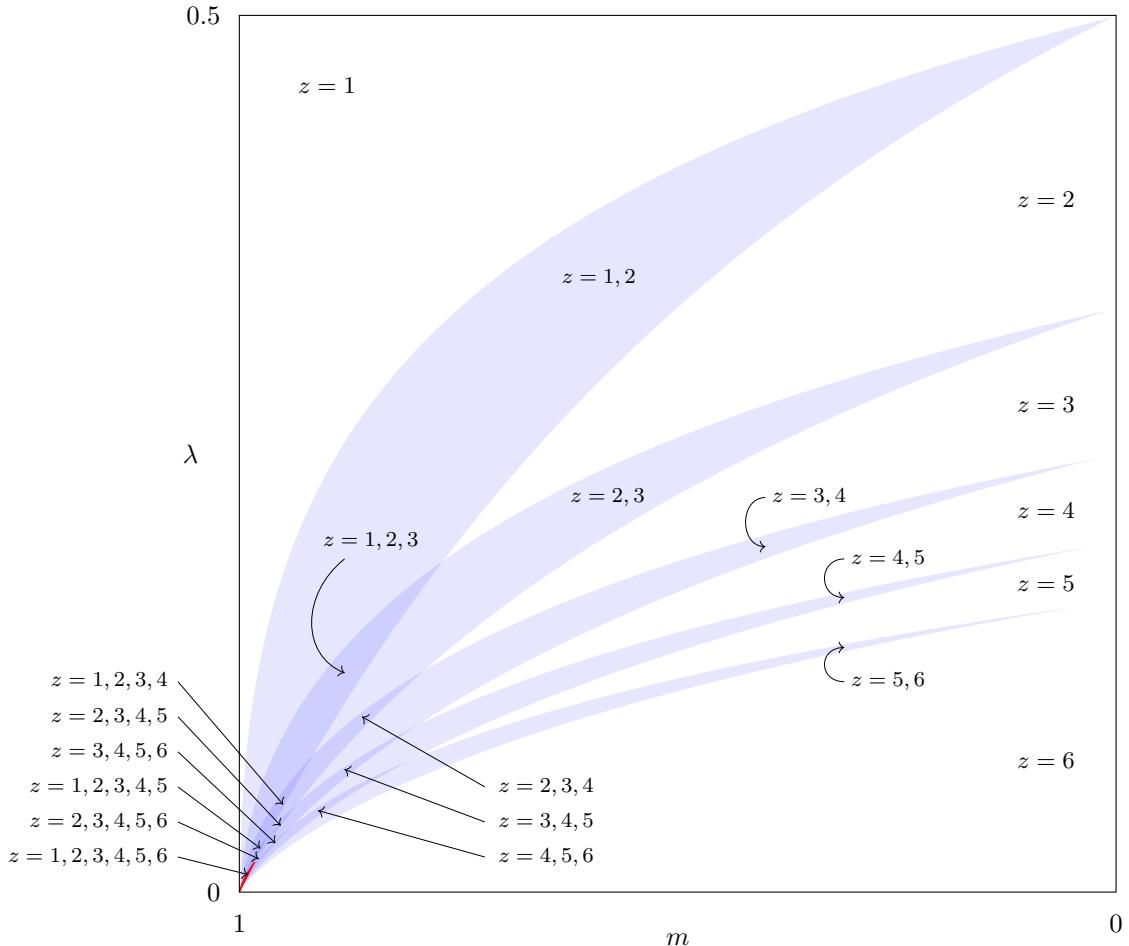
Proposition 6. *Fix $K > 1$ and $\beta < 1$. There exists $\bar{m} < 1$ such that the following is true. For every $\bar{m} < m < 1$, there exists an open interval of storage costs $\Lambda \subset (0, 1)$ such that, if $\lambda \in \Lambda$, a stationary z -storage equilibrium exists for all $z = 1, \dots, K$.*

Figure 1 further suggests that, when multiple stationary z -storage equilibria exist, the equilibrium set is “connected” in the sense that the set of equilibrium values for z consists of consecutive integers. The following proposition confirms this:

Proposition 7. *If stationary z -storage and z' -storage equilibria exist, with $z < z'$, then stationary z'' -storage equilibria exist for all integers $z \leq z'' \leq z'$.*

In our model, stockpiling incentives are driven by two forces. The first force is the fundamental, aggregate supply shortfall—that is, the fact that only $m < 1$ new units of the

Figure 1: Stationary z -storage equilibria ($\beta = .999$, $K = 6$).



good are available in each period despite households wanting to consume 1 unit. Without this shortfall, Proposition 3 implies that stockpiling would not arise in any equilibrium. However, if there is an aggregate supply shortage, it may be optimal for a household to maintain an inventory of more than one unit of the good. We call this the *direct effect* of the shortage. The second force is an *indirect effect*: If every household decides to stockpile, in-store shortages become more frequent and, as a result, further stockpiling incentives are created. Thus, there exists a feedback mechanism from stockpiling to even more stockpiling. The equilibria of the model reflect the combined effect of both direct incentive and indirect feedback.

To measure the direct effect only, we can compute the z -storage rule that is optimal for an individual household in the following situation: All other households use the minimum storage rule that would be the unique equilibrium if there was no aggregate

shortage, and that the aggregate state is $x^* = (1 - m, m, 0, \dots, 0)$; that is, the stationary state the economy would converge to under the minimum storage rule. The likelihood that a household finds the item in the store in any given period is then $p^* = m$, and using (15), the z -storage rule is a best response in this situation if

$$\lambda \leq \bar{\lambda}_z(m, m, \dots) \quad \text{and} \quad \lambda \geq \bar{\lambda}_{z+1}(m, m, \dots),$$

or, equivalently,

$$\frac{\gamma(1)^z(1 - \gamma(1))}{1 - \gamma(1)^{z+1}} \leq \lambda \leq \frac{\gamma(1)^{z-1}(1 - \gamma(1))}{1 - \gamma(1)^z}.$$

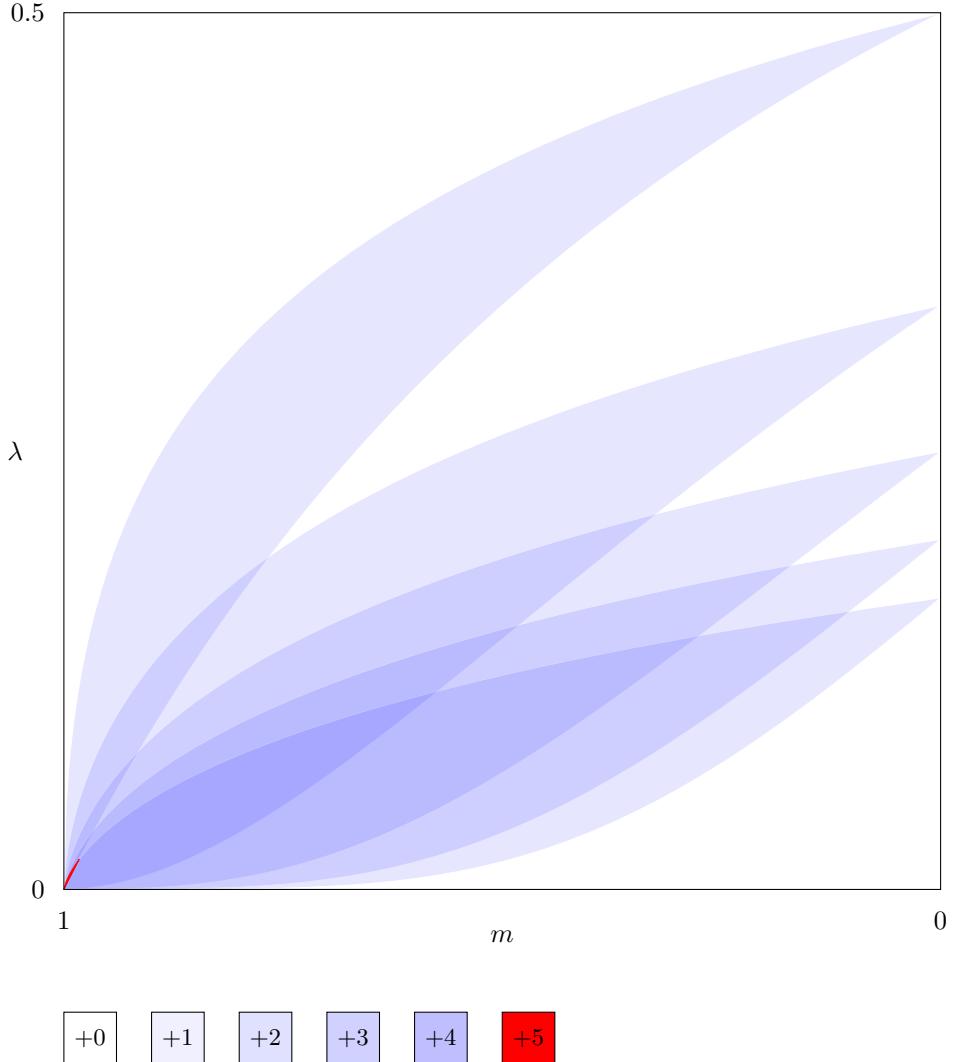
Now suppose that, for a given (m, λ) , a z -storage equilibrium exists for some value of z . We can then take the difference between the equilibrium value of z and the previously computed value that is a best response to the minimum storage rule. This difference can be interpreted as a measure of the “excess stockpiling” that arises from the aforementioned feedback loop, or as a measure of the strength of the indirect stockpiling incentive.

Figure 2 depicts the amount of excess stockpiling for the same parameter values as those used in Figure 1. When multiple stationary z -storage equilibria exist, we selected the one with the largest z in order to measure how strong the indirect incentive could be in the most extreme case. As Figure 2 shows, the feedback mechanism can generate a significant amount of excess stockpiling for certain parameter configurations. In particular, consider the set of parameter combinations for which both the minimum and maximum storage equilibrium exist (indicated in red in Figure 2 as well). If all households followed the minimum storage strategy, it would be a unilateral best response to use the minimum storage strategy as well. Therefore, if the maximum storage equilibrium is played in this case, the incentive to maintain inventories above a single unit is driven *entirely* by the fact that other households follow the same strategy. In other words, the feedback loop from stockpiling to more stockpiling accounts for $K - 1$ out of the K units in each household’s target inventory.

4.3 Transitional dynamics and non-stationary equilibria

The preceding Propositions 5, 7, and 6 described stationary equilibria in which the aggregate state had converged to its long-run limit. As such, these results do not describe stockpiling behavior that emerges as the short-run response to a supply shortage. Consider, for example, a situation in which supply and demand are balanced in each period until an unexpected supply shock reduces the per-period supply to $m < 1$. In response to this shock households may begin to accumulate inventories, thereby initiating the transition from the previous stationary state to a new one. If households have perfect foresight, this transition will itself be governed by decision rules that are best responses to each other.

Figure 2: Excess stockpiling ($\beta = .999$, $K = 6$).



Colors represent additional units stockpiled in highest z -storage equilibrium, relative to units stockpiled as best response to 1-storage rule.

A full analysis of the resulting transitional dynamics is beyond the scope of this paper; however, some insights can be established using the apparatus already developed. Specifically, we consider an economy that starts at initial state

$$x^0 = (0, 1, 0, \dots, 0),$$

that is, at the unique equilibrium steady state for $m = 1$. We then imagine that the supply of the good is unexpectedly reduced to $m < 1$, and that all households adopt

the same z -storage rule in response. A measure 1 of households will each attempt to purchase z units of the good, and a measure m/z of households will be successful. Thus, next period's state is

$$x^1 = \left(\underbrace{1 - \frac{m}{z}, 0, \dots, 0}_{x_0^1}, \underbrace{\frac{m}{z}, 0, \dots, 0}_{x_z^1} \right),$$

and if all households continue to follow the z -storage rule, the state will further evolve according to the law of motion (7) and converge to the stationary state x^* described in Proposition (1). Associated with these states is a sequence of probabilities $p^0 = p(x^0|z)$, $p^1 = p(T(x^0|z)|z)$, etc., with which a household finds the item in the store in each period. If the z -storage rule remains optimal in every period for a household that anticipates that all other households use the same z -storage rule in this period and all future periods, we call the resulting outcome a non-stationary z -storage equilibrium.

In analogy to our condition (16) for stationary equilibria, we will derive a condition on the storage cost λ under which a non-stationary z -storage equilibrium exists. It should be clear that this condition is more stringent than the corresponding condition (16) for stationary equilibria, as the same z -storage rule must be optimal in a larger set of circumstances. We begin with the following result:

Lemma 8. *Suppose $m < 0$, $x^0 = (0, 1, 0, \dots, 0)$, and all households use the same z -storage rule in every period $t = 0, 1, \dots$. Let p^t be the probability that a household finds the good in the store in period t . Then $p_{t+1} \geq p_t$ for all t , and $p_t \rightarrow p^* = 1 - (1 - m)^{1/z}$.*

As before, let $\mathbf{p}^t = (p^t, p^{t+1}, \dots)$. Lemma 8 implies that $\mathbf{p}^0 \leq \mathbf{p}^1 \leq \mathbf{p}^2 \leq \dots$; furthermore, $\mathbf{p}^t \rightarrow (p^*, p^*, \dots)$ uniformly.

Now consider whether using the z -storage rule is individual optimal in period $t = 0, 1, \dots$. As shown in Section 4.1, this is the case if and only if (15) holds for all t . 4.1 contains two conditions. The first condition applies whenever $z > 1$ and states that the storage cost cannot be so high that the household would rather store fewer than z units:

$$\lambda \leq \bar{\lambda}_z(\mathbf{p}^t), \quad (17)$$

where $\bar{\lambda}_z(\mathbf{p}^t)$ was defined in (14). By Lemma 4, $\bar{\lambda}_z(\mathbf{p}^t)$ is decreasing in \mathbf{p}^t ; and by Lemma 8, \mathbf{p}^t is increasing in t . Thus, for (17) to hold for all t , it is necessary and sufficient that it holds at $\lim_{t \rightarrow \infty} \mathbf{p}^t = (p^*, p^*, p^*, \dots)$.

$$\lambda \leq \bar{\lambda}_z(p^*, p^*, p^*, \dots) = \frac{\gamma(z)^{z-1}(1 - \gamma(z))}{1 - \gamma(z)^z}, \quad (18)$$

where the term on the right-side is the same as in (16). The second condition applies whenever $z < K$ and states that the storage cost cannot be so low that the household would rather store more than z units:

$$\lambda \geq \bar{\lambda}_{z+1}(\mathbf{p}^t). \quad (19)$$

Again applying Lemma 4 and Lemma 8, we see that (19) holds for all t if and only if it holds at $\mathbf{p}^0 = (p^0, p^1, p^2, \dots)$:

$$\lambda \geq \bar{\lambda}_z(p^0, p^1, p^2, \dots). \quad (20)$$

Combining (18) and (20) and making the dependence of p^t on z explicit, we get:

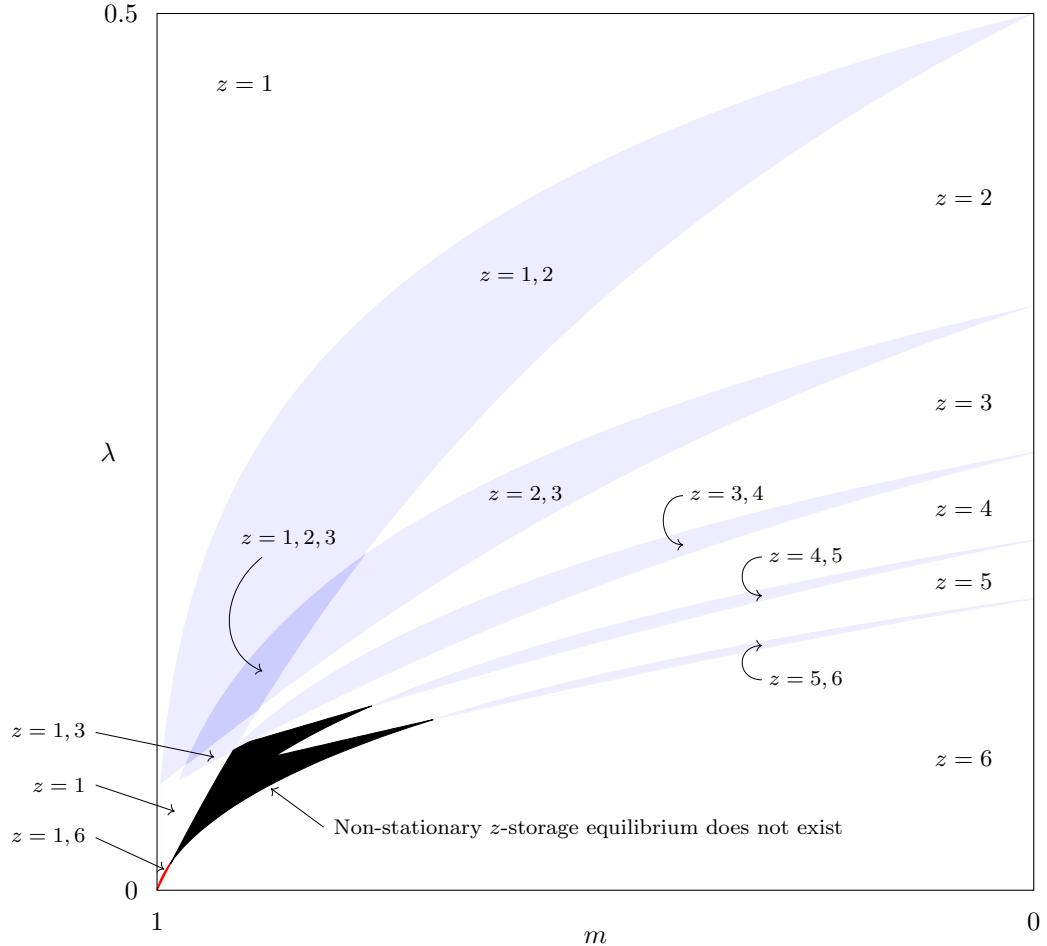
$$\bar{\lambda}_z(p(x^0|z), p(T(x_0|z)|z), p(T^2(x_0|z)|z), \dots) \leq \lambda \leq \frac{\gamma(z)^{z-1}(1-\gamma(z))}{1-\gamma(z)^z}. \quad (21)$$

A non-stationary z -storage equilibrium exists if condition (21) is satisfied (for $z = 1$, the second inequality can be ignored; and for $z = K$ the first inequality can be ignored). Note that the upper bound on the storage cost λ is the same as the previous upper bound for stationary z -storage equilibrium; however, the lower bound on the storage cost λ is larger than the previous lower bound for stationary equilibria. Therefore, a non-stationary z -storage equilibrium may fail to exist for parameter values under which a stationary z -storage equilibrium existed.

Figure 3 depicts the set of non-stationary z -storage equilibria for the same parameter values as were used in Figure 1 (i.e., $\beta = .999$ and $K = 6$), assuming initial state $x^0 = (0, 1, 0, 0, 0, 0, 0)$. As the new equilibrium condition (21) is more stringent than the previous condition (16), some of the z -storage rules that were stationary equilibria have disappeared. Moreover, Figure 3 demonstrates that none of the previous Propositions 5, 6, and 7 carries over to the non-stationary case: Non-stationary z -storage equilibria need not exist (see the black region in the graph), and when they exist the range of z -values for which the z -storage rule constitutes an equilibrium can have “holes.”

However, a non-stationary minimum storage equilibrium exists whenever a stationary minimum storage equilibria exists, and the same is true for maximum storage equilibria. For example, as shown in Figure 1 and Figure 3, in the red-colored parameter region for which the full range of stationary z -storage equilibrium exists, the minimum and maximum storage rules survive as non-stationary equilibria. To see why, note that if the initial state is $x^0 = (0, 1, 0, \dots, 0)$ and households use the minimum storage rule, the probability of finding the item in the store is m in every period. This is the same probability as in the stationary state $x^* = (1-m, m, 0, \dots, 0)$. Thus, the condition for a stationary minimum storage equilibrium (i.e., the left inequality in (16), for $z = 1$) is identical to the condition for a non-stationary minimum storage equilibrium (i.e., the left

Figure 3: Non-stationary z -storage equilibria ($\beta = .999$, $K = 6$), assuming initial state $x^0 = (0, 1, 0, 0, 0, 0, 0)$.



inequality in (21), for $z = 1$). Similarly, the condition for a stationary maximum storage equilibrium (i.e., the right inequality in (16), for $z = K$) is identical to the condition for a non-stationary maximum storage equilibrium (i.e., the right inequality in (21), for $z = K$). We summarize this observation in the following result:

Proposition 9. *Let $x^0 = (0, 1, 0, \dots, 0)$. A non-stationary minimum (maximum) storage equilibrium exists if and only if a stationary minimum (maximum) storage equilibrium exists.*

5 Welfare and Policy Implications

In any situation where multiple equilibria exist, it is natural to ask if these equilibria can be ranked by the welfare. Moreover, is welfare maximized in equilibrium, and if it is not, what policy interventions could improve it?

As shown in Section 3, in the stationary state x^* associated with any z -storage rule, the fraction of households that are able to consume the good in any period is m . Moreover, this is the maximum fraction, given that m units of the good are available in each period. Thus, in the model, welfare differs across the z -storage equilibria only insofar as households pay higher total storage costs in equilibria with higher z . Specifically, in stationary z -storage equilibrium the total storage cost incurred by households in each period is

$$E[\lambda s_i] = \lambda[x_1^* + 2x_2^* + \dots + zx_z^*] = \lambda \left[z - \frac{(1-m)^{1/z}}{1-(1-m)^{1/z}} m \right],$$

where x^* denotes the stationary state associated with the z -storage rule, characterized in Proposition 1. It is straightforward to verify that, if x^{**} is the stationary state associated with the z' -storage rule and $z' > z$, then $x^{**} \succ x^*$.¹⁴ Thus the z -storage equilibrium with the lowest z is the one with the least storage cost payment. In the applications we have in mind (i.e., the stockpiling of everyday household items), this storage cost payment is minor, as the per-unit cost λ is likely insignificant in comparison to the flow utility of consumption. Therefore, any welfare differences across different stationary equilibria are relatively minor as well.¹⁵

The non-stationary equilibria we examined explicitly account for the transition from an initial state to the stationary state, and it is along this transition that the *accumulation* of inventories (and not only their maintenance) occurs. Given a fixed per-period supply, inventory accumulation has a much more significant impact on a household's contemporaneous utility than storage costs, as it reduces consumption of the good. However, this consumption reduction is temporary and becomes less severe over

¹⁴To see this, let $z' = z + 1$. From Proposition 1, $x_0^* = 1 - m = x_0^{**}$. Moreover, for $0 < s \leq z$ and $m \in (0, 1)$,

$$x_s^* = \left(1 - (1-m)^{\frac{1}{z}}\right)(1-m)^{\frac{z-s}{z}} > \left(1 - (1-m)^{\frac{1}{z+1}}\right)(1-m)^{\frac{z-s+1}{z+1}} = x_s^{**}.$$

Thus, for all $k = 0, \dots, K$ we have $\sum_{s=0}^k x_s^* \geq \sum_{s=0}^k x_s^{**}$, with strict inequality if $k = 1, \dots, z$, and it follows that $x^{**} \succ x^*$.

¹⁵An additional welfare effect arises from the fact that household inventories bind valuable resources. In the model, we abstracted from this effect because we set the price of the stockpiled good to zero. However, in reality this price is positive. The total value of inventories is, therefore, lower in an equilibrium with less stockpiling, which means that households would have the value difference available to spend on other goods if one z -storage equilibrium was replaced by another equilibrium with less storage. This welfare improvement, too, should be minor, as items such as toilet paper or dried pasta do not account for significant expenditure shares in most households.

time, as the economy converges to the new stationary state. Storage cost payments, on the other hand, are persistent. Thus, storage costs may still account for a significant share of the overall welfare loss of stockpiling in non-stationary equilibria (even if this welfare loss is small).

To confirm this, consider the following parameter configuration:

$$\beta = 0.999, \quad K = 6, \quad m = 0.9995, \quad \lambda = 0.0005.$$

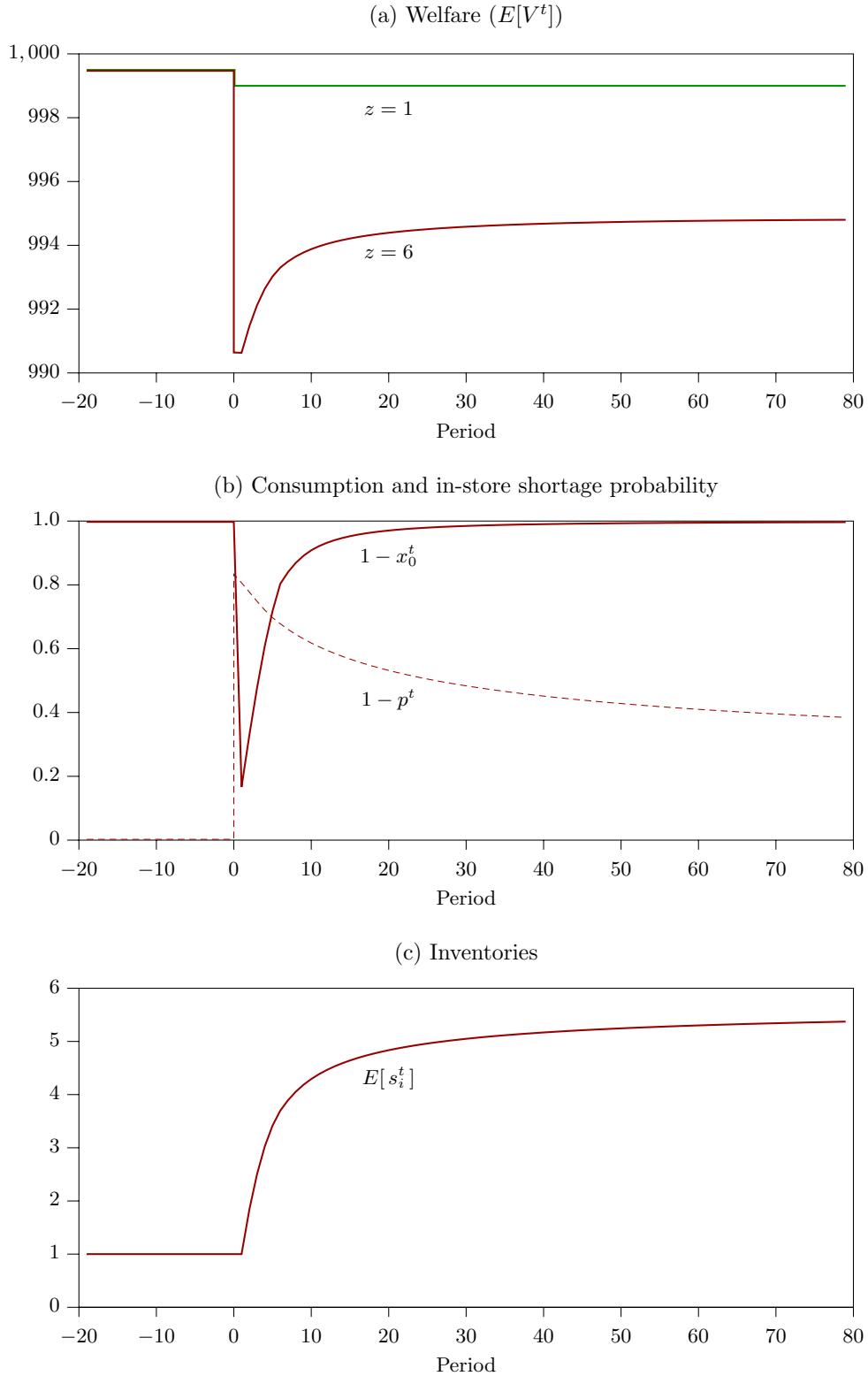
If the time period is one week, a discount factor of $\beta = 0.999$ implies an annual discount rate of approximately 5%. A per-unit storage cost of $\lambda = 0.0005$ means that the cost of storing a one week's worth of toilet paper is 0.05% of the utility the household obtains from the using toilet paper for a week (relative to consuming the next best substitute). And a supply of $m = 0.9995$ implies that—absent any stockpiling by consumers—a household experiences an in-store shortage of toilet paper once every 38 years on average. This parameterization is in the red-colored region in Figure 3, so that non-stationary minimum and maximum storage equilibria both exist. Thus, despite the negligible supply shortage, it is an equilibrium for all households to accumulate and maintain inventories lasting $K = 6$ weeks.¹⁶

For each of these extremal equilibria, Figure 4 (a) plots average continuation values $E[V^t]$ over time. Figure 4 (b) plots a household's per-period consumption probability, $1 - x_0^t$, and the probability that a household experiences an in-store shortage, $1 - p^t$. Figure 4 (c) plots average household inventories, $E[s_i^t]$. Period $t = 0$ is when the supply shortage takes effect; i.e., we assume that prior to period 0 aggregate per-period supply was $m = 1$, with the economy being in the unique minimum storage equilibrium. If the minimum storage equilibrium was maintained, welfare would drop by a small amount in period 0. If, on the other hand, a switch to the maximum storage equilibrium occurred, $E[V^t]$ would drop by a much larger amount, and this drop is accompanied by a sharp decrease in consumption probability and a sharp rise in in-store shortages. However, as the speed of inventory build-up slows, consumption rises again. In-store shortage remain frequent; however, this does not affect long-run consumption as households build inventories precisely to tide over these shortages. Finally, $E[V^t]$ recovers approximately half of the initial welfare loss. The remaining half cannot be recovered, as it is not caused by non-consumption but by the persistently higher storage costs in the maximum storage equilibrium.

Given that stockpiling creates inefficiencies both in the short and long run, welfare can be improved by imposing limits on inventory accumulation. Such restrictions are not uncommon: Many countries have anti-hoarding laws in place to prevent stockpiling

¹⁶Similar examples can be constructed for larger values of K .

Figure 4: Response to a supply shock ($\beta = .999$, $K = 6$, $m = 0.9995$, $\lambda = 0.0005$).



by businesses or households during emergencies,¹⁷ and stores, too, limit quantities per customer in times of shortages.¹⁸ In situations where a z -storage equilibrium might arise with $z > 1$, limiting sales to $z' < z$ units per household per period will result in a Pareto improvement. If the z' -storage rule is itself an equilibrium, this policy is simply an equilibrium selection device—in particular, households will not be constrained by the inventory limit in the new equilibrium.¹⁹ If the z' -storage rule is not an equilibrium (and no z'' -storage equilibrium exists for $z'' < z'$), the inventory constraints will, of course, be binding, but welfare would still be higher than in the original z -storage equilibrium.

A second policy option is to communicate that underlying supply disruptions that might induce stockpiling are small, or do not even exist. Recall that, by Proposition 3, stockpiling cannot arise when there is no aggregate supply shortage. However, aggregate supply may not be observable to households, and the (mistaken) perception of even a slight shortage could trigger a switch to stockpiling strategies. The resulting in-store shortages are directly experienced by consumers and could reinforce the perception of an aggregate shortage and, thereby, reinforce stockpiling behavior. To prevent hoarding by consumers in the early days of the Covid-19 pandemic, Dutch prime minister Mark Rutte famously told shoppers at a grocery store that the Netherlands had sufficient toilet paper for its citizens to be able to “poop for 10 years.”²⁰

6 Conclusion

I conclude this paper with two brief remarks. First, supply-demand imbalances that can generate stockpiling incentives may not only arise due to supply reductions (holding demand fixed), but also due to demand increases (holding supply fixed). This distinction may seem to be irrelevant, and it ultimately is. However, care must still be taken when adapting the formal model we set up in Section 2 to the case of a demand increase. The reason is that supply was modeled in the aggregate, while demand was modeled on the household level. Because household inventories must be integers, we cannot use the model to study the increase of household demand from one unit per period to, say, 1.1 units per period. For example, if a household had 3 units in storage and consumed 1.1 unit, it

¹⁷In the United States, for example, 50 U.S. Code § 4512 states that “[i]n order to prevent hoarding, no person shall accumulate (1) in excess of the reasonable demands of business, personal, or home consumption, or (2) for the purpose of resale at prices in excess of prevailing market prices, materials which have been designated by the President as scarce materials or materials the supply of which would be threatened by such accumulation.” 50 U.S. Code § 4512 specifies a maximum fine of \$10,000 and a maximum prison sentence of one year for violations.

¹⁸These store-imposed policies are, of course, not perfectly enforceable, as customers could simply make more than one trip to the store or visit more than one store.

¹⁹Eckert *et al.* (2017) examine the use of quantity limits as equilibrium selection devices in an antitrust context. Their model, too, has the feature that the constraints will be non-binding in the equilibrium they select.

²⁰See <https://www.reuters.com/article/us-health-coronavirus-netherlands-toilet-idUSKBN21627A>.

would be left with a remaining inventory of 1.9 units, which is not an integer. Thus, it appears that the only demand increases we could study are large increases, i.e., increases from one unit per period to at least twice this quantity.

This, however, is not the only way to think about demand shocks. A better approach is to assume that supply is always fixed $m < 1$ units per period and that each household requires one unit of the good with probability m .²¹ If these household-level demands are independent, the aggregate demand will be exactly equal to aggregate supply. Thus, the scenario where households require one unit with probability m is the “balanced scenario” and corresponds to the case $m = 1$ in the model in Section 2. Relative to this “balanced scenario,” we can now think of a demand increase as an increase in the consumption probability from $m < 1$ to 1, holding supply fixed at m , and this setting would be mathematically equivalent to the model with a supply shortage.

Second, the analysis of non-stationary equilibria in Section 4.3 left open the question how the transition from x^0 to some new long-run state looks like when a z -storage equilibrium does not exist. It is conceivable that the equilibrium is asymmetric, with a fraction of households using one z -storage rule and another fraction using a different z -storage rule. It is also conceivable that all households use the same z -storage rule but change the value of z over time. Yet another possibility is that the equilibrium does not involve simple decision rules like the z -storage rule, or that it does not involve convergence to a stationary state at all. A full examination of these possibilities is beyond the scope of this paper and a topic for future research.

²¹This could reflect a situation where the household consumes one unit in each period, but with probability $1 - m$ the consumed unit is an at-work variety provided by the employer; see footnote 1.

Appendix

Proof of Proposition 2

Take an initial state $x^0 \in \Delta_K$. For $n = 1, 2, \dots$ define

$$x^n \equiv T^n(x^0|z) \quad \text{and} \quad p^n \equiv p(x^n),$$

where $T(\cdot|z) : \Delta_K \rightarrow \Delta_K$ and $p : \Delta_K \rightarrow [0, 1]$ are defined via (5)–(7). We will construct a sequence $\underline{q}^n \rightarrow p^*$ such that $p^n \geq \underline{q}^n \forall n$. We will construct a second sequence $\bar{q}^n \rightarrow p^*$ such that $p^n \leq \bar{q}^n \forall n$. This implies that $p^n \rightarrow p^*$. Therefore, by definition of T in (7), we have $x^n \rightarrow x^*$.

It is sufficient to prove the result for the maximum storage rule. Observe that the law of motion (7) implies $z_k^n = 0$ for all $k > z$ and all $n \geq K - z$. Therefore, after at most $K - z$ iterations of $T(\cdot|z)$, we have $x^n \in \Delta_z \times \{0\}^{K-z}$, and the projection of $T(\cdot|z)$ onto Δ_z becomes

$$x_k^{n+1} = T_k(x^n|z) \equiv \begin{cases} p(x^n) & \text{if } k = z, \\ (1 - p(x^n))x_{k+1}^n & \text{if } 0 < k < z, \\ (1 - p(x^n))(x_1^t + x_0^n) & \text{if } k = 0, \end{cases}$$

which is the same as (7) when $z = K$. Without loss of generality, therefore, we can restrict attention to the maximum storage rule.²² To save on notation, for the remainder of this proof we write $T(\cdot)$ instead of $T(\cdot|z)$.

The proof is divided into a series of steps. In Step 1 we establish some preliminary results that we will apply repeatedly later on. In Step 2 we construct the sequence \underline{q}^n , in Step 3 we show that $\underline{q}^n \rightarrow p^*$, and in Step 4 we show that $p^n \geq \underline{q}^n$ for all n . Step 5 repeats Steps 2–4 to establish analogous results for \bar{q}^n . Finally, Step 6 establishes that $x^0 \succsim (\precsim) x^*$ implies convergence of p^n to p^* from above (below).

Step 1: Preliminaries

Define a function $f : \Delta_K \times [0, 1] \rightarrow \Delta_K$ as follows:

$$f_k(x, q) = \begin{cases} q & \text{if } k = K, \\ (1 - q)x_{k+1} & \text{if } 0 < k < K, \\ (1 - q)(x_1 + x_0) & \text{if } k = 0. \end{cases}$$

Note that $T(x) = f(x, p(x))$. In Step 4 and Step 6, we will apply the following result:

²²Put differently: If $z < K$, we can redefine $K := z$ and proceed with proving the result for $z = K$, thus redefined.

Lemma 10.

- (a) If $x \succsim x'$ then $p(x) \geq p(x')$.
- (b) If $x \succsim x'$ then $f(x, q) \succsim f(x', q)$ for all $q \in [0, 1]$.
- (c) If $q \geq q'$ then $f(x, q) \succsim f(x, q')$ for all $x \in \Delta_K$.

Proof. Part (a) is readily apparent from (5)–(6). To show part (b), suppose $x \succsim x'$, that is,

$$\sum_{s=0}^k x_s \leq \sum_{s=0}^k x'_s \quad \forall k = 0, \dots, K.$$

Fix $q \in [0, 1]$ and let $y = f(x, q)$ and $y' = f(x', q)$. Then we have

$$\sum_{s=0}^k y_s = (1 - q) \sum_{s=0}^{k+1} x_s \leq (1 - q) \sum_{s=0}^{k+1} x'_s = \sum_{s=0}^k y'_s$$

for all $k = 0, \dots, K - 1$, and $\sum_{s=0}^K y_s = 1 = \sum_{s=0}^K y'_s$. It follows that $y \succsim y'$. Finally, to show part (c), suppose $q \geq q'$. Fix $x \in \Delta_K$ and let $y = f(x, q)$ and $y' = f(x, q')$. Then we have

$$\sum_{s=0}^k y_s = (1 - q) \sum_{s=0}^{k+1} x_s \leq (1 - q') \sum_{s=0}^{k+1} x_s = \sum_{s=0}^k y'_s$$

for all $k = 0, \dots, K - 1$, and $\sum_{s=0}^K y_s = 1 = \sum_{s=0}^K y'_s$. It follows that $y \succsim y'$. \square

Step 2: Construction of the sequence \underline{q}^n

Associated with the sequence \underline{q}^n will be a sequence of states, $\underline{x}^n \in \Delta_K$, defined through $\underline{x}^0 = (1, 0, \dots, 0)$ and $\underline{x}^{n+1} = f(\underline{x}^n, \underline{q}^n)$. For each n , define $\underline{p}^n = p(\underline{x}^n)$. Note that $\underline{p}^0 = m/K$.

We build the sequence \underline{q}^n in pieces of K elements at a time. We begin by setting the first K values of \underline{q}^n to

$$\underline{q}^0, \dots, \underline{q}^{K-1} = \underline{p}^0.$$

Given the definition of f , in period K we have

$$\underline{x}^K = ((1 - \underline{p}^0)^K, (1 - \underline{p}^0)^{K-1} \underline{p}^0, (1 - \underline{p}^0)^{K-2} \underline{p}^0, \dots, (1 - \underline{p}^0) \underline{p}^0, \underline{p}^0).$$

Using the same formulas as in (8)–(9), we can write

$$\underline{p}^K = \frac{m \underline{p}^0}{1 - (1 - \underline{p}^0)^K}.$$

We then set the next K values of \underline{q}^n to

$$\underline{q}^K, \dots, \underline{q}^{2K-1} = \underline{p}^K.$$

Therefore, in period $2K$ we have

$$\underline{x}^{2K} = ((1 - \underline{p}^K)^K, (1 - \underline{p}^K)^{K-1}\underline{p}^K, (1 - \underline{p}^K)^{K-2}\underline{p}^K, \dots, (1 - \underline{p}^K)\underline{p}^K, \underline{p}^K)$$

and

$$\underline{p}^{2K} = \frac{m\underline{p}^K}{1 - (1 - \underline{p}^K)^K},$$

and we set the next K values of \underline{q}^n to $\underline{q}^{2K}, \dots, \underline{q}^{3K-1} = \underline{p}^{2K}$. Proceeding in the same fashion for $\ell = 3, 4, \dots$, we have

$$\underline{q}^{\ell K}, \dots, \underline{q}^{\ell K + (K-1)} = \underline{p}^{\ell K} = \frac{m\underline{p}^{(\ell-1)K}}{1 - (1 - \underline{p}^{(\ell-1)K})^K}. \quad (22)$$

Step 3: $\underline{q}^n \rightarrow p^*$ as $n \rightarrow \infty$

Denote the function on the right-hand side of (22) by

$$A(p) = \frac{mp}{1 - (1 - p)^K}.$$

The unique fixed point of $A : [0, 1] \rightarrow [0, 1]$ is $p^* = 1 - (1 - m)^{1/K}$. We will show that $0 < A'(p) < 1$ for all $p \in (0, 1)$. This implies that $\underline{p}^{\ell K} \rightarrow p^*$ as $\ell \rightarrow \infty$. Since $\underline{q}^{\ell K} = \dots = \underline{q}^{(\ell+1)K-1} = \underline{p}^{\ell K}$, it follows that $\underline{q}^n \rightarrow p^*$ as $n \rightarrow \infty$.

Note that

$$\begin{aligned} A'(p) &= m \frac{1 - (1 - p)^K - pK(1 - p)^{K-1}}{(1 - (1 - p)^K)^2} < 1 \\ &\Leftarrow \frac{1 - (1 - p)^K - pK(1 - p)^{K-1}}{(1 - (1 - p)^K)^2} < 1 \\ &\Leftrightarrow K > \frac{(1 - p) - (1 - p)^{K+1}}{p} = \sum_{k=1}^K (1 - p)^k, \end{aligned}$$

which is true if $p \in (0, 1)$. Likewise, note that

$$\begin{aligned} A'(p) &= m \frac{1 - (1 - p)^K - pK(1 - p)^{K-1}}{(1 - (1 - p)^K)^2} > 0 \\ &\Leftarrow \frac{1 - (1 - p)^K - pK(1 - p)^{K-1}}{(1 - (1 - p)^K)^2} > 0 \end{aligned}$$

$$\Leftrightarrow K < \frac{1 - (1-p)^K}{p(1-p)^{K-1}} = \sum_{k=0}^{K-1} \frac{(1-p)^k}{(1-p)^{K-1}},$$

which, too, is true if $p \in (0, 1)$. Therefore $0 < A'(p) < 1 \ \forall p \in (0, 1)$.

Step 4: $p^n \geq q^n$ for all n

It will be convenient to construct a sequence $\underline{x}^n \in \Delta_L$ as follows: $\underline{x}^0 = (1, 0, \dots, 0)$ and $\underline{x}^{n+1} = T(\underline{x}^n)$. Also define $\underline{p}^n = p(\underline{x}^n)$. Note that $\underline{x}^1 \succsim \underline{x}^0$ necessarily; then using Lemma 10 (a)–(c) we can write

$$\begin{aligned}\underline{x}^2 &= T(\underline{x}^1) = f(\underline{x}^1, \underline{p}^1) \succsim f(\underline{x}^1, \underline{p}^0) \succsim f(\underline{x}^0, \underline{p}^0) = T(\underline{x}^0) = \underline{x}^1, \\ \underline{x}^3 &= T(\underline{x}^2) = f(\underline{x}^2, \underline{p}^2) \succsim f(\underline{x}^2, \underline{p}^1) \succsim f(\underline{x}^1, \underline{p}^1) = T(\underline{x}^1) = \underline{x}^2,\end{aligned}$$

and so on. Therefore \underline{x}^n is increasing in the sense that $\underline{x}^{n+1} \succsim \underline{x}^n \ \forall n$. This implies that $\underline{p}^{n+1} \geq \underline{p}^n \ \forall n$.

We will show that $p^n \geq \underline{p}^n \geq \underline{p}^0 \geq q^n$ for all n . All four sequences are illustrated in Figure 5 below. (The same figure also shows the corresponding sequence that will bound p^n from above; see Step 5.)

First, to show that $p^n \geq \underline{p}^n \ \forall n$, note that $x^0 \succsim \underline{x}^0$ necessarily. Thus, using Lemma 10 (a)–(c), we have

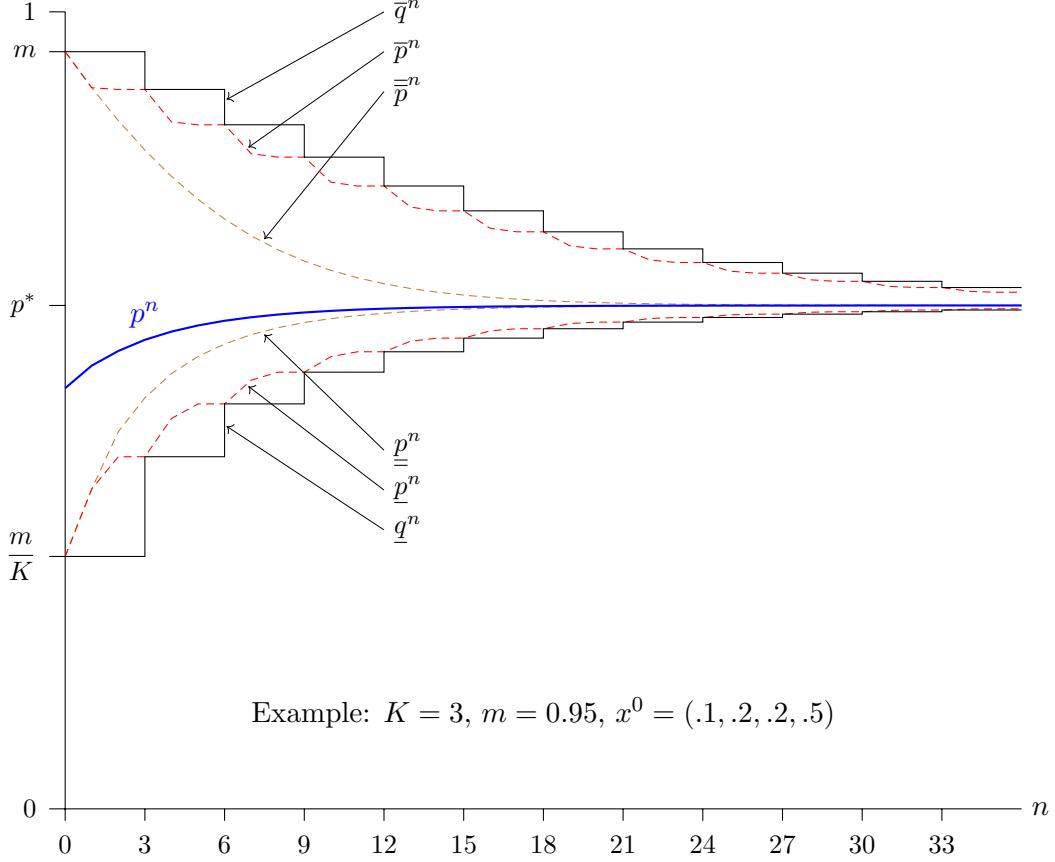
$$\begin{aligned}x^1 &= T(x^0) = f(x^0, p(x^0)) \succsim f(x^0, \underline{p}^0) \succsim f(\underline{x}^0, \underline{p}^0) = T(\underline{x}^0) = \underline{x}^1, \\ x^2 &= T(x^1) = f(x^1, p(x^1)) \succsim f(x^1, \underline{p}^1) \succsim f(\underline{x}^1, \underline{p}^1) = T(\underline{x}^1) = \underline{x}^2,\end{aligned}$$

and so on. It follows that $x^n \succsim \underline{x}^n \ \forall n$. By Lemma 10 (a), this implies $p^n \geq \underline{p}^n \ \forall n$.

Second, to show that $\underline{p}^n \geq \underline{p}^0 \ \forall n$, observe that $\underline{x}^0 = \underline{x}^0$ implies $\underline{p}^0 = \underline{p}^0 (= q^0, \dots, q^{K-1})$. Using Lemma 10 (a)–(c) and the fact that \underline{x}^n is increasing, it follows that

$$\begin{aligned}\underline{x}^1 &= T(\underline{x}^0) = f(\underline{x}^0, \underline{p}^0) \\ &= f(\underline{x}^0, \underline{p}^0) = f(x^1, \underline{p}^0) = \underline{x}^1, \\ \underline{x}^2 &= T(\underline{x}^1) = f(\underline{x}^1, \underline{p}^1) \succsim f(\underline{x}^1, \underline{p}^0) \\ &= f(\underline{x}^1, \underline{p}^0) \succsim f(x^1, \underline{p}^0) = \underline{x}^2, \\ &\vdots \\ \underline{x}^K &= T(\underline{x}^{K-1}) = f(\underline{x}^{K-1}, \underline{p}^{K-1}) \succsim f(\underline{x}^{K-1}, \underline{p}^0) \\ &= f(\underline{x}^{K-1}, \underline{p}^0) \succsim f(x^{K-1}, \underline{p}^0) = \underline{x}^K.\end{aligned}$$

Figure 5: Illustration of the proof of Proposition 2.



This implies that $\underline{\underline{p}}^K \geq \underline{p}^K (= \underline{q}^K, \dots, \underline{q}^{2K-1})$. Using Lemma 10 (a)–(c) and the fact that $\underline{\underline{x}}^n$ is increasing, it follows that

$$\begin{aligned}
\underline{\underline{x}}^{K+1} &= T(\underline{\underline{x}}^K) = f(\underline{\underline{x}}^K, \underline{\underline{p}}^K) \\
&\gtrsim f(\underline{\underline{x}}^K, \underline{p}^K) \gtrsim f(\underline{\underline{x}}^K, \underline{p}^K) = \underline{\underline{x}}^{K+1}, \\
\underline{\underline{x}}^{K+2} &= T(\underline{\underline{x}}^{K+1}) = f(\underline{\underline{x}}^{K+1}, \underline{\underline{p}}^{K+1}) \gtrsim f(\underline{\underline{x}}^{K+1}, \underline{\underline{p}}^K) \\
&\quad \vdots \\
&\gtrsim f(\underline{\underline{x}}^{K+1}, \underline{p}^K) \gtrsim f(\underline{\underline{x}}^{K+1}, \underline{p}^K) = \underline{\underline{x}}^{K+2}, \\
\underline{\underline{x}}^{2K} &= T(\underline{\underline{x}}^{2K-1}) = f(\underline{\underline{x}}^{2K-1}, \underline{\underline{p}}^{2K-1}) \gtrsim f(\underline{\underline{x}}^{2K-1}, \underline{\underline{p}}^K)) \\
&\quad \vdots \\
&\gtrsim f(\underline{\underline{x}}^{2K-1}, \underline{p}^K) \gtrsim f(\underline{\underline{x}}^{2K-1}, \underline{p}^K) = \underline{\underline{x}}^{2K}.
\end{aligned}$$

Proceeding in the same fashion (i.e., in blocks of K elements at a time) we can establish that $\underline{\underline{x}}^n \succsim \underline{x}^n \forall n$. By Lemma 10 (a), this implies $\underline{\underline{p}}^n \geq \underline{p}^n \forall n$.

Finally, we show $\underline{p}^n \geq \underline{q}^n \forall n$. As before, we proceed in blocks of K elements at a time. Note that $\underline{q}^0, \dots, \underline{q}^{K-1} = \underline{p}^0$. Furthermore $\underline{x}^1 \succsim \underline{x}^0$ necessarily. Using Lemma 10 (b)–(c), we have

$$\begin{aligned}\underline{x}^2 &= f(\underline{x}^1, \underline{q}^1) \succsim f(\underline{x}^0, \underline{q}^1) = f(\underline{x}^0, \underline{p}^0) = \underline{x}^1, \\ &\vdots \\ \underline{x}^K &= f(\underline{x}^{K-1}, \underline{q}^{K-1}) \succsim f(\underline{x}^{K-2}, \underline{q}^{K-1}) = f(\underline{x}^{K-2}, \underline{p}^0) = \underline{x}^{K-1}.\end{aligned}$$

This implies that $\underline{x}^{K-1} \succsim \dots \succsim \underline{x}^0$, and hence $\underline{p}^{K-1} \geq \dots \geq \underline{p}^0 = \underline{q}^0, \dots, \underline{q}^{K-1}$. Next, note that $\underline{q}^K, \dots, \underline{q}^{2K} = \underline{p}^K$. Furthermore, by (22) and Step 3 we have $\underline{p}^K = A(\underline{p}^0) > \underline{p}^0$, and Lemma 10 (b)–(c) implies

$$\underline{x}^{K+1} = f(\underline{x}^K, \underline{q}^K) \succsim f(\underline{x}^{K-1}, \underline{q}^K) \succsim f(\underline{x}^{K-1}, \underline{p}^0) = \underline{x}^K.$$

Using Lemma 10 (b)–(c) again, we then have

$$\begin{aligned}\underline{x}^{K+2} &= f(\underline{x}^{K+1}, \underline{q}^{K+1}) \succsim f(\underline{x}^K, \underline{q}^{K+1}) = f(\underline{x}^K, \underline{p}^K) = \underline{x}^{K+1}, \\ &\vdots \\ \underline{x}^{2K} &= f(\underline{x}^{2K-1}, \underline{q}^{2K-1}) \succsim f(\underline{x}^{2K-2}, \underline{q}^{2K-1}) = f(\underline{x}^{K-2}, \underline{p}^K) = \underline{x}^{2K-1}.\end{aligned}$$

This implies that $\underline{x}^{2K-1} \succsim \dots \succsim \underline{x}^K$, and hence $\underline{p}^{2K-1} \geq \dots \geq \underline{p}^K = \underline{q}^K, \dots, \underline{q}^{2K-1}$. Proceeding in the same fashion, we can establish that $\underline{p}^n \succsim \underline{q}^n \forall n$.

Step 5: Construction of $\bar{q}^n \rightarrow p^*$ such that $p^n \leq \bar{q}^n$ for all n

This step is analogous to Steps 2–4. In Step 2, the sequences $\underline{x}^n, \underline{p}^n, \underline{q}^n$ are replaced with $\bar{x}^n, \bar{p}^n, \bar{q}^n$, where $\bar{x}^0 = (0, \dots, 0, 1)$, $\bar{x}^{n+1} = f(\bar{x}^n, \bar{q}^n)$, $\bar{p}^n = p(\bar{x}^n)$,

$$\begin{aligned}\bar{q}^0, \dots, \bar{q}^{K-1} &= \bar{p}^0, \\ \bar{q}^K, \dots, \bar{q}^{2K-1} &= \bar{p}^K, \\ \bar{q}^{2K}, \dots, \bar{q}^{3K-1} &= \bar{p}^{2K},\end{aligned}$$

and so on. Because we can write $\bar{p}^{\ell K} = A(p^{(\ell-1)K})$, where A was defined via the right-hand side of (22), we can apply the previous Step 3 to establish that $\bar{q}^n \rightarrow p^*$. In Step 4, the sequences $\underline{\underline{x}}^n$ and $\underline{\underline{p}}^n$ are replaced with $\bar{\bar{x}}^n$ and $\bar{\bar{p}}^n$, where $\bar{\bar{x}}^0 = (0, \dots, 0, 1)$, $\bar{\bar{x}}^{n+1} = T(\bar{\bar{x}}^n)$ and $\bar{\bar{p}}^n = p(\bar{\bar{x}}^n)$. One can then show that $p^n \leq \bar{\bar{p}}^n \leq \bar{p}^n \leq \bar{q}^n$ for all n .

Step 6: $x^0 \succsim (\precsim) x^*$ implies $p^n \rightarrow p^*$ from above (below)

Suppose $x^0 \succsim x^*$. Using Lemma 10 (a)–(c), we have

$$\begin{aligned} x^1 &= T(x^0) = f(x^0, p(x^0)) \succsim f(x^0, p(x^*)) = f(x^0, p^*) \succsim f(x^*, p^*) = T(x^*) = x^*, \\ x^2 &= T(x^1) = f(x^1, p(x^1)) \succsim f(x^1, p(x^*)) = f(x^0, p^*) \succsim f(x^*, p^*) = T(x^*) = x^*, \end{aligned}$$

and so on. It follows that $x^n \succsim x^* \forall n$. By Lemma 10 (a) this implies $p^n \geq p^* \forall n$. The argument when $x^0 \precsim x^*$ is analogous. \square

Proof of Proposition 3

Existence of an equilibrium follows from our Proposition 5. We need to show that the equilibrium is unique and consists of the use of the minimum storage strategy in every period.

Recall that the assumption $\lambda < \beta$ ensures that each household wants to store at least one unit in every period. Furthermore, if $\lambda > \beta^2/(1 + \beta)$ then no household wants to store more than one unit in every period,²³. In this case, the 1-storage rule is strictly dominant in every period, and the proof is complete. Thus, assume $\lambda \leq \beta^2/(1 + \beta)$.

In the following, we use the notation developed in Section 4.1. Fix period $t > 0$ and consider *any* sequence of states x^t, x^{t+1}, \dots (associated with p^t, p^{t+1}, \dots). Suppose that

$$D_2(\mathbf{p}^t) > 0 \Leftrightarrow p^t > \delta \equiv 1 - \frac{\lambda}{\beta(1 - \lambda)} \in (0, 1)$$

(where $\delta \in (0, 1)$ follows from $0 < \lambda \leq \beta^2/(1 + \beta) < \beta/(1 + \beta)$). In Section 4.1 we showed that

- (i) $D_2(\mathbf{p}^t) \leq D_3(\mathbf{p}^t) \leq \dots$; and
- (ii) $V_{k-1}^t - V_k^t \geq D_k(\mathbf{p}^t) \forall k \geq 2$.

Therefore, $p^t > \delta$ implies $D_2(\mathbf{p}^t) > 0$, which implies $V_1^t > V_2^t > \dots > V_K^t$. This means that the 1-storage rule becomes strictly optimal in period $t - 1$. If every household applies the minimum storage rule in period $t - 1$, the demand for toilet paper in period $t - 1$ is $\theta^{t-1} \leq 1$. Since $m = 1$, this implies $p^{t-1} = 1$, which means that it is optimal for every household to apply the minimum storage rule in period $t - 2$; and so on. Thus, if $p^t > \delta$ for some t , then every household follows the minimum storage rule in every period $t' < t$.

It follows that, in any equilibrium in which some household does *not* follow the minimum storage rule in every period, there must exist t^* such that $p^t \leq \delta \forall t \geq t^*$.

²³The marginal cost of storing the second unit is $\lambda + \beta\lambda$ (as the unit would be stored for two periods) and the marginal benefit is β^2 (as the unit would be consumed after it was stored for two periods); thus, if $\lambda > \beta^2/(1 + \beta)$ it is not optimal to store more than one unit.

Suppose this is the case. Then, in any period $t \geq t^* + K$, the fraction of households who have experienced K or more in-store shortages in a row, and hence enter period t with no toilet paper in storage, is

$$(1 - p^{t-1})(1 - p^{t-2}) \cdots (1 - p^{t-K}) \geq (1 - \delta)^K > 0.$$

Since households with a zero inventory cannot consume; starting in period $t^* + K$ at most a measure $1 - (1 - \delta)^K$ of the good is being consumed in each period. Since $p^t \leq \delta < 1$ $\forall t \geq t^*$, the store sells the entire supply in period t (i.e., the does not dispose of any excess supply at the end of period t ; if it did then $p^t = 1$). Because households cannot resell or dispose of the good, the unused amount must end up in storage. Because each household can store at most K units, at the latest in period $t^{**} = t^* + K + \lfloor K/(1 - \delta)^K \rfloor$ every household must have K units in storage. But this means that $\theta^{t^{**}} \leq 1$ which implies $p^{t^{**}} = 1$, a contradiction.

It follows that there cannot exist an equilibrium in which some household does not follow the minimum storage rule in every period. \square

Proof of Lemma 4

Fix $z \in \{2, \dots, K\}$. Let $\hat{p}^{t+s} \leq p^{t+s} \forall s = 0, \dots, K-2$. For $k = 0, \dots, z-2$ define

$$\alpha^k = \prod_{s=0}^k \frac{1 - \hat{p}^{t+s}}{1 - p^{t+s}}.$$

Note that $\alpha^{z-2} \geq \dots \geq \alpha^0 \geq 1$. Therefore,

$$\begin{aligned} \bar{\lambda}(\hat{\mathbf{p}}^t) &= \frac{\beta^{z-1} \prod_{s=0}^{z-2} (1 - \hat{p}^{t+s})}{1 + \sum_{k=1}^{z-1} \beta^k \prod_{s=0}^{k-1} (1 - \hat{p}^{t+s})} = \frac{\beta^{z-1} \alpha^{z-2} \prod_{s=0}^{z-2} (1 - p^{t+s})}{1 + \sum_{k=1}^{z-1} \beta^k \alpha^{k-1} \prod_{s=0}^{k-1} (1 - p^{t+s})}, \\ &\geq \frac{\beta^{z-1} \alpha^{z-2} \prod_{s=0}^{z-2} (1 - p^{t+s})}{\alpha^{z-2} + \sum_{k=1}^{K-1} \beta^k \alpha^{z-2} \prod_{s=0}^{k-1} (1 - p^{t+s})} = \frac{\beta^{z-1} \prod_{s=0}^{z-2} (1 - p^{t+s})}{1 + \sum_{k=1}^{z-1} \beta^k \prod_{s=0}^{k-1} (1 - p^{t+s})} = \bar{\lambda}(\mathbf{p}^t). \quad \square \end{aligned}$$

Proof of Proposition 5

For $z < K$, let $L^*(z)$ be equal to the left-hand side of (16), and set $L^*(K) = \beta$. Similarly, for $z > 1$ let $U^*(z)$ be equal to the right-hand side of (16), and set $U^*(1) = \beta$. A stationary z -storage equilibrium exists if $\lambda \in [L^*(z), U^*(z)]$ for some $z \in \{1, \dots, K\}$.

The argument is in three parts:

- (i) $L^*(z) \leq U^*(z) \forall z = 1, \dots, K$. Therefore, the interval $[L^*(z), U^*(z)]$ is well-defined and non-empty for each z .
- (ii) $L^*(z+1) \leq L^*(z)$ and $U^*(z+1) \leq U^*(z) \forall z = 1, \dots, K-1$. Therefore, the interval $[L^*(z+1), U^*(z+1)]$ is “below” the interval $[L^*(z), U^*(z)]$.
- (iii) $L^*(z) \leq U^*(z+1) \forall z = 1, \dots, K-1$. Therefore, $\cup_{z=1, \dots, K} [L^*(z), U^*(z)] = [0, \beta]$, which implies $\lambda \in [L^*(z), U^*(z)]$ for at least one $z \in \{1, \dots, K\}$.

Part (i) is immediate from (16), noting that $\gamma(z) \in (0, 1)$. Part (ii) follows from Lemma 4, noting that $L^*(z) = \bar{\lambda}_{z+1}((1-m)^{1/z}, (1-m)^{1/z}, \dots)$ and $U^*(z) = \bar{\lambda}_z((1-m)^{1/z}, (1-m)^{1/z}, \dots)$, and $(1-m)^{1/z}$ increases in z . For part (iii), note that

$$\begin{aligned} L^*(z) \leq U^*(z+1) &\Leftrightarrow \frac{\gamma(z)^z(1-\gamma(z))}{1-\gamma(z)^{z+1}} \leq \frac{\gamma(z+1)^z(1-\gamma(z+1))}{1-\gamma(z+1)^{z+1}} \\ &\Leftrightarrow \frac{1}{\gamma(z+1)^z} \frac{1-\gamma(z+1)^{z+1}}{1-\gamma(z+1)} \leq \frac{1}{\gamma(z)^z} \frac{1-\gamma(z)^{z+1}}{1-\gamma(z)} \\ &\Leftrightarrow \frac{1}{\gamma(z+1)^z} \sum_{k=0}^z \gamma(z+1)^k \leq \frac{1}{\gamma(z)^z} \sum_{k=0}^z \gamma(z)^k \\ &\Leftrightarrow \sum_{k=0}^z \gamma(z+1)^{-k} \leq \sum_{k=0}^z \gamma(z)^{-k}, \end{aligned}$$

which is true since $\gamma(z+1) \leq \gamma(z) \leq 1$. □

Proof of Proposition 6

By Proposition 7 (which does not depend on this result), we only need to verify the conditions for which the minimum and maximum storage equilibria exist at the same time. Using (16), the minimum storage equilibrium exists if

$$\lambda \geq \frac{\gamma(1)(1-\gamma(1))}{1-\gamma(1)^2} = \frac{\beta(1-m)(1-\beta(1-m))}{1-\beta^2(1-m)^2} > 0,$$

and the maximum storage equilibrium exists if

$$\lambda \leq \frac{\gamma(K)^{K-1}(1-\gamma(K))}{1-\gamma(K)^K} = \frac{\beta^{K-1}(1-m)^{(K-1)/K}(1-\beta(1-m)^{1/K})}{1-\beta^K(1-m)} < 1.$$

Thus, both equilibria coexist as long as

$$\frac{\beta(1-m)(1-\beta(1-m))}{1-\beta^2(1-m)^2} \leq \lambda \leq \frac{\beta^{K-1}(1-m)^{(K-1)/K}(1-\beta(1-m)^{1/K})}{1-\beta^K(1-m)}. \quad (23)$$

We need to show that, if m is sufficiently large (but smaller than one), the left-hand side of (23) is strictly smaller than the right-hand side. After rearranging, this condition becomes

$$\frac{1-\beta^K(1-m)}{1-\beta^2(1-m)^2} < \frac{\beta^{K-2}(1-\beta(1-m)^{1/K})}{(1-m)^{1/K}(1-\beta(1-m))}. \quad (24)$$

As $m \rightarrow 1$ from below, the left-hand side in (24) converges to 1 and the right-hand side converges to $+\infty$. Thus, for fixed K and b , there exists $\bar{m} < 1$ such that (24) holds for all $\bar{m} < m < 1$. (The open interval Λ is then defined by replacing the weak inequalities in (23) with strict inequalities.) \square

Proof of Proposition 7

The result follows from property (ii) in the proof of Proposition 5. Suppose that a stationary z -storage equilibrium exists and that a z' stationary storage equilibrium exists, with $z < z'$. Let z'' be an integer such that $z < z'' < z'$. Because a stationary z -storage equilibrium exists, $L^*(z) \leq \lambda \leq U^*(z)$; similarly, because a stationary z' -storage equilibrium exists, $L^*(z') \leq \lambda \leq U^*(z')$. Since $z'' > z$, by property (ii) we have $L^*(z'') \leq L^*(z) \leq \lambda$. Similarly, since $z'' < z'$, by property (ii) we have $U^*(z'') \geq U^*(z') \leq \lambda$. Therefore, $L^*(z'') \leq \lambda \leq U^*(z'')$, which means that a stationary z'' -storage equilibrium exists. \square

Proof of Lemma 8

Fix $z \in \{1, \dots, K\}$ and $x^0 = (0, 1, 0, \dots, 0)$, and let x^1 be defined as in the text. Note that, from (5)–(6),

$$p^0 = \frac{m}{z} \leq \frac{m}{z - (m/z)(z-1)} = \frac{m}{m/z + z(1-m/z)} = p^1.$$

We will show that $x^* \succsim x^1$; Proposition 2 then implies that $p^1 \leq p^2 \leq \dots \rightarrow p^* = 1 - (1-m)^{1/z}$, and the result follows.

For $k = 0, \dots, z-1$, we have

$$\sum_{s=0}^k x_s^* \leq \sum_{s=0}^{z-1} x_s^* = (1-m)^{1/z} \quad \text{and} \quad \sum_{s=0}^k x_s^1 = 1 - \frac{m}{z}.$$

Thus, we need to show that

$$(1-m)^{1/z} \leq 1 - \frac{m}{z}. \quad (25)$$

If $m = 0$, then (25) holds as an equality. Therefore, it is sufficient to show that

$$\frac{\partial}{\partial m} \left[(1-m)^{1/z} \right] = -(1-m)^{1/z-1} \leq -\frac{1}{z} = \frac{\partial}{\partial m} \left[1 - \frac{m}{z} \right]. \quad (26)$$

The left-hand side of (26) decreases in m , and the right-hand side is independent of m . Therefore, it is sufficient that (26) holds at $m = 0$. This is satisfied, as $-1 \leq -1/z$. \square

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