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Equilibrium Indeterminacy and Extreme Outcomes: A Fat Sunspot Ta(i)l(e)

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Equilibrium Indeterminacy and Extreme Outcomes: A Fat Sunspot Ta(i)l(e)*

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Abstract

Competing explanations for the fat-tailed empirical distribution of aggregate time series range from exogenous stochastic volatility, boundedly rational agents reflecting a lot of structural change or that exogenous structural shocks are themselves extreme. We build on this literature and show that sunspots in dynamic models can accumulate as linear recursions with multiplicative noise. Thus, using known results from the large deviations literature allows us to conclude that even small sunspot shocks can lead to large movements in endogenous variables. We apply these results to models that admit indeterminacies to investigate the empirical relevance of sunspots in accounting for observed fat-tails in output.

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1. Introduction

A main objective of linear macroeconometric modeling in the dynamic stochastic general equilibrium (DSGE) tradition is for model variables to replicate the statistical properties of data, with a focus usually on first and second moments. In both the rational expectations (RE) econometrics tradition, and its adaptive learning cousin, small exogenous Gaussian structural shocks cause model variables to cycle around a trend. Absent further assumptions on the distributions of shocks or other model characteristics, the fixed coefficients linear recursion that characterizes equilibrium of a given model imparts a Gaussian distribution for the model, which in turn should match that of data.

However, a recent literature has questioned the Gaussian nature of the statistical properties of data itself, e.g. Christiano (2007), Fagiolo et al. (2008) and Cúrdia et. al. (2014).[1] Moreover, in an important contribution, Ascari et al. (2015) provide numerical results showing that the two standard workhorse models - the Real Business Cycle (RBC) model and the medium-scale New Keynesian (NK) monetary framework - both lack an endogenous mechanism able to deliver non-Normality and fat-tailed behavior for growth-rate macroeconomic time-series distributions. The literature then leaves open the issue of what sort of models and/or assumptions would admit non-Normality of macroeconomic aggregates.

Here we argue that models exhibiting indeterminacies, and thus admitting (thin-tailed) sunspot shocks, would allow for a replication of fat-tailed behavior observed in macroeconomic time series. We demonstrate that this happens since indeterminate equilibrium models generically admit reduced form representations that can be written as linear recursions with multiplicative noise (LRMN), and thus the tools of large deviations theory can be brought to bear upon the problem of reconciling aggregate data characteristics with DSGE models *without assuming* the existence of exogenous large shocks *or* departing from the conventional RE paradigm.

[1] See Dave and Malik (2017) for a review.

As mentioned, previous work in applied business cycle theory has attempted to reconcile the higher moments of data with model counterparts. One avenue consists of fat-tailed specifications and/or stochastic volatility of innovations (e.g. Chib and Ramamurthy, 2014; Ascari et al., 2015), i.e. the assumption of an appropriate non-Normal distribution from which exogenous shocks are drawn, which, holding model representation constant as a fixed coefficient recursion, imparts non-Normal characteristics to data-effectively a "fat in-fat out" approach. Other approaches have explored the role of state dependent and exogenous parameter drifting specifications (e.g. Auerbach and Gorodnichenko, 2012; Cogley and Sargent, 2001, 2005), as well as the consequence of bounded rationality alternatives to RE in otherwise standard settings, e.g. variance adjusted adaptive learning (Dave and Feigenbaum, 2018) or endogenous parameter drifting under constant gain learning (Benhabib and Dave, 2014; Dave and Tsang 2014; Dave and Malik, 2017). While under RE, linear model dynamics are described by a fixed coefficient recursion, adaptive learning specifications allow for model dynamics to follow a LRMN, where shock accumulation over time delivers, under a number of regularity conditions, fat-tailed behavior for endogenous variables even with Normal innovations - effectively a "thin in-fat out" approach albeit having abandoned the RE construct. In the same spirit, a recent strand of behavioral macroeconomics literature has set out models based on the idea that cognitive limitations may force economic agents to exploit simple heuristic rules of behavior, that in turn allow animals spirits to become an engine for business cycle dynamics and extreme events (e.g. De Grauwe and Ji, 2019).

Our contribution is to put forward a novel theoretical framework that does not relinquish the RE paradigm while explicitly appealing to equilibrium indeterminacy, which is known to obtain under reasonable parameterizations of DGSE models (e.g. Lubik and Schorfheide, 2004). By creating room for self-fulfilling (rational) expectations to arise, we formally show that indeterminacy allows a given DSGE model to endogenously reproduce fat-tailed behavior for endogenous variables from standard i.i.d. Gaussian shocks. It is well known that, in the presence of an infinite number of admissible equilibrium paths, rational forecast errors made by forward-looking agents are not uniquely pinned down by the economy's fundamentals (e.g. Benhabib and Farmer, 1999; Sims, 2001; Lubik and Schorfheide, 2003; Farmer et al., 2015). The main intuition behind Lubik and Schorfheide (2003)'s approach to indexing indeterminate solutions is that endogenous forecast errors - which need to be serially uncorrelated per RE requirements - react in a predictable way to current (structural and sunspot) shocks only, disregarding past observed shocks as a source of expectation revisions over time. As a result, while possibly driven by non-structural (sunspot) noise and allowing for ambiguous responses of the model's dynamics to structural shocks, such an approach precludes by construction the possibility of random variation in the equilibrium reduced form's coefficients and/or shock volatility. We show how forecast revisions in RE environments can be conditioned on current and past observables via randomly varying weights, that need not be related to fundamentals (sunspots). When such a revision process coordinates into an RE stable trajectory, small i.i.d. sunspot shocks can produce fat-tailed distributions for the endogenous variables, which can thereby take on extreme values with a higher probability than under a Normal distribution.

Our exploration of fat tails within the realm of DSGE models is inspired by Ascari et al. (2019), who provide a martingale-based equilibrium representation of multivariate DSGE models to explore the empirical plausibility of temporarily unstable paths, thus focusing only on the determinate region of the parameter space, for which a (mild) relaxation of the RE hypothesis is over-imposed in order to meet the asymptotic stability requirement. The two approaches in fact share the view that RE solutions under indeterminacy can be constructed by randomizing on the weights that economic agents attach to current and past observables when building their expectations; the way such expectations are formed then affect the model's dynamics. This is akin to the seminal Muth (1961) contribution on RE as weighted average of past observations, and resembles a generalized adaptive belief formation process, yet fully coherent with the RE requirement about absence of correlated forecast errors and optimality according to the minimum variance criterion (e.g. Sorge, 2013). However, the two approaches produce different properties for the ensuing equilibrium law of motion, in terms of (i) the statistical dependence between the multiplicative and the additive noise components, (ii) the way sunspot weights enter the endogenous process of forecast revision, and most importantly (iii) the degree of adherence to the building blocks of the RE theoretical construct, for ours fully complies with the latter.

A sketch of our analysis facilitates sailing over technical details. Let y_t be any endogenous variable with equilibrium dynamics governed by

$$y_t = \frac{1}{\theta} E_t(y_{t+1}) + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma_\epsilon^2), \tag{1}$$

where ϵ_t is a structural shock (defined on the same probability space as y_t), $E_t(\cdot)$ denotes the conditional expectation operator and $\theta \in \Re$. E.g., y_t can be the inflation rate in a Fisherian model of inflation augmented with an interest rate rule (e.g. Cochrane, 2011). Any solution to (1) satisfies

$$y_t = \theta y_{t-1} - \theta \epsilon_{t-1} + \eta_t, \tag{2}$$

where $\eta_t = y_t - E_{t-1}(y_t)$ is the RE forecast error.

When $\theta \leq 1$ (indeterminacy), the RE forecast error is not constrained by stability requirements, hence any covariance-stationary martingale difference sequence (MDS) η_t will deliver a non-explosive RE solution of the form (2). Lubik and Schorfheide (2003) express η_t as a linear (time invariant) combination of the model's current structural disturbance and a reduced form sunspot shock ξ_t^*, i.e. $\eta_t = \tilde{m}\epsilon_t + \xi_t^*$, where $\tilde{m}$ is an arbitrary constant unrelated to θ. The full set of non-explosive solutions under indeterminacy is thus described by the ARMA(1,1) process

$$y_t = \theta y_{t-1} + \tilde{m}\epsilon_t - \theta \epsilon_{t-1} + \xi_t^*. \tag{3}$$

It is possible to show that the very same model (1) generically admits randomly varying solutions in the indeterminacy region which display the LRMN form

$$y_t = \alpha_t y_{t-1} + \beta_t \tag{4}$$

where (α_t, β_t) satisfy $E_{t-1}(\alpha_t) = \theta$, $E_{t-1}(\beta_t) = \tilde{m}$, and emerge when η_t follows

$$\eta_t = \xi_{1,t} g(\xi_{2,t-i}, \epsilon_{t-j}, \xi_{1,t-k}; \theta) + (\tilde{m} + \xi_{2,t})\epsilon_t, \ \ i, j, k \geq 1,$$

where $g(\cdot) \in H_{\epsilon,\xi}(t)$ is a deterministic transformation of past shocks weighted by the sunspot shock $\xi_{1,t}$, which is orthogonal to the t-dated information set. Thus $E_{t-1}(\eta_t) = 0$ for all t as the RE construct requires. Most importantly, the multiplicative component α_t in the LRMN (4) is such that $E(\alpha_t) < 1$ (thus the process for y_t contracts on average) and satisfies $Pr(\alpha_t > 1) > 0$ when the distribution of the sunspot shock $\xi_{1,t}$, though possibly thin-tailed, displays some mass to the right of $(1 - \theta) \in (0, 1)$. This property makes the recurrence (4) expand with positive probability, i.e. it allows small i.i.d. shocks to accumulate over time so as to lead to high-frequency large movements in the endogenous variable.[2]

In multidimensional settings, constructing solutions in LRMN form requires decoupling the (linearized) equilibrium conditions into their stable and unstable components, and identifying the dimension of indeterminacy by imposing restrictions on the endogenous forecast errors so as to satisfy the asymptotic stability requirement for the unstable block (Sims, 2001). RE forecast errors can then be taken to depend linearly on both lagged values of the observed endogenous variables and structural shocks, where the time-varying, random loading matrices are selected in order for the t-dated RE forecast errors (i) to be measurable with respect to the current information set, as well as orthogonal to information available at period $t-1$ (martingale difference property), and (ii) to fulfill the asymptotic stability

[2] Dave and Sorge (2020) presents numerical results for the simple model (1) establishing that, as the model's parameterization enters the indeterminacy territory, the ensuing LRMN equilibrium representation delivers a remarkably lower Pareto tail index relative to its determinate counterpart, thereby suggesting fat tails for the model-implied distribution.

requirement providing for existence of stable solutions.

Here, we offer a general algorithm to compute LRMN solutions for indeterminate equilibrium DSGE frameworks. We then investigate the ability of LRMN representations to replicate observed patterns of aggregate data, with a particular focus on output in our estimation exercise. We show that both NK and RBC models under indeterminacy do allow fat-tailed behavior stemming from sunspot-driven forecast revisions, in contrast to the results in Ascari et al. (2015) that those workhorse models fail to replicate such statistical regularities due to weak propagation mechanisms. We offer insights into the role of sunspot shocks in amplifying the propagation mechanisms embedded in standard DSGE models, and contribute to the quantitative implications of equilibrium indeterminacy for business cycle dynamics and the realizations of extreme macroeconomic outcomes.

The paper is organized as follows. In Section 2 we formally show that multivariate models that admit indeterminacies, and thus the occurrence of sunspot noise, can be written as LRMNs that can qualify as generalized Kesten processes (e.g. Kesten, 1973) and thus allow application of standard results on proportional random growth models in order to characterize properties of the upper tail of the ensuing time-invariant distribution (e.g. Gabaix, 2009). In Section 3 we illustrate our solution algorithm by means of a standard small-scale NK model, that admits a closed-form solution under either equilibrium regime. While allowing a direct comparison between the standard way of computing sunspot solutions (Lubik and Schorfheide, 2003) and our approach, this exercise clearly shows that the LRMN equilibrium representation does indeed deliver lower tail indices (characteristic of non-Normality as argued in Section 2) in indeterminate regions of the parameter space, irrespective of how sunspot solutions *à la* Lubik and Schorfheide (2003) are selected. In Section 4 we provide simulated moments estimates in an RBC framework, which suggests that in matching data tail index estimates for aggregate output the data may prefer model indeterminacy over determinacy. Section 5 offers concluding remarks.

2. General Framework

We consider Sims (2001)'s canonical form for linear rational expectations (LRE) models

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi\eta_t,\ t \geq 1, \tag{5}$$

where y_t is an n-dimensional vector of endogenous variables, ε_t is an l-dimensional vector of exogenous (structural) shocks, satisfying $E_t(\varepsilon_t) = \varepsilon_t$ and $E_{t-1}(\varepsilon_t) = 0$, and the vector $\eta_t = y_t - E_{t-1}(y_t)$ collects $k \leq n$ non-zero endogenous forecast errors. Here, $E_t(\cdot)$ denotes the expectation operator conditional on the information set $H_{\varepsilon,\zeta}(t)$, i.e. the closure of the span of present and past components of ε_t and ζ_t, the p-dimensional martingale difference sequence of sunspot shocks, which are taken to be orthogonal to the structural ones at all lags and leads (Lubik and Schorfheide, 2003). All random variables are defined with respect to a common probability space. The matrices Γ_0, Γ_1, Ψ and Π have as elements a models' parameters and are of dimension $(n \times n)$, $(n \times n)$, $(n \times l)$ and $(n \times k)$, respectively.

An RE equilibrium is a *stable and causal solution* to the LRE model (5), i.e. any square integrable process (y_t) included in $H_{\varepsilon,\zeta}(t)$ which, for given initial conditions $y_0 = \bar{y}$, satisfies the structural relationship (5) for all $t \geq 1$ as well as the asymptotic growth restriction

$$E_t(\xi^{-h} y_{t+h}) \overset{h\to\infty}{\to} 0,\ \xi \geq 1. \tag{6}$$

When such a solution is non-unique, then the LRE model is said to be *indeterminate*.

The matrix Γ_0 can be non-singular and so the generalized Schur (QZ) decomposition (e.g. Moler and Stewart, 1973) is exploited to decouple the system into its stable and unstable components, see Sims (2001). Formally, one has

$$\begin{aligned} Q'\Lambda Z' &= \Gamma_0, \\ Q'\Omega Z' &= \Gamma_1, \end{aligned}$$

where Q and Z are orthogonal matrices, and Λ and Ω are upper-triangular matrices.[3] Since Q is non-singular, the system (5) can be equivalently rewritten as

$$\Lambda w_t = \Omega w_{t-1} + Q\left(\Psi\epsilon_t + \Pi\eta_t\right), \tag{7}$$

where $w_t = Z'y_t$. Rearranging the system so that the $n_2 \leq n$ unstable generalized eigenvalues $\{\omega_{ii}/\lambda_{ii}\}$ - i.e. those larger than ξ - correspond to the lower right blocks of Λ and Ω yields

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix}\begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix}\begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}\left(\Psi\epsilon_t + \Pi\eta_t\right), \tag{8}$$

where Q_1 and Q_2 are of dimension $(n - n_2 \times n)$ and $(n_2 \times n)$, respectively.

As shown in Sims (2001), provided $w_{2,0} = 0$, a non-explosive solution of a model in (5) exists if and only the column space of $Q_2\Psi$ is contained in that of $Q_2\Pi$, i.e.

$$span\left(Q_2\Psi\right) \subseteq span\left(Q_2\Pi\right), \tag{9}$$

which requires $n_2 \leq k$. For the purposes of our analysis, let the existence condition (9) be fulfilled. Then LRMN solutions can be constructed by assuming that the RE forecast errors depend linearly on both lags of the endogenous variables y_{t-1} and on the structural shocks ε_t as follows We let

$$\eta_t = A_{1,\zeta_t}\varepsilon_t + A_{2,\zeta_t}y_{t-1}, \tag{10}$$

where the time-varying (random) matrices $\left(A_{1,\zeta_t}, A_{2,\zeta_t}\right)$ are selected in order for η_t to fulfill (i) the RE orthogonality requirement $E_{t-1}(\eta_t) = 0$ for all t, and (ii) the actual stability restrictions imposed by the existence condition (9).

[3] Thus, $QQ' = I_n = ZZ'$, where the $'$ symbol denotes transposition. The QZ-decomposition always exists, and is unique up to the ordering of the generalized eigenvalues. The matrices Q, Z, Λ and Ω can always be chosen so that the absolute values of the generalized eigenvalues are displayed in descending order. If $\lambda_{ii} = 0$, then the corresponding generalized eigenvalue is infinity.

As for the former, by causality of any RE equilibrium process (y_t), it is sufficient that the entries of the matrices A_{1,ζ_t} and A_{2,ζ_t} be taken to be t-dated i.i.d. sunspot variables which are orthogonal to any other variable in the information set $H_{\varepsilon,\zeta}(t-1)$. In particular, we can simply let

$$A_{1,\zeta_t} = \underbrace{A_1}_{k\times l}\underbrace{\zeta_{1,t}}_{l\times l},\ A_{2,\zeta_t} = \underbrace{A_2}_{k\times n}\underbrace{\zeta_{2,t}}_{n\times n} \tag{11}$$

where - letting $p = n + l$ - $\zeta_{1,t}$ and $\zeta_{2,t}$ are diagonal random matrices whose entries are distinct elements of the sunspot vector ζ_t, and (A_1, A_2) are conformable matrices. This is without loss of generality, since the number p of sunspot variables ζ_t is arbitrary, provided they are all $H(t)$-measurable.

As for the stability requirement, following Lubik and Schorfheide (2003) it is crucial to take into account potential rank deficiencies in the $(n_2 \times k)$ matrix $Q_2\Pi$, which imposes only $r \leq n_2$ restrictions on the RE forecast errors η_t that have to be fulfilled for the unstable part of the LRE system (8) to admit an asymptotically stable solution. Specifically, consider the singular value decomposition (SVD)

$$Q_2\Pi = UDV' = \underbrace{U_1}_{n_2\times r}\underbrace{D_{11}}_{r\times r}\underbrace{V_1'}_{r\times k} \tag{12}$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. Notice that existence of (at least one) stable solution implies that the span condition (9) be equivalent to existence of a (real) $k \times l$ matrix Υ such that $Q_2\Psi = Q_2\Pi\Upsilon$. Thus the stability requirement can be rewritten as

$$U_1 D_{11} V_1' \left(\Upsilon\varepsilon_t + \eta_t\right) = 0 \tag{13}$$

or, given (10) and (11) and labeling as $\tilde{\zeta}_t$ the $(l \times 1)$ random vector $\zeta_{1,t}\varepsilon_t$, one has

$$U_1 D_{11} V_1' \Upsilon\varepsilon_t + U_1 D_{11} V_1' A_1 \tilde{\zeta}_t + U_1 D_{11} V_1' A_2 \zeta_{2,t} y_{t-1} = 0 \tag{14}$$

which must hold for any ε_t and any arbitrary selection of (conditionally mean zero) sunspot shocks $(\zeta_{1,t}, \zeta_{2,t})$; this in turn requires that both A_1 and A_2 belong to the right null space of V_1, which is spanned by the columns of V_2. Since U is orthonormal, if $Q_2\Pi$ is not of full (row) rank, then the (internal) dimension of indeterminacy is $k-r$ and the full set of solutions for the forecast errors (10) can be written as

$$\eta_t = -V_1 D_{11}^{-1} U_1' Q_2 \Psi \varepsilon_t + V_2 \underbrace{M_1}_{(k-r)\times l} \tilde{\zeta}_t + V_2 \underbrace{M_2}_{(k-r)\times n} \zeta_{2,t} y_{t-1} \tag{15}$$

where M_1 and M_2 are arbitrary matrices, whose entries do not depend on the structural parameters of the LRE model (5).

Notice that any stable and causal solution for $w_{2,t}$ in the unstable block in (8) is identically zero by force of the existence requirement. We can therefore obtain a causal and stable solution for $w_{1,t}$ by exploiting the upper part of (8) and the equilibrium process for the RE forecast errors (15) as follows

$$w_{1,t} = \Lambda_{11}^{-1}\Omega_{11} w_{1,t-1} + \Lambda_{11}^{-1} Q_1 \Psi \varepsilon_t + \Lambda_{11}^{-1} Q_1 \Pi \left(-V_1 D_{11}^{-1} U_1' Q_2 \Psi \varepsilon_t + V_2 M_1 \tilde{\zeta}_t + V_2 M_2 \zeta_{2,t} Z w_{t-1} \right) \tag{16}$$

where $w_{t-1} = [w_{1,t-1}, 0]$ because of the stability requirement. Thus one has

$$\begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} = \begin{pmatrix} \Lambda_{11}^{-1} \left(\Omega_{11} + Q_1 \Pi \left[V_2 M_2 \zeta_{2,t} Z \right]_{n-n_2} \right) \\ 0 \end{pmatrix} w_{1,t-1} + \begin{pmatrix} \Lambda_{11}^{-1} Q_1 \\ 0 \end{pmatrix} \left[\Psi - \Pi \left(V_1 D_{11}^{-1} U_1' Q_2 \Psi - V_2 M_1 \zeta_{1,t} \right) \right] \varepsilon_t \tag{17}$$

where $\left[V_2 M_2 \zeta_{2,t} Z \right]_{n-n_2}$ is the selection of the first $n-n_2$ columns of the $k \times n$ matrix $V_2 M_2 \zeta_{2,t} Z$. This system can then easily be solved for the original variables by using $y_t = Z w_t = Z_1 w_{1,t}$, where Z_1 is the sub-matrix collecting the first $n-n_2$ columns of Z, so that the process for y_t will inherit the same properties as that for $w_{1,t}$.

Given that with indeterminacy and attendant sunspot shocks, LRE models can be written as LRMNs, our next step is to investigate their relationship to the processes studies in Kesten (1973). The LRMN (16) is in the form of a random difference equation of the type

$$X_{t+1} = (\rho + \alpha_{t+1})X_t + \beta_t \tag{18}$$

where $X_t := w_{1,t}$ and

$$\rho := \Lambda_{11}^{-1}\Omega_{11},$$

$$\alpha_{t+1} := \Lambda_{11}^{-1}Q_1\Pi\left[V_2M_2\zeta_{2,t+1}Z\right]_{n-n_2}$$

and

$$\beta_{t+1} := \Lambda_{11}^{-1}Q_1\left[\Psi - \Pi\left(V_1D_{11}^{-1}U_1'Q_2\Psi - V_2M_1\zeta_{1,t}\right)\right]\varepsilon_t.$$

It is apparent that, provided sunspot shocks are mean zero i.i.d. and also independent of structural shocks at all lags and leads, then the sequences of random matrices (α_t) and vectors (β_t) are themselves i.i.d., with $E[\rho + \alpha_t] = \rho$ being a stable matrix since it consists of stable generalized eigenvalues. More generally, there exist families of distributions for the arbitrary sunspot shocks $(\zeta_{1,t}, \zeta_{2,t})$ under which the random difference equation complies with a set of restrictions that characterize Kesten processes (Kesten, 1973).

To see this, let $|v|$ denote any norm of a vector $v \in \mathbb{R}^d$, where the dependence on the dimension of v is suppressed for notational convenience. For any square $d \times d$ matrix M the norm $||M||$ is defined as

$$||M|| = \sup_{v\in R^d, |v|=1} |Mv|$$

If $E\left[\log^+ ||\rho + \alpha_0||\right] < \infty$ (where $x^* = \max\{0, x\}$) then the Lyapunov exponent $\lambda = \lim_{t\to\infty} \frac{1}{t}\log||(\rho + \alpha_0)\cdot(\rho + \alpha_1)\cdots(\rho + \alpha_t)||$ exists and it is constant almost surely (see Furstenberg and Kesten, 1960). Notice this condition follows from ρ being stable and α_t being a mean zero i.i.d. sequence, by Jensen's inequality. If $\lambda < 0$ and $E\left[\log^+ |\beta_0|\right] < \infty$, then the process (18) converges in distribution to the random vector

$$X = \sum_{t=0}^{\infty} (\rho + \alpha_0) \cdots (\rho + \alpha_{t-1}) \cdot \beta_t$$

whose law μ is the unique stationary measure of the process $\{X_t\}$. Kesten (1973) establishes that, if a number of mild conditions hold, the main being that first

$$\lim_{t \to \infty} \frac{1}{t} E\left[\| \left((\rho + \alpha_0) \cdot (\rho + \alpha_1) \cdots (\rho + \alpha_t) \right)^{\kappa} \| \right] = 1, \tag{19}$$

and that $E\left[|\beta_t|^{\kappa}\right] < \infty$ for some $\kappa > 0$, then the measure μ is regularly varying at infinity with index κ. That is, there exists a positive constant C such that $x^{\kappa} \cdot \mu(X > x) \to C$ as $x \to \infty$, i.e. the upper tail of the stationary distribution for X is asymptotically equivalent to a Pareto law

$$\mu(X > x) \sim C x^{-\kappa}.$$

3. A New Keynesian Model: Simulations

The previous section formalized that models with indeterminacies and attendant sunspot shocks can take the form of LRMNs. Thus, as a function of those shocks along with innovations to usual structural processes, model variables can, in theory, exhibit fat tails, but do they reliably do so? In this section we provide simulation results based on an analytical New Keynesian model (see Lubik and Schorfheide (2004)) to demonstrate the value of the LRMN approach. The model we adopt is described by three equations

$$y_t = E_t(y_{t+1}) - \tau(r_t - E_t(\pi_{t+1})) + \varepsilon_{1t}, \tag{20}$$

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa(y_t - \varepsilon_{2t}), \tag{21}$$

$$r_t = \psi \pi_t + \varepsilon_{3t}, \tag{22}$$

in which the first is a dynamic IS curve, the second a Phillips curve and the third a simple inflation rule. The parameter ψ governs indeterminacy ($\psi < 1$) and determinacy ($\psi > 1$), and structural innovations (ε's) as well as sunspot shocks are all assumed to be Normally distributed. Appendix A describes how we derive both the conventional solution in the form of Lubik and Schorfheide (2003) and our LRMN representation, both as a function of the (in)determinate regions.

To conduct simulations we first drew (and subsequently held fixed) $M = 500$ standard Normal shocks of length $T = 1100$ (with the first 100 discarded) for all structural innovations and sunspot shocks; the parameters $\{\beta, \tau, \kappa\}$ were held fixed at $\{0.99, 1, 0.5\}$. (Since these were standard Normal innovations, we were able to vary their standard deviations as part of the simulations.) Our next step given a shock series ($m \in M$) was to recursively construct the LRMN representation of the model above, and given the recursion, estimate the tail indices of simulated output (y_t), inflation (π_t) and interest rates (r_t) series using the maximum likelihood methods of Clauset et al. (2009). Third, indices were estimated for the width of a simulation M for each value of the simulation parameter so that averages could be plotted as the main simulation parameter ψ varied (see Dave and Sorge (2020) for a univariate analog of the procedure).

Our first set of simulations are presented in Figure 1 below in which all standard deviations were unitary (i.e. all structural innovations and sunspot shocks were standard Normal).

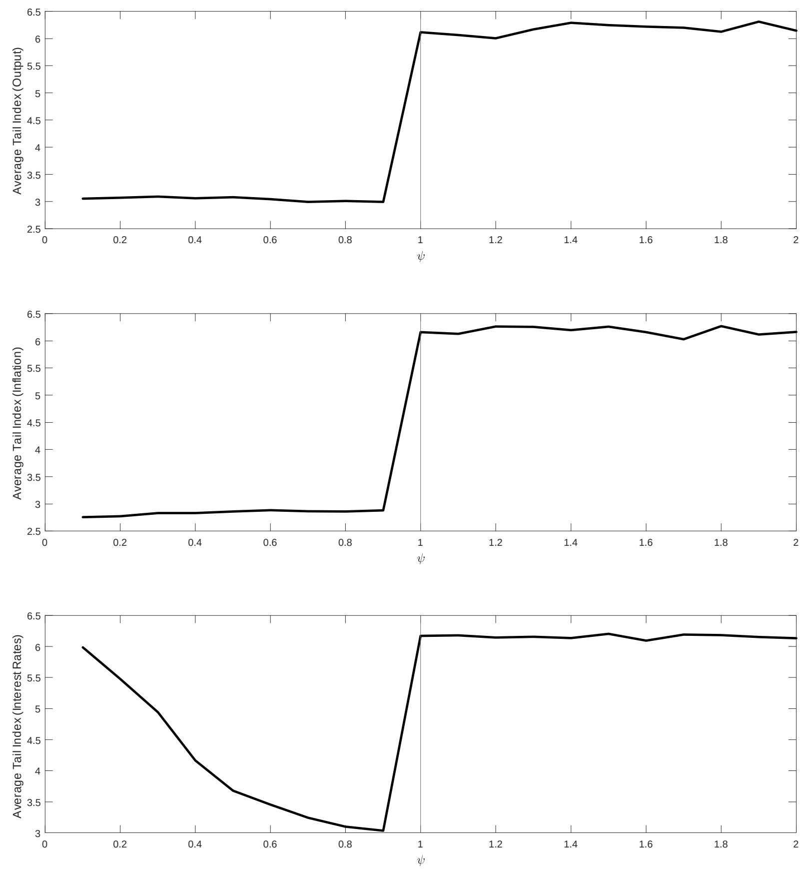

Figure 1. All Innovations as Standard Normal.

The results plotted in Figure 1 confirm that in the indeterminate region ($\psi < 1$) the average estimated tail index falls to levels closer to empirical counterparts (as reported for instance in Dave and Malik (2017)). In the determinate region ($\psi > 1$) the average estimated tail index rises. Since the results plotted in Figure 1 held all shocks to be standard Normal, we also reduced the standard deviation of shocks to 0.5 and 0.005 with plots of those respective results in Figures 2 and 3 below.

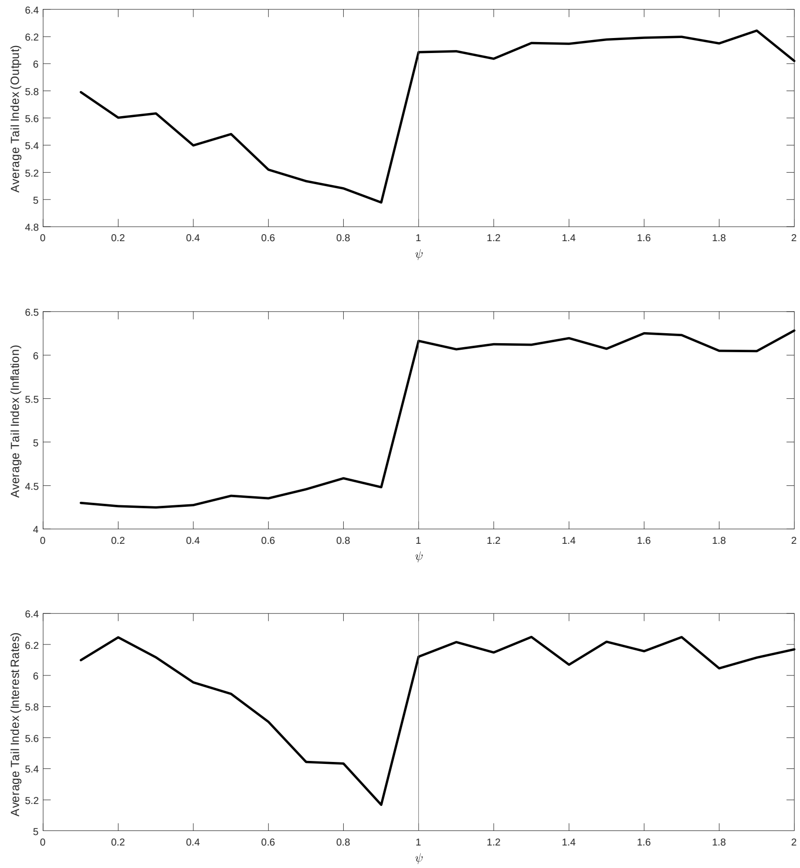

Figure 2. Innovations with 0.5 Standard Deviations.

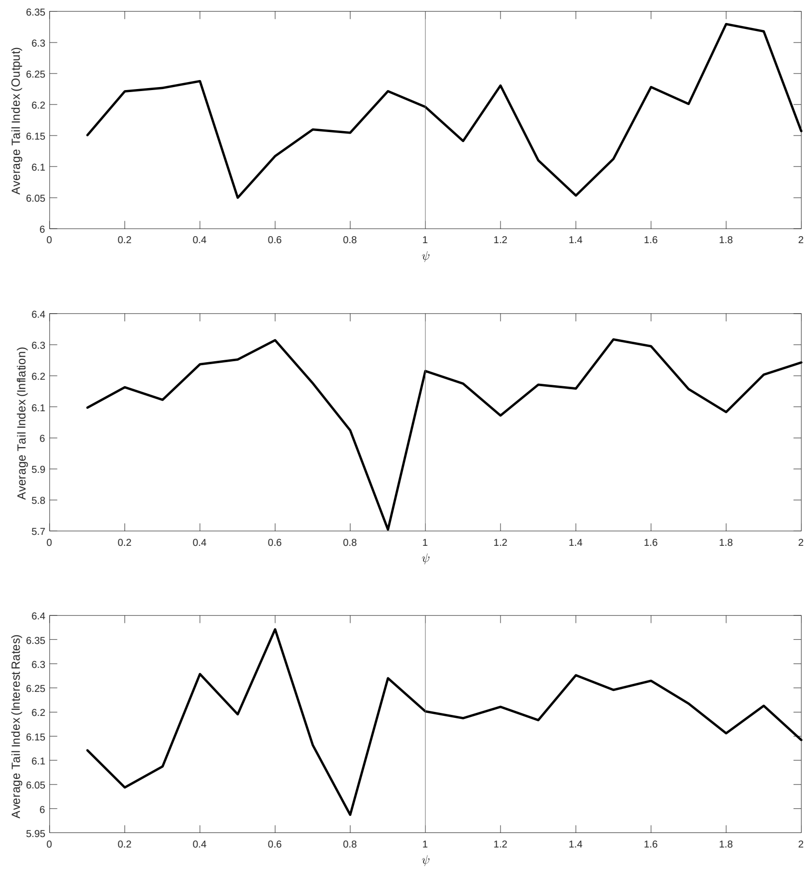

Figure 3. Innovations with 0.005 Standard Deviations.

Figures 2 and 3 demonstrate that as the standard deviation of the structural and sunspot shock deviations tend to zero, and the LRMN begins to increasingly resemble a fixed coefficient recursion, that tail indices rise. This is of course unsurprising as we have shown in the previous section that a model with indeterminacies and thus a role for sunspot shocks can be written as a LRMN when those sunspot shocks are in operation in the multiplicative portion of the recursion.

Finally, in order to compare our LRMN representation with the conventional fixed coefficient representation of Lubik and Schorfheide (2004) we conducted a final set of simulations in which we employed the fixed coefficient representation. The prior of course is that without the LRMN model representation, the simulations should result in high values for average estimated tail coefficients for all model variables, the results plotted in Figure 4 below confirm that prior.

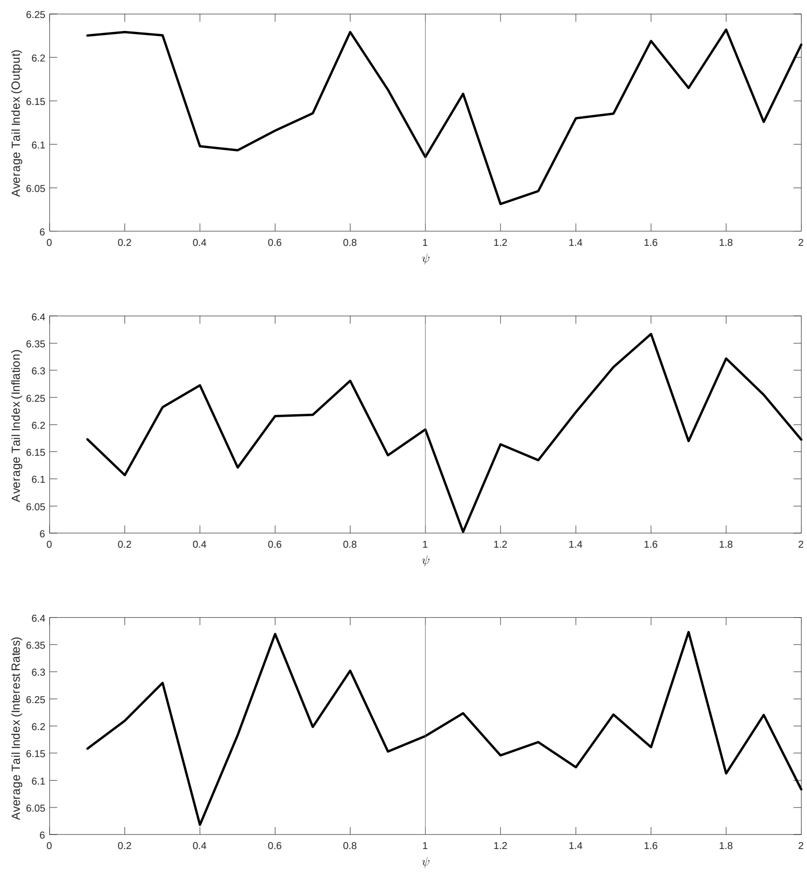

Figure 4. Simulations with a Fixed Coefficient Model Representation.

In summary, simulations for a simple three equation new Keynesian model in which the (in)determinate region is governed by a single parameter suggest two results. First, tail indices are lower in the indeterminate region when the model representation takes the form of a LRMN. Second, as the model representation tends to a fixed coefficient variety (either by diminishing the importance of sunspot shocks or by explicitly employing a fixed-coefficient model representation) tail indices rise. These simulation results lead us to our final exercise in the next section: an empirical assessment of the contributions of sunspot shocks in accounting for the tail behavior of output in a real business cycle model featuring indeterminacy.

4. A Real Business Cycle Model: Estimates

A main thrust of our analysis is that models that admit indeterminacy and thus space for sunspot shocks can, under an LRMN representation, account for fat-tailed behavior of aggregate time series. While the methods of Clauset et al. (2009) can be used to estimate empirical tail indices (see Dave and Malik (2017)) a particularly useful framework to evaluate empirical relevance is the workhorse RBC model. This model was previously rejected by Ascari et al. (2015) as lacking the necessary propagation mechanisms to replicate empirical fat tails. Here we adapt such a model, following Benhabib and Wen (2004), to allow for indeterminate solutions and thus LRMN representations in which sunspot shocks could help account for fat tails in aggregate output. The key difference to extant analyses (e.g. Ascari et al. (2015)) being of course that our structural innovations and sunspot shocks are thin tailed with the LRMN representation leading to fat tails for model variables, a "thin in, fat out" approach afforded by the results in Kesten (1973).

The representative agent of our model solves the following program

$$\max_{\{C_t,\ N_t,\ e_t,\ K_{t+1}\}} \Gamma = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t\right) - \frac{N_t^{1+\gamma}}{1+\gamma} \right],\ \gamma \geq 0, \tag{23}$$

$$s.t.\ Y_t = C_t + I_t, \tag{24}$$

$$K_{t+1} = [1 - \delta\left(e_t\right)] K_t + I_t, \tag{25}$$

$$Y_t = Z_t \Phi_t \left[e_t K_t\right]^{\alpha} N_t^{1-\alpha},\ \alpha \in (0,1), \tag{26}$$

$$\delta\left(e_t\right) = \frac{\nu}{\theta} e_t^{\theta},\ \theta > 1,\ 0 < \nu < \theta, \tag{27}$$

$$Z_t \sim CSSP(\rho, \sigma^2), \tag{28}$$

where $\Phi_t = \left[\left[e_t K_t\right]^{\alpha} N_t^{1-\alpha}\right]^{\eta}$ (with $\eta \geq 0$) is taken as given by the representative agent. In this environment, the parameter η governs indeterminacy stemming from production. We calibrate all parameters in the linear system of expectational difference equations characterizing the model, except for η and the standard deviation of sunspot shocks (σ_ζ), at usual values: $\beta = 0.99$, $\alpha = 0.36$, $\gamma = 0.001$ (so that we have near linearity in hours worked), $\theta = 1.2$, $\rho = 0.97$ and $\sigma = 0.007$. Appendix B provides further detail on the linear representation of this model.

In order to estimate, we again draw and fix a set of structural innovations and sunspot shocks and conduct a simulated minimum distance exercise, as follows. Given fixed shocks, the linear version of the model admits a LRMN representation as discussed formally above. Using this representation we construct the output series and estimate its' tail index using the methods of Clauset et al. (2009). Thus, for a given parametrization (η, σ_ζ) we can calculate the squared distance between the empirical tail index of output (set at 4, see Dave and Malik (2017)) and the corresponding simulated output tail index. We then minimize this distance by choice of various parametrizations for (η, σ_ζ); since this surface will have curvature, standard errors can be calculated as measures of how sharp the estimates are (see Benhabib and Dave (2014), Dave and Tsang (2014), Dave and Malik (2017) and DeJong and Dave (2011)). Our estimation results are provided in Table 1 below along with other relevant statistics for the cyclical component of data.

Table 1. Tail Index Estimates For Cyclical Components

	HP-Filtered Data		RBC Model
Variable (Frequency)	Tail index estimate (s.e.)	Parameter	Estimate (s.e.)
Output (Quarterly)	3.6395 (0.7147)	η	0.1201 (0.0003)
Output (Annual)	3.5418 (1.7982)	σ_ζ	0.0046 (0.0079)

In Table 1 above we note that the estimated tail index of the cyclical component of aggregate output, irrespective of frequency, is approximately 4; estimation carried out using the methods of Clauset et al. (2009). Under an LRMN assumption on the underlying data generating process, this estimate suggests that the tail of the stationary distribution of data only has its' first 3 moments. Were the data Normal in nature, a much larger tail index estimate would have resulted as, for instance, the Normal distribution has all of its' moments. Next, the estimates in Table 1 suggest that our estimate of η is within the range of indeterminacy with sunspot shocks helping to account for the empirical tail index of output. This result is comfortably close to the calibration employed in Benhabib and Wen (2004) and is in fact was expected to be; we demonstrated how the LRMN representation can produce thick tails and the estimation does indicate the same given the results in Benhabib and Wen (2004).

5. Conclusion

That aggregate economic data cannot always be characterized as Gaussian (cf. Ascari et al. (2015)) has led to much change in otherwise standard DSGE macroeconometric modeling. The extant literature has considered "fat in-fat out" options in which fat-tailed structural shocks, when fed through otherwise standard workhorse models, yield fat-tails for endogenous variables represented with a model written as a fixed coefficient recursion. The literature has also considered adaptive learning versus rational expectations to yield a "thin in-fat out" option in which thin-tailed shocks accumulate in a model written as a stochastic coefficients recursion with the result that model variables can exhibit fat tails. The drawback of course is that rational expectations, a workhorse assumption in of itself, needs to be sacrificed. Our contribution here is to open up the possibility that models featuring indeterminacy and thus a role for sunspot shocks may help reconcile models with fat-tailed behavior of macroeconomic aggregates, *without* abandoning the assumption of rational expectations. Our results are driven by the fact that such models can also be written as linear recursions with multiplicative noise and thus we are able to use certain tools from the large deviations theory.

We conducted simulations with an explicit version of a New Keynesian model to demonstrate the contribution in practice. We also estimated key parameters in a Real Business Cycle model featuring indeterminacies to again show the empirical plausibility of our approach. These exercises suggest continuing down the path of investigating models featuring indeterminacy as the sunspots they allow for can help account for non-Normal behavior in macroeconomic aggregates. Increasing the fit of workhorse models to data then allows for potentially improved forecasting as global economies face severe fluctuations due to otherwise low probability shocks like COVID-19.

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Appendices for Online Publication

Appendix A

Consider the model in Section 4 of Lubik and Schorfheide (2004), hereafter, LS,

$$\begin{aligned} y_t &= E_t(y_{t+1}) - \tau(r_t - E_t(\pi_{t+1})) + \varepsilon_{1t}, && (29)\\ \pi_t &= \beta E_t(\pi_{t+1}) + \kappa(y_t - \varepsilon_{2t}), && (30)\\ r_t &= \psi \pi_t + \varepsilon_{3t}. && (31) \end{aligned}$$

For this simplified version of their model, LS exploit an equivalent canonical representation as in Sims (2001) which features a block-triangular structure: the upper block simply relates the endogenous variables (y_t, π_t, r_t) to the endogenous forecast errors $\eta_t = [\eta_t^y, \eta_t^\pi]$ where $\eta_t^y = y - E_{t-1}[y_t]$ and $\eta_t^\pi = \pi_t - E_{t-1}[\pi_t]$; the lower block features the recursion of the one-step ahead conditional expectations $\xi_t = [\xi_t^y,\, \xi_t^\pi] = [E_t[y_{t+1}],\, E_t[\pi_{t+1}]]$ as forced by the structural shocks ε_t and the forecast errors η_t.

Formally, the upper block reads as

$$\begin{bmatrix} y_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \xi_{t-1}^y \\ \xi_{t-1}^\pi \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix}, \qquad (32)$$

whereas the lower block reads as

$$\begin{bmatrix} 1 & \tau \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \xi_t^y \\ \xi_t^\pi \end{bmatrix} = \begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} \xi_{t-1}^y \\ \xi_{t-1}^\pi \end{bmatrix} + \begin{bmatrix} \tau & -1 & 0 \\ 0 & 0 & \kappa \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix}, \tag{33}$$

which, upon inversion of the the square coefficient matrix on the LHS, can be equivalently rewritten as

$$\xi_t = \underbrace{\begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau(\psi - \frac{1}{\beta}) \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \tau & -1 & \frac{-\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix}}_{\Psi^*} \varepsilon_t + \underbrace{\begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau(\psi - \frac{1}{\beta}) \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \tag{34}$$

To solve the model under either equilibrium regime, LS exploit this block-triangular structure by (i) first focusing on the lower block (34) to solve for the forecast errors η_t by imposing the appropriate existence condition in order to eliminate the explosive dynamics, if any; (ii) then using the equilibrium forecast errors to pin down, again within the lower block (34), the equilibrium law of motion for the conditional expectations ξ_t; (iii) finally plugging equilibrium values of conditional expectations as well as equilibrium forecast errors into the upper block (32) in order to derive the equilibrium law of motion for the endogenous variables (y_t, π_t, r_t).

Step (i) above requires decoupling stable from unstable components, which can be easily done here by means of the Jordan decomposition of $\Gamma_1^* = J\Lambda J^{-1}$ where

$$J = \begin{bmatrix} \frac{1}{\kappa}(1 - \beta l_1 + \beta l_2) & -\frac{1}{\kappa}(1 - \beta l_1 - \beta l_2) \\ 1 & 1 \end{bmatrix} \tag{35}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{36}$$

$$J^{-1} = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1 + \beta l_1 + \beta l_2) \\ -\kappa & 1 - \beta l_1 + \beta l_2 \end{bmatrix} \tag{37}$$

where the eigenvalues (λ_1, λ_2) are

$$\lambda_1 = l_1 - l_2; \quad \lambda_2 = l_1 + l_2 \tag{38}$$

with

$$l_1 = \frac{1}{2}\left(1 + \frac{1+\kappa\tau}{\beta}\right); \quad l_2 = \frac{1}{2}\sqrt{\left(\frac{1+\kappa\tau}{\beta} + 1\right)^2 + \frac{4\kappa\tau}{\beta}(1-\psi)} \tag{39}$$

Notice that $\psi > 1$ implies $|\lambda_2| > |\lambda_1| > 1$, whereas $\psi \in (0,1)$ implies $|\lambda_2| > 1 > |\lambda_1|$. Letting $\omega_t = J^{-1}\xi_t$ the lower block (34) can be thus diagonalized as

$$\omega_t = \Lambda\omega_{t-1} + J^{-1}\Psi^*\varepsilon_t + J^{-1}\Pi^*\eta_t \tag{40}$$

Determinacy

When both eigenvalues are larger than one in absolute value, then (40) explodes over time for any given initial condition. In order to prevent that, it must be the case that $\omega_0 = 0$ (which is equivalent to requiring $\xi_0 = 0$ due to non-singularity of J) and

$$J^{-1}\Psi^*\varepsilon_t + J^{-1}\Pi^*\eta_t = 0, \quad t \geq 1 \tag{41}$$

which is a square system leading to uniquely determined equilibrium forecast errors

$$\eta_t^D = -{\Pi^*}^{-1}\Psi^*\varepsilon_t = -\frac{1}{1+\kappa\psi\tau}\begin{bmatrix} \tau & -1 & -\tau\kappa\psi \\ \kappa\tau & -\kappa & \kappa \end{bmatrix}\varepsilon_t \tag{42}$$

where the apex D stands for *determinacy.*

Plugging (42) into (32) and recalling that $\xi_0 = 0$ joint with (41) implies $\xi_t = 0$ for all t, one has

$$\begin{bmatrix} y_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} - \frac{1}{1+\kappa\psi\tau} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \tau & -1 & -\tau\kappa\psi \\ \kappa\tau & -\kappa & \kappa \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{43}$$

which simplifies to

$$\underbrace{\begin{bmatrix} y_t \\ \pi_t \\ r_t \end{bmatrix}}_{x_t} = \frac{1}{1+\kappa\tau\psi} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi \\ -\kappa\tau & \kappa & -\kappa \\ 1 & \kappa\psi & -\kappa\psi \end{bmatrix} \underbrace{\begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t} \tag{44}$$

Indeterminacy

When only one eigenvalue (here λ_2) is larger than one in absolute value, then only the second row of (40) defines an explosive autoregressive process, so that η_t must be chosen in order to eliminate it; this requires $\omega_{2,0} = J_2^{-1}\xi_0 = 0$ (where J_2^{-1} is the second row of the matrix J^{-1}) and

$$\left(J^{-1}\Psi^*\right)_2 \varepsilon_t + \left(J^{-1}\Pi^*\right)_2 \eta_t = 0, \quad t \geq 1 \tag{45}$$

where $(\cdot)_2$ denotes the second row of the matrix in square brackets. Since

$$J^{-1}\Psi^* = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix} \begin{bmatrix} \tau & -1 & \frac{-\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix} \tag{46}$$

and

$$J^{-1}\Pi^* = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix} \begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi-\frac{1}{\beta}) \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \tag{47}$$

the existence condition (45) is equivalent to the following restriction

$$\begin{bmatrix} -\kappa\tau & \kappa & \kappa(\lambda_2 - 1) \end{bmatrix} \epsilon_t = \begin{bmatrix} \kappa\lambda_2 & 1 + \kappa\tau\psi - \lambda_2 \end{bmatrix} \eta_t \tag{48}$$

i.e. an indeterminate system: for any given vector of structural shocks ε_t, there exists an infinity of pairs (η_t^y, η_t^π) solving (48), for the latter always admits the solution η_t^D as defined in (42). In order to find out all such solutions, LS exploit the Singular Value Decomposition (SVD) of $(J^{-1}\Pi^*)_2$ i.e.

$$\left(J^{-1}\Pi^*\right)_2 = U_{.1} D_{11} V_{.2}' \tag{49}$$

where $U_1 = 1$, $D_{11} = d = \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\tau\psi)^2}$ and

$$V_2' = \frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\tau\psi & \kappa\lambda_2 \end{bmatrix} \tag{50}$$

so that, following Lubik and Schorfheide (2003), the full set of equilibrium forecast errors solving (48) is

$$\eta_t^I = \eta_t^D + V_2 \tilde{M}_1 \varepsilon_t + V_2 \tilde{M}_2 \zeta_t \tag{51}$$

where $\tilde{M}_1 = dM_1$ with M_1 being a 1×3 matrix with arbitrary (real) entries, and $\tilde{M}_2 = dM_2$ with M_2 being $1 \times p$ matrix with arbitrary (real) entries; p denotes the (arbitrary) dimension of the vector of sunspot shocks ζ_t which behave as martingale difference sequences with respect to the t-dated information set (notice we have w.l.o.g. normalized both arbitrary matrices by d in order for the latter not to show up in the equilibrium representation below). Let $\zeta_t^* = M_2\zeta_t$ be the reduced form sunspot shock and denote by η_t^* the forecast error component which arises only under indeterminacy, i.e.

$$\eta_t^* = V_2 \left(\tilde{M}_1 \varepsilon_t + \zeta_t^*\right) = \begin{bmatrix} \lambda_2 - 1 - \kappa\tau\psi \\ \kappa\lambda_2 \end{bmatrix} (M_1 \varepsilon_t + \zeta_t^*) \,. \tag{52}$$

One has $\eta_t^I = \eta_t^D + \eta_t^*$ and thus the stable (first) row of the diagonalized system (40) becomes (recall that by the existence condition $\omega_{2,t} = 0$ for all t)

$$\omega_{1,t} = \lambda_1 \omega_{1,t-1} + \left(J^{-1}\Psi^*\right)_1 \varepsilon_t + \left(J^{-1}\Pi^*\right)_1 \eta_t^I. \tag{53}$$

Recalling that η_t^D is such that $\Psi^*\varepsilon_t + \Pi^*\eta_t^D = 0$ the above recursion then reduces to the following

$$\begin{aligned}
\omega_{1,t} &= \lambda_1 \omega_{1,t-1} + (J^{-1}\Psi^*)_1 \varepsilon_t + (J^{-1}\Psi^*)_1 \eta_t^* && (54)\\
&= \lambda_1 \omega_{1,t-1} + \frac{1}{2\beta l_2} \begin{bmatrix} \kappa\tau \\ -\kappa \\ -\frac{\kappa^2\tau}{\beta} + \frac{\kappa}{\beta}\left(-1 + \beta l_1 + \beta l_2\right) \end{bmatrix}' \varepsilon_t + && (55)\\
&\quad \frac{1}{2\beta l_2} \begin{bmatrix} \kappa\left(1 + \frac{\kappa}{\beta}\right) + \frac{\kappa}{\beta}(1 - \beta l_1 - \beta l_2) \\ \kappa\tau\left(\psi - \frac{1}{\beta} - \frac{1}{\beta}\left(1 - \beta l_1 - \beta l_2\right)\right) \end{bmatrix}' \begin{bmatrix} \lambda_2 - 1 - \kappa\tau\psi \\ \kappa\lambda_2 \end{bmatrix} (M_1\varepsilon_t + \zeta_t^*) && (56)
\end{aligned}$$

Finally, recalling that

$$\xi_t = J\omega_t = J_1\omega_{1,t} = \begin{bmatrix} \frac{1}{\kappa}(1 - \beta l_1 + \beta l_2) \\ 1 \end{bmatrix} \omega_{1,t} \tag{57}$$

where J_1 is the first column of the matrix J, one can use the upper block (32) together with the equilibrium recursion for the conditional expectations ξ_t as well as the equilibrium forecast errors η_t^I to recover the equilibrium dynamics for the original endogenous variables, i.e.

$$\begin{bmatrix} y_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \frac{1}{\kappa}(1-\beta l_1+\beta l_2) \\ 1 \end{bmatrix} \omega_{1,t-1} \tag{58}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi \end{bmatrix} \left(\frac{-1}{1+\kappa\psi\tau} \begin{bmatrix} \tau & -1 & -\tau\kappa\psi \\ \kappa\tau & -\kappa & \kappa \end{bmatrix} \varepsilon_t + \eta_t^* \right) \tag{59}$$

which, upon slight rearrangement, delivers

$$x_t = \frac{1}{1+\kappa\tau\psi} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi & \lambda_2 - 1 - \kappa\tau\psi \\ -\kappa\tau & \kappa & -\kappa & \kappa\lambda_2 \\ 1 & \kappa\psi & -\kappa\psi & \psi\kappa\lambda_2 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \zeta_t \end{bmatrix} + \begin{bmatrix} \frac{\beta(\lambda_2-1)-\tau\kappa}{\kappa} \\ 1 \\ \psi \end{bmatrix} \omega_{1,t-1} \tag{60}$$

where

$$\omega_{1,t-1} = \lambda_1 \omega_{1,t-2} + \mu_1(\theta)\eta_{t-1}^* \tag{61}$$

and $\mu_1(\theta)$ is the following function of the parameter vector θ:

$$\mu_1(\theta) = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa\left(1+\frac{\kappa\tau}{\beta}\right) + \frac{\kappa}{\beta}(1-\beta l_1-\beta l_2) \\ \kappa\tau\left(\psi - \frac{1}{\beta} - \frac{1}{\beta}(1-\beta l_1 - \beta l_2)\right) \end{bmatrix}'. \tag{62}$$

LRMN representation

Let $\psi < 1$, i.e. let the model be in the indeterminacy regime. Existence of non-explosive solutions is still governed by condition (45). However, we now assume that the forecast errors are a function of the (past) actual states $X_{t-1} = [y_{t-1},\ \pi_{t-1}]'$ as well as structural shocks ε_t with random loadings i.e.

$$\eta_t = A_{1,\zeta_t}\varepsilon_t + A_{2,\zeta_t}X_{t-1} \tag{63}$$

where

$$A_{1,\zeta_t} = \underbrace{A_1}_{2\times 3} \underbrace{\zeta_{1,t}}_{3\times 3}, \qquad A_{2,\zeta_t} = \underbrace{A_2}_{2\times 2} \underbrace{\zeta_{2,t}}_{2\times 2} \tag{64}$$

where $\zeta_{1,t}$ and $\zeta_{2,t}$ are diagonal random matrices whose entries are diagonal random matrices whose entries are mutually independent (and independent of the structural shocks ϵ_t) MDSs with respect to the t-dated information set, and (A_1, A_2) are conformable matrices whose coefficients have to be determined.

Following the general procedure described in Dave and Sorge (2020), the equilibrium forecast errors under indeterminacy are fully characterized by the following

$$\eta_t^I = \eta_t^D + V_{\cdot 2} \underbrace{M_1}_{1\times 3} \begin{bmatrix} \zeta_{1,t}^1 & 0 & 0 \\ 0 & \zeta_{1,t}^2 & 0 \\ 0 & 0 & \zeta_{1,t}^3 \end{bmatrix} \varepsilon_t + V_{\cdot 2} \underbrace{M_2}_{1\times 2} \begin{bmatrix} \zeta_{2,t}^1 & 0 \\ 0 & \zeta_{2,t}^2 \end{bmatrix} X_{t-1} \tag{65}$$

where M_1 and M_2 are arbitrary matrices, whose entries do not depend on the structural parameters of the RE model.

Denote by η_t^* the forecast error component which arises only under indeterminacy, one has $\eta_t^I = \eta_t^D + \eta_t^*$, while conditional expectations ξ_t under indeterminacy evolve along the equilibrium path according to the following

$$E_t\left(X_{t+1}\right) = J_1 \omega_{1,t} = J_1 \left[\lambda_1 \omega_{1,t-1} + \left(J^{-1}\Psi^*\right)_1 \varepsilon_t + \left(J^{-1}\Pi^*\right)_1 \eta_t^*\right] \tag{66}$$

Notice now the RE model (29)-(31) can be rewritten (upon inserting the third equation into the first one) as

$$\begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \tau \\ 0 & \beta \end{bmatrix} E_t \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} + \begin{bmatrix} -\tau & 1 & 0 \\ 0 & 0 & -\kappa \end{bmatrix} \varepsilon_t \tag{67}$$

or in more compact form

$$AX_t = BE_t\left(X_{t+1}\right) + C\varepsilon_t \tag{68}$$

Plugging (66) into (68) and using the definition of η_t^I one has

$$AX_t = BJ_1\left[\lambda_1\omega_{1,t-1} + \left(J^{-1}\Psi^*\right)_1\varepsilon_t + \left(J^{-1}\Pi^*\right)_1\eta_t^*\right] + C\varepsilon_t \tag{69}$$

Using the fact that $\omega_{1,t-1} = J_1^{-1}\left(X_t - \eta_t^I\right)$, where J_1^{-1} is the first row of the matrix J^{-1}, one finally has the LRMN representation

$$X_t = \alpha_t X_{t-1} + \beta_t \tag{70}$$

where

$$\alpha_t := (A-\Theta_0)^{-1}\Theta_1; \quad \beta_t := (A-\Theta_0)^{-1}\Theta_2\varepsilon_t \tag{71}$$

$$\Theta_0 := BJ_1\lambda_1 J_1^{-1} \tag{72}$$

$$\Theta_1 := \left[-\Theta_0 + BJ_1\left(J^{-1}\Pi^*\right)_1\right]V_{\cdot 2}M_2\zeta_{2,t} \tag{73}$$

$$\Theta_2 := C + \Theta_0\left[{\Pi^*}^{-1}\Psi^* + V_{\cdot 2}M_1\zeta_{1,t}\right] + BJ_1\left[\left(J^{-1}\Psi^*\right)_1 + \left(J^{-1}\Pi^*\right)_1 V_{\cdot 2}M_1\zeta_{1,t}\right] \tag{74}$$

For convenience we report below all the vectors and matrices that are needed for computing α_t, β_t, Θ_0, Θ_1 and Θ_2:

$$A = \begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & \tau \\ 0 & \beta \end{bmatrix}; \quad J_1 = \begin{bmatrix} \frac{1}{\kappa}(1-\beta l_1+\beta l_2) \\ 1 \end{bmatrix} \tag{75}$$

$$\lambda_1 = l_1 - l_2; \quad J_1^{-1} = \frac{1}{2\beta l_2}\begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \end{bmatrix} \tag{76}$$

$$\left(J^{-1}\Pi^*\right)_1 = \frac{1}{2\beta l_2}\begin{bmatrix} \kappa\left(1+\frac{\kappa\tau}{\beta}\right) - \frac{\kappa}{\beta}(-1+\beta l_1+\beta l_2) & \kappa\tau\left(\psi-\frac{1}{\beta}\right) + \frac{1}{\beta}(-1+\beta l_1+\beta l_2) \end{bmatrix} \tag{77}$$

$$V_{\cdot 2} = \frac{1}{d}\begin{bmatrix} \lambda_2 - 1 - \kappa\tau\psi \\ \kappa\lambda_2 \end{bmatrix}; \quad C = \begin{bmatrix} -\tau & 1 & 0 \\ 0 & 0 & -\kappa \end{bmatrix} \tag{78}$$

$$\Pi^{*^{-1}}\Psi^* = \frac{1}{1+\kappa\psi\tau}\begin{bmatrix} \tau & -1 & -\tau\kappa\psi \\ \kappa\tau & -\kappa & \kappa \end{bmatrix} \tag{79}$$

$$\left(J^{-1}\Psi^*\right)_1 = \frac{1}{2\beta l_2}\begin{bmatrix} \kappa\tau & -\kappa & -\frac{\tau\kappa^2}{\beta} + \frac{\kappa}{\beta}(-1+\beta l_1+\beta l_2) \end{bmatrix} \tag{80}$$

where M_1, M_2, $\zeta_{1,t}$ and $\zeta_{2,t}$ are all defined in (65).

Appendix B

The optimization problem is

$$\max_{\{C_t,\ N_t,\ e_t,\ K_{t+1}\}} \quad \Gamma = E_0\sum_{t=0}^{\infty}\beta^t\left[\log\left(C_t\right) - a\frac{N_t^{1+\gamma}}{1+\gamma}\right], \quad \gamma \geq 0, \quad a > 0, \tag{81}$$

$$s.t. \quad Y_t = C_t + I_t, \tag{82}$$

$$K_{t+1} = [1 - \delta\left(e_t\right)]K_t + I_t, \tag{83}$$

$$Y_t = Z_t\Phi_t\left[e_tK_t\right]^{\alpha} N_t^{1-\alpha},\ \alpha \in (0,1), \tag{84}$$

$$\delta\left(e_t\right) = \frac{\nu}{\theta}e_t^{\theta},\ \theta > 1,\ 0 < \nu < \theta, \tag{85}$$

$$Z_t \sim CSSP(\rho, \sigma^2), \tag{86}$$

where $\Phi_t = \left[[e_t K_t]^\alpha N_t^{1-\alpha} \right]^\eta$ (with $\eta \geq 0$) is taken as parametric by the representative agent. We let the Lagrange multiplier be denoted as Λ_t to obtain equations

$$\Lambda_t = \frac{1}{C_t} \tag{87}$$

$$(1-\alpha)\Lambda_t \frac{Y_t}{N_t} = a N_t^\gamma \tag{88}$$

$$\alpha Z_t \Phi_t [e_t K_t]^{\alpha-1} N_t^{1-\alpha} = \alpha \frac{Y_t}{[e_t K_t]} = \nu e_t^{\theta-1} \rightarrow \alpha \frac{Y_t}{K_t} = \nu e_t^\theta \tag{89}$$

$$\Lambda_t = \beta \Lambda_{t+1} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \frac{\nu}{\theta} e_{t+1}^\theta \right] \tag{90}$$

$$C_t = Y_t - K_{t+1} + \left[1 - \frac{\nu}{\theta} e_t^\theta \right] K_t \tag{91}$$

$$K_{t+1} = \left[1 - \frac{\nu}{\theta} e_t^\theta \right] K_t + I_t \tag{92}$$

$$Y_t = Z_t \Phi_t [e_t K_t]^\alpha N_t^{1-\alpha} \tag{93}$$

$$\Phi_t = \left[[e_t K_t]^\alpha N_t^{1-\alpha} \right]^\eta \tag{94}$$

$$Z_t \sim CSSP(\rho, \sigma^2) \tag{95}$$

which constitute a 9×9 system in $\{Y_t, C_t, I_t, N_t, e_t, K_t, Z_t, \Lambda_t, \Phi_t\}$ with parameter vector $\mu = \{\alpha, a, \gamma, \theta, \beta, \nu, \eta, \rho, \sigma\}$. Without loss of generality we can reduce the system by 2 variables that are otherwise redundant: Λ_t and Φ_t. Doing so yields the following 7×7

system in $\{Y_t, C_t, I_t, N_t, e_t, K_t, Z_t\}$ with parameter vector μ,

$$
\begin{aligned}
(1-\alpha)\frac{Y_t}{C_t} &= aN_t^{1+\gamma} && (96)\\
\alpha\frac{Y_t}{K_t} &= \nu e_t^{\theta} && (97)\\
\frac{1}{C_t} &= \beta E_t\left\{\frac{1}{C_{t+1}}\left[\alpha\frac{Y_{t+1}}{K_{t+1}}+1-\frac{\nu}{\theta}e_{t+1}^{\theta}\right]\right\} && (98)\\
C_t &= Y_t - K_{t+1} + \left[1-\frac{\nu}{\theta}e_t^{\theta}\right]K_t && (99)\\
K_{t+1} &= \left[1-\frac{\nu}{\theta}e_t^{\theta}\right]K_t + I_t && (100)\\
Y_t &= Z_t\left[\left[e_tK_t\right]^{\alpha}N_t^{1-\alpha}\right]^{1+\eta} && (101)\\
Z_t &\sim CSSP(\rho,\sigma^2) && (102)
\end{aligned}
$$

Note the lack of a deterministic trend in the model specification (that is, no balanced growth). We therefore assume that the stochastic process for Z_t is such that eventually all linearized variables will be interpreted as logarithmic deviations from a HP-filtered trend, and move directly to the steady state derivation.

The Nonstochastic Steady State

We now derive the nonstochastic steady state around which we log-linearize the system for eventual use in Sims (2001) in order to solve the model. In a nonstochastic steady state we begin by assuming that the steady state value of Z_t $(\overline{Z})$ is in hand. Then using (98) we know that

$$\frac{1}{C}=\beta\left\{\frac{1}{C}\left[\alpha\frac{Y}{K}+1-\frac{\nu}{\theta}e^{\theta}\right]\right\}\rightarrow\frac{1-\beta}{\beta}+\frac{\nu}{\theta}e^{\theta}=\alpha\frac{Y}{K} \qquad (103)$$

which itself can be inserted into (97) to yield

$$
\begin{aligned}
\alpha\frac{Y}{K} &= \nu e^{\theta}\rightarrow\frac{1-\beta}{\beta}+\frac{\nu}{\theta}e^{\theta}=\nu e^{\theta} && (104)\\
&\rightarrow\ \overline{e}=\left[\frac{\theta(1-\beta)}{\nu\beta(\theta-1)}\right]^{\frac{1}{\theta}}. && (105)
\end{aligned}
$$

Now assume that we have $\overline{Y}$ in hand then we know from (98)

$$\frac{1-\beta}{\beta}+\frac{\nu}{\theta}e^{\theta} = \alpha\frac{Y}{K} \to \frac{Y}{K}=\frac{1-\beta}{\alpha\beta}+\frac{\nu}{\alpha\theta}e^{\theta} \quad (106)$$

$$\to \overline{K}=\left(\frac{\alpha\beta(\theta-1)}{\theta(1-\beta)}\right)\overline{Y} \quad (107)$$

which in turn implies from (100) that

$$K = [1-\frac{\nu}{\theta}\overline{e}^{\theta}]K+I \to \frac{I}{K}=\frac{\nu\overline{e}^{\theta}}{\theta} \to \overline{I}=\frac{\nu\overline{e}^{\theta}}{\theta}\overline{K} \to \overline{I}=\frac{1-\beta}{\beta(\theta-1)}\overline{K} \quad (108)$$

$$\overline{I} = \frac{1-\beta}{\beta(\theta-1)}\left(\frac{\alpha\beta(\theta-1)}{\theta(1-\beta)}\right)\overline{Y} \to \overline{I}=\frac{\alpha}{\theta}\overline{Y} \quad (109)$$

Next, use the previous relations in (99) to yield

$$C = Y-K+[1-\frac{\nu}{\theta}e^{\theta}]K \to C=Y-K+K-\frac{\nu}{\theta}e^{\theta}K \quad (110)$$

$$\to \overline{C}=\frac{\theta-\alpha}{\theta}\overline{Y} \quad (111)$$

The steady state value of labor is now readily obtained using (96) as

$$(1-\alpha)\frac{Y}{C} = aN^{1+\gamma} \to \frac{(1-\alpha)}{a}\frac{\overline{Y}}{\overline{\overline{C}}}=N^{1+\gamma} \quad (112)$$

$$\to \overline{N}=\left(\frac{\theta(1-\alpha)}{a(\theta-\alpha)}\right)^{\frac{1}{1+\gamma}} \quad (113)$$

Now, to obtain $\overline{Y}$ we insert all elements into (101) keeping in mind that $\overline{e}$ and $\overline{N}$ are purely functions of parameters,

$$Y = Z\left[[eK]^{\alpha}N^{1-\alpha}\right]^{1+\eta} \to \overline{Y}=\overline{Z}\left[\left[\overline{e}\overline{K}\right]^{\alpha(1+\eta)}\overline{N}^{(1-\alpha)(1+\eta)}\right] \quad (114)$$

$$\to \overline{Y}=\left[\overline{Z}\left(\frac{\alpha\beta(\theta-1)}{\theta(1-\beta)}\overline{e}\right)^{\alpha(1+\eta)}\overline{N}^{(1-\alpha)(1+\eta)}\right]^{\frac{1}{1-\alpha(1+\eta)}} \quad (115)$$

Linearized System

In terms of notation let $\widehat{x}_t = \log X_t - \log \overline{X}$ and then linearize each equation individually to obtain

$$-\widehat{c}_t - (1+\gamma)\widehat{n}_t + \widehat{y}_t = 0 \tag{116}$$

$$\widehat{y}_t = \theta \widehat{e}_t + \widehat{k}_t \tag{117}$$

$$\widehat{y}_t - (1+\eta)(1-\alpha)\widehat{n}_t = \widehat{z}_t + \alpha(1+\eta)\widehat{e}_t + \alpha(1+\eta)\widehat{k}_t \tag{118}$$

$$\theta(1-\beta)E_t(\widehat{e}_{t+1}) + \theta(1-\beta)\widehat{k}_{t+1} = (\theta-1)\widehat{c}_t - (\theta-1)E_t(\widehat{c}_{t+1}) + \theta(1-\beta)E_t(\widehat{y}_{t+1}) \tag{119}$$

$$\alpha\beta(\theta-1)\widehat{k}_{t+1} = \theta(1-\beta)\widehat{y}_t - \alpha\theta(1-\beta)\widehat{e}_t - (\theta-\alpha)(1-\beta)\widehat{c}_t + \alpha(\beta\theta-1)\widehat{k}_t \tag{120}$$

$$\beta(\theta-1)\widehat{k}_{t+1} = -\theta(1-\beta)\widehat{e}_t + (\beta\theta-1)\widehat{k}_t + (1-\beta)\widehat{i}_t \tag{121}$$

$$\widehat{z}_t = \rho \widehat{z}_{t-1} + \varepsilon_t. \tag{122}$$

We can further reduce the dimensionality of the system by noting that investment (I_t) is a redundant variable in the nonlinear system, thereby reducing the linear system to

$$\widehat{y}_t = E_{t-1}(\widehat{y}_t) + \iota_t^y \tag{123}$$

$$\widehat{c}_t = E_{t-1}(\widehat{c}_t) + \iota_t^c \tag{124}$$

$$\widehat{e}_t = E_{t-1}(\widehat{e}_t) + \iota_t^e \tag{125}$$

$$-\widehat{c}_t - (1+\gamma)\widehat{n}_t + \widehat{y}_t = 0 \tag{126}$$

$$\widehat{y}_t - \theta \widehat{e}_t = \widehat{k}_t \tag{127}$$

$$\widehat{y}_t - (1+\eta)(1-\alpha)\widehat{n}_t - \widehat{z}_t - \alpha(1+\eta)\widehat{e}_t = \alpha(1+\eta)\widehat{k}_t \tag{128}$$

$$\theta(1-\beta)E_t(\widehat{e}_{t+1}) + \theta(1-\beta)\widehat{k}_{t+1} - (\theta-1)\widehat{c}_t + (\theta-1)E_t(\widehat{c}_{t+1}) - \theta(1-\beta)E_t(\widehat{y}_{t+1}) = 0 \tag{129}$$

$$\alpha\beta(\theta-1)\widehat{k}_{t+1} - \theta(1-\beta)\widehat{y}_t + \alpha\theta(1-\beta)\widehat{e}_t + (\theta-\alpha)(1-\beta)\widehat{c}_t = \alpha(\beta\theta-1)\widehat{k}_t \tag{130}$$

$$\widehat{z}_t = \rho \widehat{z}_{t-1} + \varepsilon_t \tag{131}$$

where ι_t is an 'expectations error' requiring identities to be added to the system so as to match the notation of Sims (2001).

Estimation

We denote the empirical tail index from Table 1 as $\varkappa = 4$. The RBC model can be written as a LRMN recursion and for fixed draws of the sunspot and structural shocks the implied $T = 250$ long simulated series for endogenous variables created, given a candidate parametrization $\mu = [\eta \ \sigma_\varsigma]'$. Thus for a candidate μ the tail index of model implied output, estimated using the maximum likelihood methods of Clauset et al. (2009), is denoted as $\varkappa(\mu)$. We then search over the parameter space to minimize the squared difference between $\varkappa$ and $\varkappa(\mu)$ in order to estimate values for μ; i.e., our estimates are delivered by

$$\min_{\mu} \quad \digamma = [\varkappa - \varkappa(\mu)]'[\varkappa - \varkappa(\mu)] \tag{132}$$

with standard errors computed using the Hessian of the above objective function at the parameter estimates. This simulated minimum distance estimation method is not just distribution free but also does not necessarily entail the matching of any particular set of moments if the empirical targets are not moments but tail indices, see Dave and Malik (2017) for further details albeit in a different context.

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