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Public Private Competition

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Abstract

We examine competition between a private and a public provider in markets for "merit goods" such as education, healthcare, housing, recreation, or culture. The private firm provides a high-price/high-quality variety of the good and serves richer individuals, while the public firm provides a low-price/low-quality variety and serves poorer individuals. We derive the private competitor's best response to changes in the public firm's price and quality level. This enables us to examine the distributional effects of government policies aimed at making publicly provided goods more affordable or increase their quality, and of changes to the government budget constraint that make publicly provided goods more expensive or decrease their quality. Our results have implications for the financing of the public supply of such goods, and for whether additional resources, if available, should be spent on reducing the price or enhancing the quality of publicly provided goods.

Keywords: Mixed duopoly; quality differentiation; public provision of private goods; private responses to public policy; crowding-out/in; funding of public services.

JEL codes: D21; D43; H11; H42; H44; I00; L38

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1 Introduction

Many goods and services are provided by private firms as well as public entities. Education, healthcare, health insurance, housing, housing-related financial services, as well as recreational and cultural amenities are important examples of markets in which private and public suppliers compete. Governments are active suppliers of these goods not only—and in many cases not even primarily—to correct some market failure. Public provision of private goods is also an indirect way of redistributing incomes. The public supply of education and healthcare, for example, accounts for a significant share of income redistribution in developed countries (Poterba 1996), and major public investments in these sectors are often made with the stated goal of expanding affordable access to low-income segments of the population.

This paper investigates how private firms respond to public supply in these markets. We develop a mixed duopoly model in which one private firm and one public firm compete by offering differentiated varieties of a good. We focus on market outcomes in which the the private firm provides a high-price/high-quality variety and serves higher-income individuals, while the public firm provides a low-price/low-quality variety and serves lower-income individuals.[1] Our analysis will have the following structure. We always start with some change in the price or quality of the public provider's good. This change may occur because of a change in the government's objective, or because of a change to the government's budget constraint. When this happens, the private firm will re-optimize and adjust its own price and quality, and our goal is to investigate this private response. We then argue that the nature of the private response matters when evaluating the distributional effects of government activity in the market.

To give a sense of our results, consider a policy initiative to expand access to some good, by lowering the price the public provider charges. Clearly, low-income individuals who could not afford the good before will benefit from this change. If the good is to be offered at the same quality quality, the affordability initiative requires an increase in the public firm's funding level. Absent such a subsidy, the only way for the public firm to meet its new mandate is to cut costs by lowering quality. We will demonstrate that these two scenarios—a fully funded affordability initiative, and an unfunded one—can lead to opposite responses by the private firm. For certain specifications of the consumer preferences and the income distribution, the former scenario results in a reduced private supply (i.e., crowding-out) at a higher price and higher quality, while the latter results

[1] This pattern is not universal, but common in many developed market economies, especially when the goal of public provision is to offer affordable access to certain goods. For example, public universities charge lower tuition rates than elite private universities, but have larger classes and employ less famous professors. Public healthcare providers charge lower treatment fees than private clinics, but have longer wait times and less luxurious facilities. Municipal golf courses charge lower membership or green fees than private country clubs, but are less well maintained and more crowded.

in an increased private supply (i.e., crowding-in) at a lower price and lower quality. Consumers with sufficiently high incomes benefit in the first case but are hurt in the second.

We also examine how the private firm reacts to changes in the distribution of disposable income. This enables us to examine the effect of income taxes to fund the public firm. When the public firm runs a deficit, financing this deficit through taxing the consumers in the market generally results in a lower price and a lower quality by the private firm, relative to financing the deficit externally (i.e., through an injection of funds from outside). However, an external funding source may not always be available. When compared to the remaining alternative of *not* subsidizing the public firm, the tax system may reduce the private firm's price and increase its quality. This outcome can be Pareto-superior to the no-subsidy case; in particular, it may be preferred by high-income individuals who do not buy from the public firm but subsidize it through their taxes.

The welfare gains from such policy changes can then be evaluated using some social welfare function. However, our model is general and does not require the specification of a *particular* social welfare function used by the policy maker, or objective pursued by the public firm. The cost of this generality is that we do not compute an "optimal" or "equilibrium" quantity, price, or quality for the public firm. Yet, it has two important advantages. Note that our analytical approach is variational in nature: We are concerned with the private response to *changes* in public policy. Many of the policy changes we examine—e.g., a mandate to improve affordability—reflect underlying changes in social objectives—e.g., an increased concern for the welfare of low-income individuals. The first advantage of our approach is that it allows us to analyze the effects of such changes in the objective without knowledge of the precise objective function that has changed. Other changes in public policy may not reflect changes in social objectives at all, but changes in the government's budget constraint—for example, government may reduce the subsidy paid to a public provider during a recession, resulting in a reduction of the public firm's quality or in an increase in its price. The second advantage of our approach is that it also allows us to characterize the distributional effects of such changes in constraints, again without knowing the precise objective function of the government.[2]

Our paper brings together the literature on quality differentiation and the literature on mixed markets. The model we develop combines elements from each strand; however, it differs from existing models in important conceptual aspects, which we discuss below.

[2]As an analogy, consider the "law of demand," i.e., the statement that consumption of a normal good is decreasing in its price. We can prove this statement even without knowing precisely which utility function the consumer maximizes. Knowledge of the utility function would allow us to compute the actual bundle the consumer purchases, but it would not change the law of demand.

Quality differentiation. A number of authors have examined how firms choose the quality of their products as a strategic variable (Shaked and Sutton 1982; Choi and Shin 1992; Motta 1993; Ferreira and Thisse 1996; Wauthy 1996; Lehmann-Grube 1997; Wang and Yang 2001; Garella and Petrakis 2008). One of the main findings is that endogenous quality differentiation divides consumers into market segments and hence relaxes the competitive pressure faced by each firm. With very few exceptions (specifically, Shaked and Sutton 1982), the papers in the quality choice literature model consumer heterogeneity as differences in a "taste for quality" parameter. The advantage of this approach is that it yields very tractable models in which to examine firm competition in quality. We depart from the assumption of taste heterogeneity and examine, instead, a framework with homogenous preferences but income inequality. This is technically more challenging but provides the foundation for the analysis of distributional effects of public policies: Governments provide goods such as education and healthcare not because they are concerned about the welfare of consumers who have little taste for education or healthcare, but because they are concerned about the welfare of consumers who value these goods but cannot afford them (or can afford only an insufficient amount). To analyze policies that redistribute resources to those consumers, a model with income differences is necessary.

Mixed markets. The literature on mixed markets examines the interaction between private firms and public enterprises (De Fraja and Delbono 1990; Cremer *et al.* 1991; Barros 1995; Anderson *et al.* 1997; Matsumura 1998; Pal 1998; Bárcena-Ruiz 2007; Ishida and Matsushima 2009; Matsumura and Ogawa 2014). In this literature, the firms' products are typically modelled as either perfect or imperfect substitutes with a fixed demand function, which implies that quality is not an endogenous choice made by the firms. In contrast, our our model allows both the private firm and the public firm to set the quality of their products. More importantly, the mixed markets literature has in large part been motivated by the liberalization of industries that used to be comprised of state-owned monopolies, such as the telecommunications, railroad, or utilities industries. The goal in this context is to establish equal competition between the former public monopolies and new private entrants, by eliminating subsidies and other advantages enjoyed by the public firms. This is not the case in the markets we consider. To the contrary, public providers of education, healthcare, and the like often enjoy significant financial subsidies that are not available to their private rivals, and providing such advantages is an explicit part of redistributive government policy. Historically, governments have increased subsidies to public firms, resulting in an overall growth of the public sector.

The remainder of the paper is organized as follows. In Section 2 we present our model of mixed oligopoly. In Section 3 we develop our analytical results concerning the private firm's response to changes in the public firm's price and quality and in the income distribution. In Section 4 we apply these results to a number of different policy scenarios. We then illustrate the possible distributional implications of these scenarios in Section 5, using a numerical example. Section 6 concludes. An Appendix contains proofs that are too long to be included in the main text.

2 Model

2.1 The market

We study a market with two goods, X and Y. Good Y is a good such as education, healthcare, or recreation, while good X is a numeraire good, or "everything else." Every individual demands one unit of Y. The price of a unit of good X is one; thus, if an individual with income m purchases one unit of Y at price p she consumes $m - p$ units of X.

The quality of good Y is denoted by $\theta \geq 0$. An individual who consumes x units of good X and one unit of good Y of quality θ obtains utility $u(x, \theta)$. We assume that the utility function u is strictly quasi-concave, twice continuously differentiable, and satisfies the following properties for $(x, \theta) \gg (0, 0)$: $u_x > 0$, $u_\theta > 0$, $u_{xx} \leq 0$, $u_{x\theta} \leq 0$, $u_{x\theta} > 0$. The last property ensures that higher-income consumers have a higher willingness to pay for quality.

Good Y is available in the following price-quality combinations:$(p_h, \theta_h) \gg (p_l, \theta_l) \gg (0, 0)$. The first option, (p_h, θ_h), is offered by a private firm, and the second option, (p_l, θ_l), is offered by a public firm. The prices and qualities of both suppliers will be made endogenous later. The third option, $(0, 0)$, is an "outside option" and interpreted as not consuming good Y. For both providers, the cost of producing one unit of Y at quality θ is $c(\theta) = \theta$.[3]

Individuals differ in their incomes. We assume that incomes are distributed according to an atomless distribution F with convex support, density f, and non-decreasing hazard rate $\lambda(m) = f(m)/[1 - F(m)]$. An individuals cannot spend more than her income. Thus, high-income individuals, i.e., those with $m \geq p_h$, can buy good Y from the public or the private firm. Middle-income individuals, i.e., those with $m \in [p_l, p_h)$, can buy Y from the public provider only. Individuals with incomes $m < p_l$ are unable to buy Y.

2.2 The individual's choice

Each individual chooses wether to purchase one unit of good Y, and if so, from which of the two providers. Depending on (p_l, θ_l) and (p_h, θ_h), one or both providers can have a

[3]This functional form is without loss of generality. As long as $c'(\theta) > 0$ and $c''(\theta) \geq 0$, one can always adjust the utility function to translate the model into one where $c(\theta) = \theta$.

zero market share. For example, if the private firm's price-quality combination is very unattractive compared to the price-quality combination offered by its public competitor, all individuals will prefer to buy from the public firm (if they buy good Y at all).

For the purpose of this paper, this is not an interesting case.[4] We want to focus on outcomes in which some individuals buy from the public firm, others buy from the private firm, and yet others do not buy good Y at all. In such situations, the two firms are engaged in meaningful competition; at the same time there is scope for government policy aimed at increasing the number of consumers who purchase good Y ("expanding access"). The following result states that, in this case, the market is segmented by income:

Lemma 1. *Suppose that a positive mass of individuals purchase good Y from the public provider, a positive mass purchase Y from the private provider, and a positive mass do not purchase Y. There exist thresholds $0 < \underline{m} < \overline{m}$ such that the individuals with incomes $m \leq \underline{m}$ are the ones who do not consume Y, the individuals with incomes $m \in (\underline{m}, \overline{m}]$ are the ones who buy Y from the public provider, and individuals with incomes $m > \overline{m}$ are the one who buy Y from the private provider.*[5]

Note that this market segmentation result mirrors similar results in Gabszewicz and Thisse (1979) and Shaked and Sutton (1982), derived for the specific utility function $u(x, \theta) = x\theta$. In the former model, the qualities θ_l and θ_h are exogenous, while in the latter, they are chosen by the firms. However, in neither model is there a cost associated with producing a good of higher quality.

2.3 The private firm's problem

We assume that the private provider is a single firm that maximizes its profit,

$$\pi(\theta_h, p_h) \;=\; \big(1 - F(\overline{m})\big)\big(p_h - \theta_h\big) \tag{1}$$

by choosing quality θ_h and price p_h. The term $1 - F(\overline{m})$ is the number of units sold by the public firm, and the term $p_h - \theta_h$ is the per-unit profit margin. The first order conditions with respect to p_h and θ_h can be written (in one line) as follows:

$$\frac{\partial \overline{m}}{\partial p_h}\big(p_h - \theta_h\big) \;=\; \frac{1}{\lambda(\overline{m})} \;=\; -\frac{\partial \overline{m}}{\partial \theta_h}\big(p_h - \theta_h\big). \tag{2}$$

[4]It also cannot be a long-run equilibrium of the market, as the private firm would either exit the market or would adjust its price and quality so as to gain a positive measure of consumers, and the same is true for a public firm that has no customers.

[5]The marginal individual of income $\underline{m}$ could also purchase Y from the public provider, and the marginal individual of income $\overline{m}$ could also purchase Y from the private provider. These changes would not affect the results.

It follows that, at the private firm's optimal price-quality combination,

$$\frac{\partial \overline{m}}{\partial p_h} \;=\; -\frac{\partial \overline{m}}{\partial \theta_h}. \tag{3}$$

If this condition did not hold, the firm could change quality and price by identical amounts and gain customers (decrease $\overline{m}$) while leaving its profit margin unchanged. This would increase the firm's profit, which is impossible at the profit maximum.

Note that the income threshold $\overline{m}$, at which the a consumer is indifferent between purchasing from the public or private provider, is defined by the indifference condition

$$u(\overline{m} - p_l, \theta_l) \;=\; u(\overline{m} - p_h, \theta_h). \tag{4}$$

Differentiating the right-hand side of (4) with respect to p_h and θ_h, and treating $\overline{m}$ as a function of these variables, we have

$$\begin{aligned} \frac{\partial \overline{m}}{\partial p_h} &= \frac{u_x(\overline{m} - p_h, \theta_h)}{u_x(\overline{m} - p_h, \theta_h) - u_x(\overline{m} - p_l, \theta_l)} \;>\; 0, \\ \frac{\partial \overline{m}}{\partial \theta_h} &= \frac{u_\theta(\overline{m} - p_h, \theta_h)}{u_x(\overline{m} - p_l, \theta_l) - u_x(\overline{m} - p_h, \theta_h)} \;<\; 0, \end{aligned}$$

and substituting these expressions back in (3) we obtain

$$u_x(\overline{m} - p_h, \theta_h) \;=\; u_\theta(\overline{m} - p_h, \theta_h). \tag{5}$$

That is, the marginal utility of the quantity of X equals the marginal utility of the quality of Y, for the individual located at the public-private income threshold $\overline{m}$ and assuming this individual purchases from the private provider. This is also easy to interpret: If, for example, $u_\theta(\overline{m} - p_h, \theta_h)$ was larger than $u_x(\overline{m} - p_h, \theta_h)$, the firm could raise its quality by some amount and raise price by a larger amount, thereby increasing its profit margin while leaving the number of customers unchanged.

2.4 The public firm's problem

Most of our results examine the private firm's response to changes in the public firm's price and quality. As we discussed in the introduction, these changes themselves can be viewed as resulting from underlying changes either in broadly specified government objectives (e.g., an initiative to "enhance access" to good Y) or in the public firm's fiscal constraints (e.g., a cut in an operating subsidy), or a combination of these.

We assume that the public firm maximizes some objective function W by choice of p_l and θ_l, subject to a budget constraint. In order to examine how changes in p_l and

θ_l induce changes in p_h and θ_h, a precise formal specification of W is not needed—in fact, it would complicate matters unnecessarily.[6] What is more important is the budget constraint, which can take two forms. First, the public firm may face a constraint on its per-customer profit,

$$p_l - \theta_l \ \geq \ -b,$$

where $b \geq 0$ is a subsidy per customer served (e.g., a grant per student enrolled at a public university). Alternatively, it may face a constraint on its total profit,

$$\big(F(\overline{m}) - F(\underline{m})\big)\big(p_l - \theta_l\big) \ \geq \ -B,$$

where $B \geq 0$ is a fixed operating subsidy (e.g., a block grant paid to a healthcare provider). In both cases, if the budget constraint binds we have $\theta_l \geq p_l$, that is, the public firm sets price below cost. Furthermore, if b or B decreases, then the public firm must either raise its price to increase operating revenue, or lower its quality to decrease operating costs, or both.

An implicit assumption underlying our analysis is that, by choosing p_l and θ_l, the public firm anticipates the optimal choice of p_h and θ_h by the public competitor. In other words, we assume a Stackelberg model of competition, with the public firm being the leader and the private firm being the follower. This choice reflects that idea that the public firm adjusts its price and quality in response to changes in policy objectives and fiscal constraints, while the private firm adjusts its price and quality in response to changes in its competitive environment (which consists of the public firm).

2.5 Remarks

In the model presented above, we made several assumptions that deserve a brief discussion. First, we assume that the private firm supplies a higher quality at a higher price than does the public firm. Our model is not intended to answer the question why or how

[6] Knowledge of the social welfare function would be required, of course, if we wanted to pin down the actual equilibrium values of p_l, θ_l, p_h, and θ_h. In this case, the "standard approach" would be to let

$$U(m) \ = \ \begin{cases} u(m, 0) & \text{if } m < \underline{m}, \\ u(m - p_l, \theta_l) & \text{if } \underline{m} \leq m < \overline{m}, \\ u(m - p_h, \theta_h) & \text{if } m \geq \overline{m} \end{cases}$$

denote the indirect utility of an individual with income m, and then assume that the government maximizes the Bergson-Samuelson social welfare function

$$W \ = \ \int_m \Psi(U(m))dF(m),$$

where Ψ is some strictly increasing and weakly concave function. If Ψ is strictly concave, a higher weight is placed on increasing the utility of poorer individuals. This specification puts no weight on the the private firm's profits, which is reasonable in the contexts we consider.

equilibria with this property might arise. Instead, our focus is on comparative statics of such equilibria. We point out, however, that the sorting we assume is commonly observed in markets in which public entities enter with the explicit goal of providing affordable choices to lower-income segments of the population.

Second, we assume that the private firm maximizes profits. In reality, of course, many private firms are non-profit firms. For example, most private universities in the United States are non-profit institutions, and the few that are for-profit operate at the very low end of the price-quality spectrum. However, even non-profit private firms may have a short-term objective to maximize the difference between revenues and costs—for instance, to pay for capital investments or to build up their endowments. Since our results do not depend on whether the private firm's profit is distributed to shareholders or retained within the organization, the model captures the behavior of these non-profit firms as well.

Third, we assume that there is exactly one private and one public firm and that each firm supplies exactly one variety. Since consumers are heterogeneous in their incomes, and hence in their willingness to pay for quality, new niche firms could enter profitably[7] or existing firms could increase their profits if they price-discriminated by offering additional price-quality combinations. We sidestep these issues in order to keep our model tractable.

3 The Private Firm's Best Response

In this section, we take a closer look at the profit maximizing price-quality choice of the private firm and examine how this choice adjusts to changes in public policy, i.e., to changes in p_l and θ_l. In order to do so, it will be convenient to first transform the private firm's problem—described formally in Section 2.3—into a dual. This dual problem closely resembles a conventional monopolistic pricing setup, in which the firm chooses a location on a demand curve. Changes in p_l and θ_l alter this demand curve, so that studying private responses to public policy changes becomes a problem of a monopolist re-optimizing after a demand change.

3.1 Dual formulation

Fix a given public price-quality pair (p_l, θ_l). Also fix a marginal individual $\overline{m}$ and suppose the private firm serves all individuals with incomes $\overline{m}$ and above. The (p_h, θ_l)-pair that maximizes the firm's profit, conditional on selling to individuals with incomes $m \geq \overline{m}$, satisfies the following two conditions.

[7]This depends, however, on the particular assumptions made on the timing of price and quality choices; see, e.g., Shaked and Sutton (1982).

First, individual $\overline{m}$ is indifferent between purchasing from the private and public firm. This is condition (4), which we rewrite as

$$I(p_h, \theta_h|\overline{m}) \;\equiv\; u(\overline{m} - p_h, \theta_h) - u(\overline{m} - p_l, \theta_l) \;=\; 0.$$

The locus $I(\cdot|\overline{m}) = 0$ is the indifference curve consisting of all (p_h, θ_h)-pairs for which individuals $\overline{m}$'s utility is constant and equal to $u(\overline{m} - p_l, \theta_l)$. Second, for individual $\overline{m}$ the marginal utilities of the two goods are equal. This is condition (5), which we rewrite as

$$H(p_h, \theta_h|\overline{m}) \;\equiv\; \frac{u_x(\overline{m} - p_h, \theta_h)}{u_\theta(\overline{m} - p_h, \theta_h)} \;=\; 1.$$

The locus $H(\cdot|\overline{m}) = 1$ is individual $\overline{m}$'s "iso-MRS curve." In Figure 1, the left graph plots both curves for a fixed public price-quality pair $A = (p_l, \theta_l)$ and a marginal individual with income $\overline{m}_1$. Note that our assumptions on u implies an upward sloping and convex indifference curve and a downward sloping iso-MRS curve. The intersection of these curves lies at the point B, meaning that the point B maximizes the private firm's profit conditional on a marginal buyer with income $\overline{m}_1$.

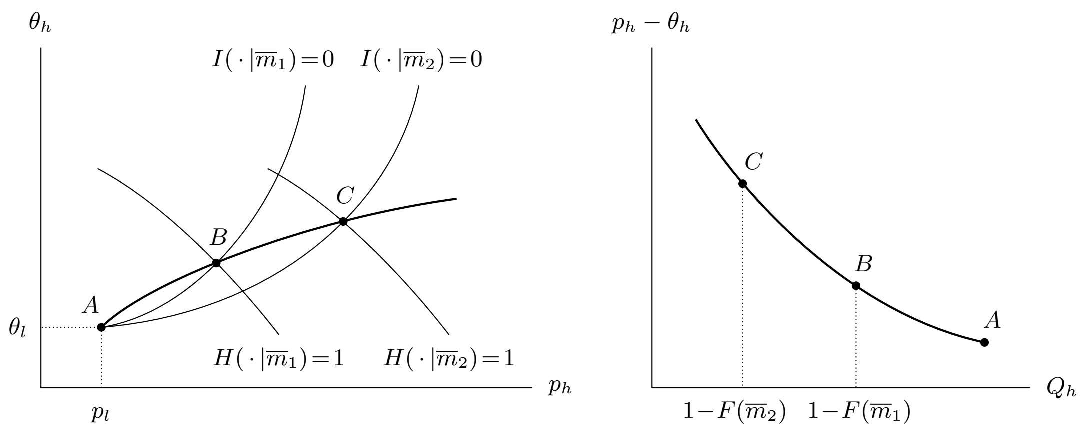

Figure 1: The private firm's price-quality locus (left) and resulting demand curve (right).

Now construct a new intersection using a different marginal individual, with income $\overline{m}_2 > \overline{m}_1$, say. The new iso-MRS curve, $H(\cdot|\overline{m}_2) = 1$, is a copy of the previous curve shifted to the right by exactly $\overline{m}_2 - \overline{m}_1$. The new indifference curve, $I(\cdot|\overline{m}2) = 0$, is a "stretched out" copy of the previous curve, with the horizontal distance between the two curves always being less than $\overline{m}_2 - \overline{m}_1$.[8] Thus, the intersection of the new indifference

[8]Suppose the curve stayed in place. Then individual $\overline{m}_2$ would strictly prefer the points on that curve to A (see Lemma 1). Therefore, to make him indifferent, p_h must increase given θ_h. On the other hand, if the curve moved to the right by exactly $\overline{m}_2 - \overline{m}_1$, the utility level associated with the new curve would be exactly the same as that associated with the previous one, which is $u(\overline{m}_1 - p_l, \theta_l)$. Since this is

curve and the new iso-MRS curve, C, lies to the right and above the previous intersection, B. Repeating these steps for other values of $\overline{m}$, one can trace out an upward sloping curve containing all (p_h, θ_h)-pairs that can be constructed in this fashion. In the diagram, this is the thick curve passing through B and C. We shall call this curve the firm's price-quality locus. Each point on the locus is a profit-maximizing price-quality pair for some value of $\overline{m}$. Because the slope of each indifference curve is exactly one at that intersection, and the price-quality locus crosses the indifference curves from above, it must have a slope strictly between zero and one. Thus, as one moves along the locus from A toward C, the variables $\overline{m}$, p_h, and θ_h all increase, as does the difference $p_h - \theta_h$.

In a final step, we can translate the price-quality locus into a "demand curve" that plots the quantity of the private firm, $Q_h = 1 - F(\overline{m})$ against its net price, or profit margin, $p_h - \theta_h$. This demand curve is depicted in the right diagram of Figure 1. Similar to a monopolist choosing a price-quantity pair from a given demand curve, we can think of the private firm as choosing a combination of a quantity and profit margin from the curve shown in the right diagram of Figure 1. The total profit the private firm earns is the area of rectangle between the origin and the chosen point on the curve, and the firm selects the point at which this area is maximized.

3.2 Effects of changes in p_l and θ_l

Suppose the public firm changes its price-quality combination from (p_l, θ_l) to $(\hat{p}_l, \hat{\theta}_l)$. Denote by

$$\Delta(m) \equiv u(m - \hat{p}_l, \hat{\theta}_l) - u(m - p_l, \theta_l) \tag{6}$$

the resulting utility change for an income-m consumer when purchasing good Y from the public firm. Pick a point on the private firm's price-quality locus, S say. Let this point be associated with marginal individual $\overline{m}$ and assume that this individual is made worse off by the change (i.e., $\Delta(\overline{m}) < 0$). Note that the change in the public firm's offering does not affect the iso-MRS curve $H(\cdot|\overline{m}) = 1$, as this curve does not depend on p_l and θ_l. However, it does affect the indifference curve $I(\cdot|\overline{m}) = 0$: Since $\Delta(\overline{m}) < 0$, the new indifference curve is associated with a lower utility than the original curve, and is thus located below and to the right of the original curve. Thus, the policy change moves point S along the iso-MRS curve in a south-easterly direction to S', as depicted in the left diagram in Figure 2. By the same logic, had we assumed that $\Delta(\overline{m}) > 0$ point S' would be located to the north-west of S.

less than $u(\overline{m}_2 - p_l, \theta_l)$, individual $\overline{m}_2$ would strictly prefer A to points on the new curve. Therefore, to make him indifferent, the increase in p_h must be less than $\overline{m}_2 - \overline{m}_1$, given θ_h. It follows that the new indifference curve lies to the right of the old one, with a horizontal distance between the two curves that is always less than $\overline{m}_2 - \overline{m}_1$.

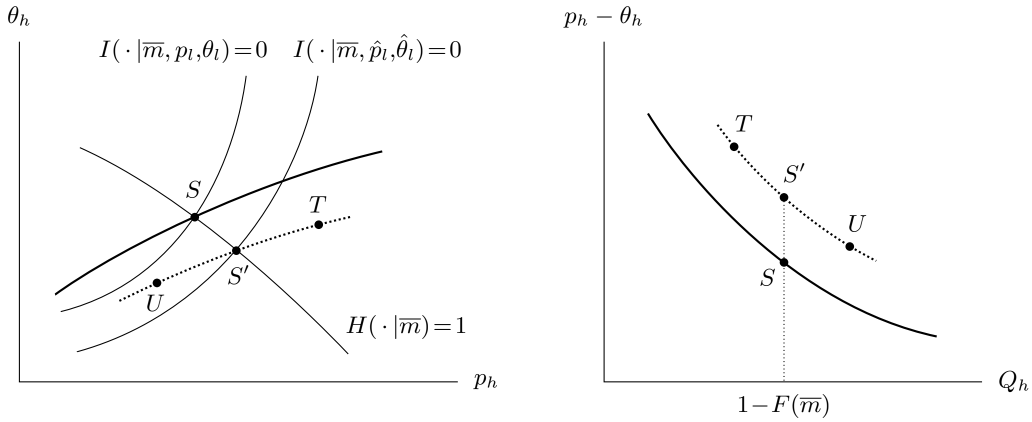

Figure 2: Local effects of changes in (p_l, θ_l) on the private firm's price-quality locus (left) and the demand curve (right).

If the private firm did not change its output, the above argument implies that the firms should increase p_h and decrease θ_h if $\Delta(\overline{m}) < 0$; and that it should decrease p_h and increase θ_h if $\Delta(\overline{m}) > 0$. This is intuitive: If $\Delta(\overline{m}) < 0$, the public firm's product variety becomes a less attractive substitute for the marginal individual at the public-private threshold, thus reducing the competitive pressure faced by the private firm. It can hence increase its price and reduce its quality without losing customers. (In fact, if it did not increase price or reduced quality the firm would attract new customers with incomes just below the original cutoff $\overline{m}$.) The opposite holds if $\Delta(\overline{m}) > 0$.

Of course, following a change in the public firm's price and quality the private firm will generally adjust its output, by selecting a different marginal consumer $\overline{m}$. In Figure 2, this corresponds to a move along the new price-quality locus starting at S', either to north-east (increasing $\overline{m}$/decreasing Q_h) or to the south-west (decreasing $\overline{m}$/increasing Q_h). If the private firm reduces its output, e.g., by moving from S' to T, its price at the new optimum T must be higher price than at the original optimum S, as T is unambiguously to the right of S. However, the effect on quality is ambiguous—it depends on whether the downward move from S to S' is outweighed by the upward move from S' to T . Conversely, if the private firm increases its output by moving from S' to U, its quality at the new optimum U is unambiguously lower than at the original optimum S, while the price effect is ambiguous.

Similar arguments can be made if $\Delta(\overline{m}) > 0$. Thus, we obtain a total of four cases, depending on (i) whether the public policy change makes the private firm's marginal consumer better or worse off, and (ii) whether the private firm expands or contracts in response. In each case, we can determine the direction of the total change in either the private firm's price or its quality, but not both. The four cases are summarized in the following Proposition:

Proposition 2. *Consider a change in public policy from (p_l, θ_l) to $(\hat{p}_l, \hat{\theta}_l)$ and define $\Delta(m)$ as in (6). Let $\overline{m}$ be the income threshold at which a consumer was indifferent between the two firms before the change.*

(a) Suppose $\Delta(\overline{m}) < 0$. If the private firm reduces its output, p_h increases while the effect on θ_h is uncertain. If the private firm expands its output, θ_h decreases while the effect on p_h is uncertain.

(b) Suppose $\Delta(\overline{m}) > 0$. If the private firm reduces its output, θ_h increases while the effect on p_h is uncertain. If the private firm expands its output, p_h decreases while the effect on θ_h is uncertain.

Proposition 2 places some constraints on the adjustments of the private firm's price, quality, and quantity, following a change in public policy. To say more, however, we need to examine the firm's quantity response in more detail.

Consider again the case $\Delta(\overline{m}) < 0$. As discussed already, if the firm were to maintain its quantity it would increase its price and reduce its quality, resulting in a higher profit margin and thus an upward move of the firm's demand curve at the current quantity $Q_h = 1 - F(\overline{m})$. This is illustrated in the right diagram in Figure 2. However, whether the firm responds to this change with an increase or decrease in output depends not on whether demand has increased or decreased, but on whether demand has become more or less elastic. If it is more elastic after the change—as would be the case, for example, if the demand curve simply shifted upward with no change to its slope—the firm expands; Proposition 2 then implies that θ_h must be lower at the firm's new optimum. Similarly, if elasticity decreases the firm reduces output and, therefore, increases p_h.

It is difficult to determine the effect of changes in p_l and θ_l on the elasticity of the private firm's demand in general. For the special case of homothetic preferences, we can state the following (the proof is in the Appendix):

Proposition 3. *Suppose the consumers' preferences are homothetic; without loss of generality in this case, assume that u is homogeneous of degree 1. Consider a change in public policy from (p_l, θ_l) to $(\hat{p}_l, \hat{\theta}_l)$, and define $\Delta(m)$ as in (6). Let $\overline{m}$ be the income threshold at which a consumer was indifferent between the two firms before the change.*

(a) If $\Delta'(\overline{m}) > \lambda(\overline{m})\Delta(\overline{m})$, the private firm's demand becomes more elastic at $1 - F(\overline{m})$ after the policy change, resulting in a higher quantity supplied by the private firm.

(b) If $\Delta'(\overline{m}) < \lambda(\overline{m})\Delta(\overline{m})$, the private firm's demand becomes less elastic at $1 - F(\overline{m})$ after the policy change, resulting in a lower quantity supplied by the private firm.

The determination as to which case of Proposition 3 applies is straightforward when $\Delta'(\overline{m})$ and $\Delta(\overline{m})$ are of opposite sign. This is the case, in particular, when the change in public policy concerns a change in the public firm's price without a change in quality (see Proposition 5 below). Generally, whether case (a) or (b) of the result applies depends not only on the policy change but also on the consumers' preferences and the income distribution.

3.3 Effects of changes in the income distribution

In the preceding Section 3.2 we examined how the private firm adjusts its output, price, and quality in response to changes in the public firm's price and quality, holding constant the the income distribution stayed the same. We can also examine how the private firm responds to changes in the income distribution, holding constant the public firm's price and quality. This information is important, for instance, when we evaluate the effects of income taxation to pay for subsidies to the public firm.

We say that income distribution $\hat{F}$ is *less dispersed* than income distribution F if there exists a function $\sigma : [0, \infty) \to [0, \infty]$ with $\sigma' \geq 0$ such that $\hat{F}(m) = F(m + \sigma(m))$. For example, subjecting all individuals to a weakly increasing income tax schedule result in a less dispersed distribution of net incomes, compared to the distribution of gross incomes. Everything else equal, this change does not alter the private firm's price-quality locus, because this locus is independent of F. However, the translation of the price-quality locus into the demand curve is affected. Recall that for each profit margin $p_h - \theta_h$ and associated threshold income $\overline{m}$, the firm sells a quantity of $1 - F(\overline{m})$ units. If F is replaced by $\hat{F}$, and $\hat{F}$ is less dispersed than F, then $1 - \hat{F}(\overline{m}) = 1 - F(\overline{m} + \sigma(\overline{m})) < 1 - F(\overline{m})$. That is, for a given profit margin the private firm now sells a lower quantity, meaning that the demand curve moves inward. The following result provides conditions under which this inward shift leads the private firm to reduce both its price and its quality:

Proposition 4. *Consider a change in the income distribution from F to $\hat{F}$, where $\hat{F}(m) = F(m + \sigma(m))$ for some $\sigma : [0, \infty) \to [0, \infty)$ with $0 \leq \sigma'(m) < \infty$ (i.e., $\hat{F}$ is less dispersed than F). Let $\lambda(m) = f(m)/(1 - F(m))$ be the hazard rate of the income distribution F at m. Let $\overline{m}$ be the income of the consumer who indifferent between the two firms before the change.*

Suppose that at least one of the following holds: (i) $\lambda'(\overline{m}) > 0$ and $\sigma(\overline{m}) > 0$; (ii) $\sigma'(\overline{m}) > 0$. Then, as a result of the change in the income distribution, the private firm's reduces both its price p_h and its quality θ_h.

In Section 4.4 below we will apply this result to examine the effects of an income tax put in place to finance losses that the public firm makes.

4 Applications

The results developed in Section 3 permit several insights into the effects of changes in the public firm's policy mandates or in its funding on market outcomes.

For example, a public healthcare provider whose operating subsidy is cut but that is required by law to provide a certain level of care will have no choice but to increase the price for its services, and our model can predict the market effect of this subsidy cut. Likewise, consider a public university that is faced with a government mandate to enroll more low-income students, but without additional subsidies to fund this effort. For the university to fulfill its new mandate, it needs lower its tuition to attract the low-income students and it needs to find some way of saving costs to offset the the reduced revenue per student (e.g., by increasing class size or reducing expenditures on facilities). The results of the previous section can be used to inform us about the response of the public firm's private competitor, and about who gains and who loses from the policy change.

In the following, we consider several specific policy changes: A fully funded price reduction, a fully funded quality improvement, an unfunded price reduction, and a price reduction funded by a progressive income tax scheme.

4.1 Fully funded price reductions

As discussed in the introduction, many government agencies have an expressly stated objective of providing affordable access to the goods they produce. Thus, we now consider a change in public policy that makes good Y more affordable via a price decrease. If the price reduction is funded by an appropriate subsidy paid to the public provider, the public firm's quality will stay the same. In our price-quality diagrams, this corresponds to a horizontal move of the point (p_l, θ_l) to the left.

What is the effect of this policy change on the private firm? Note that the price decreases makes all individuals unambiguously better off if they purchase the public variety; thus, we have $\Delta(m) > 0$ and part (b) in Proposition 2 applies. Furthermore, because $u_{xx} < 0$, the utility gain is less for richer individuals. Thus, $\Delta'(m) < 0$ and part (b) of Proposition 3 applies. Together, the two results imply an increase in $\overline{m}$ and an increase in θ_h, i.e., the private firm reduces its output and increases its quality, as summarized in the following result:

Proposition 5. *Assume that the consumers' preferences are homothetic. If the public firm's price decreases without a change in the public firm's quality, then case (b) of Proposition 2 and Proposition 3 applies. That is, the private firm's quantity decreases and its quality increases.*

At the same time, the lower threshold income $\underline{m}$, at which a consumer is indifferent between not purchasing good Y and purchasing one unit of good Y from the public provider, decreases. To see this, differentiate the condition $u(\underline{m}, 0) = u(\underline{m} - p_l, \theta_l)$ with respect to p_l, to get

$$\frac{\partial \underline{m}}{\partial p_l} = \frac{u_x(\underline{m} - p_l, \theta_l)}{u_x(\underline{m} - p_l, \theta_l) - u_x(\underline{m}, 0)} > 0.$$

Thus, an externally funded price reduction for the publicly provided good causes $F(\overline{m}) - F(\underline{m})$ and $1 - F(\overline{m})$ to move in opposite directions: An increase in the public supply of Y is accompanied by a decrease in the private supply, thereby generating a crowding-out effect.[9]

Corollary: Cutting subsidies while maintaining quality. Note that the same scenario "in reverse" describes a situation in which government reduces subsidies to the public firm while requiring it to maintain its current level quality. In this case, we have a horizontal move of the point (p_l, θ_l) to the right, and the same arguments as above imply that the private firm grows and reduces its quality.

4.2 Fully funded quality improvements

Next, consider an increase in the public provider's quality θ_l. If the quality improvement is fully funded by an appropriate subsidy to the public firm, the firm can maintain its current price level. This means we are considering a vertical upward move of the point (p_l, θ_l).

Since the quality improvement makes all individuals unambiguously better off if they purchase the public bundle, we have $\Delta(m) > 0$ for all m and part (b) in Proposition 2 applies. At the same time, since $u_{x\theta} > 0$, the quality increase benefits richer individuals relatively more; that is, we have $\Delta'(m) > 0$ for all m. Without further assumptions on the consumers' preferences and the distribution of incomes, either case (a) or (b) of Proposition 3 may apply. The following result considers the specific case of a Cobb-Douglas utility function and uniform income distribution:

Proposition 6. *Assume that the consumers' preferences can be represented by a Cobb-Douglas utility function (i.e., $u(x, \theta) = x^{\alpha}\theta^{1-\alpha}$ for some $0 < \alpha < 1$) and that incomes are uniformly distributed on $[0, M]$. If the public firm's quality increases without a change in the public firm's price, then case (b) of Proposition 2 and Proposition 3 applies. That is, the private firm's quantity decreases and its quality increases.*

[9] We cannot say anything about the response in the private firm's price following a reduction in the public firm's price. Proposition 5 states that the private firm sells a lesser quantity of a higher quality after the change. This suggests an accompanying increase in p_h, which in fact is what happens in our numerical example in Section 5. However, the lower output of the private firm could also be the direct result of the public firm's variety now being more attractive to all customers. In other words, even if it decreased p_h slightly, the private firm would have sold a lower quantity after the policy change.

Thus, as was the case with a fully funded price reduction, the private firm responds to a fully funded quality improvement by increasing $\overline{m}$ and increasing θ_h. Unlike in the previous scenario, the public firm will not grow at the lower end (with Cobb-Douglas preferences, the lower threshold income level at which an individual is indifferent between not buying Y and buying one unit from the public firm is $\underline{m} - p_l$, regardless of θ_l).[10] However, it is still the case that $F(\overline{m}) - F(\underline{m})$ increases and $1 - F(\overline{m})$ decreases, so we have a crowding-out effect again.

Corollary: Cutting subsidies while maintaining affordability. As before, we can reverse the scenario to describe a situation in which government reduces subsidies to the public firm while requiring it to maintain its current price level. In this case, the public firm must reduce its quality and we have a vertical downward move of the point (p_l, θ_l). With Cobb-Douglas utility and uniformly distributed incomes, this implies that the private firm grows in size but reduces its quality.

4.3 Unfunded price reductions

Next, consider a scenario in which the public firm relies on a per-customer subsidy $b \geq 0$ to operate, and that government mandates that good Y become more affordable. Assume that no additional subsidy is provided to the public firm to finance the price reduction. This means that that public provider must cut costs per customer by the same amount as it cuts price. This cost reduction can only be achieved by lowering quality by an amount equal to the price cut, leading to a diagonal move of the point (p_l, θ_l) downward and to the left.

Note that, unlike in the previous two scenarios, the effect of a simultaneous price and quality reduction on the public firm's customers will not be uniform: Low-income customers of the public firm, whose marginal utility of income is higher than that of quality, will benefit from the change. The opposite holds for high-income customers of the public firm, who will be hurt by the change.

The individual who matters for our results is the marginal individual with income $\overline{m}$. This individual is made worse off by the change. We know this because of the private firm's profit maximizing behavior: The private firm sets price and quality so that the marginal utility of both goods is equal for an income-$\overline{m}$ consumer. Since $(p_l, \theta_l) \ll (p_h, \theta_h)$, $u(\overline{m}-p_h, \theta_h) = u(\overline{m}-p_l, \theta_l)$, and since u is strictly quasi-concave, individual $\overline{m}$'s marginal

[10] In general, $\underline{m}$ decreases when θ_l increases. To see this, differentiate $u(\underline{m}, 0) = u(\underline{m} - p_l, \theta_l)$, with respect to θ, to get

$$\frac{\partial \underline{m}}{\partial p_l} = -\frac{u_\theta(\underline{m} - p_l, \theta_l)}{u_x(\underline{m} - p_l, \theta_l) - u_x(\underline{m}, 0)},$$

which is negative if the marginal utility of another quality unit is positive for the individual with income $\overline{m}$.

utility of income must be lower than that of quality at the public provider's bundle (p_l, θ_l). This implies that lowering p_l and θ_l by the same amount decreases $u(\overline{m} - p_l, \theta_l)$. Thus, we have $\Delta(\overline{m}) < 0$ and case (a) of Proposition 2 applies.

The utility loss is even larger for consumers with income $m > \overline{m}$, who care relatively more about quality and relatively less about price, so $\Delta'(\overline{m}) < 0$. Thus, as in the previous scenario, without knowing more about preferences or the income distribution, it is impossible to say which of the two cases in Proposition 3 applies. The following result considers, again, the specific case of a Cobb-Douglas utility function and uniform income distribution:

Proposition 7. *Assume that the consumers' preferences can be represented by a Cobb-Douglas utility function (i.e., $u(x, \theta) = x^{\alpha}\theta^{1-\alpha}$ for some $0 < \alpha < 1$) and that incomes are uniformly distributed on $[0, M]$. If the public firm's quality and price decrease by the same amount, then case (a) of Proposition 2 and Proposition 3 applies. That is, the private firm's quantity increases and its quality decreases.*

Note that this response is the *opposite* of the private firm's response to a fully funded price reduction, analyzed in Section 4.1. In terms of the two firms' relative size, an unfunded affordability improvement can result in a crowding-out effect or a crowding-in effect, and an example of the latter is given later.

Corollary: Unfunded quality improvements. The same scenario "in reverse" describes a situation in which government mandates that the public firm increase its quality, while not providing an additional subsidy per customer to offset the increased cost of providing higher quality. In this case, the public firm must increases its price, so that we have a diagonal move of the point (p_l, θ_l) upward and to the right. The argument above then implies that, under suitable assumptions on the consumer's preferences, the private firm reduces its quantity and increases its quality. Interestingly, this is now the *same* response as that to a fully funded quality improvement, described in Section 4.2.

4.4 Funding the public firm through taxation

When the public firm makes a negative profit—that is, if it sets $p_l < \theta_l$—we assumed that it receives a government subsidy $B > 0$ to cover the loss. The financing of this subsidy occurred outside of the model, however. We now extend our model to incorporate income taxation in order to generate the revenue necessary to pay for such subsidies.

To fund the public firm, consider a tax schedule $T : [0, \infty) \to [0, \infty)$, where $T(m)$ is the tax paid by an income-m individual. Total tax revenue raised is

$$R = \int_0^{\infty} T(m) dF(m).$$

Individuals pay their taxes $T(m)$ and can then allocate their net income $m - T(m)$ to the purchase of goods X and Y. The tax schedule T finances the public firm if there exist $(p_h, \theta_h, \underline{m}, \overline{m})$ such that

$$u(\overline{m} - T(\overline{m}) - p_l, \theta_l) = u(\overline{m} - T(\overline{m}) - p_h, \theta_h), \tag{7}$$

$$u(\underline{m} - T(\underline{m}) - p_l, \theta_l) = u(\underline{m} - T(\underline{m}), 0), \tag{8}$$

$$\frac{1}{\lambda(\overline{m})(p_h - \theta_h)} = \frac{u_x(\overline{m} - T(\overline{m}) - p_h, \theta_h)}{u_x(\overline{m} - T(\overline{m}) - p_h, \theta_h) - u_x(\overline{m} - T(\overline{m}) - p_l, \theta_l)}, \tag{9}$$

$$\frac{1}{\lambda(\overline{m})(p_h - \theta_h)} = \frac{u_\theta(\overline{m} - T(\overline{m}) - p_h, \theta_h)}{u_x(\overline{m} - T(\overline{m}) - p_l, \theta_l) - u_x(\overline{m} - T(\overline{m}) - p_h, \theta_h)}, \tag{10}$$

$$(\theta_l - p_l)\big(F(\overline{m}) - F(\underline{m})\big) = R. \tag{11}$$

(7)–(8) are the indifference conditions defining the the thresholds $\underline{m}$ and $\overline{m}$ in terms of the marginal individual's after-tax incomes. (9)–(10) are the private firm's profit-maximizing conditions, also taking into account the altered income distribution after taxation. (11) states that the tax revenue raised by the income tax scheme must equal the deficit the public firm makes. For a given public price-quality pair (p_l, θ_l) where θ_l is not too large relative to p_l, one can generally find a tax rate τ that finances the public firm, assuming that the private firm supplies the market with its own variety in a profit-maximizing way.

If $T'(m) \in [0, 1)$ for all m, the proportion of individuals in the population with net income less than or equal to $m - T(m)$ can be expressed as $\hat{F}(m - T(m)) = F(m)$. Moreover, the mapping from m to $m - T(m)$ is invertible, which means there exists a function $\sigma : [0, \infty) \to [0, \infty)$ with $\sigma'(m) \in [0, \infty)$ such that $\hat{F}(m) = F(m + \sigma(m))$. In other words, the net income distribution is less dispersed than the gross income distribution, according to our definition in Section 3.3. This enables us to compare the private firm's price and quality under the tax scenario to an alternative scenario in which the subsidy is paid for externally. In particular, Proposition 4 implies the following:

Proposition 8. *Fix (p_l, θ_l) with $p_l > \theta_l$. Let (p_h, θ_h) be the private firm's price and quality, assuming the public firm receives an external subsidy to cover its losses. Let (p'_h, θ'_h) be the private firm's price and quality under a tax schedule T that finances the public firm, with $T'(m) \in [0, 1)$ $\forall m$. If the individual who is indifferent between the two firms in the external-subsidy scenario faces a positive marginal tax rate in the tax scenario, then $p'_h < p_h$ and $\theta'_h < \theta_h$.*

5 Distributional Impact of Policy Changes

We now illustrate the distributional effects of several of the policy changes we discussed in Section 4 by means of a numerical example. Suppose that the utility function is $u(x, \theta) = x^{1/3}\theta^{2/3}$ and that incomes are uniform on $[0, 1]$. We will explore how individuals across the income spectrum are affected when either the price or quality of the public firm are altered.

To begin, consider a "baseline" policy scenario in which the public firm supplies its variety at price-quality pair $(p_l, \theta_l) = (0.25, 0.25)$. This baseline scenario is shown in the first row of Table 1. If the private firm maximizes its profit, it supplies its own variety at price-quality pair $(p_h, \theta_h) = (0.584, 0.396)$ and the marginal individual who is indifferent between purchasing from the private or public firm is the individual has income 0.746. This means that the private firm sells to the individuals in the top 25.4% of the income distribution, the public firm sells to the middle 49.6%, and the bottom 25% cannot afford good Y. Because the public firm's price equals its quality, it makes exactly a zero profit and thus does not require a subsidy (i.e., $B = 0$).

Policy	p_l	θ_l	p_h	θ_h	$\underline{m}$	$\overline{m}$	Q_l	Q_h	B	τ
Baseline scenario	.25	.25	.584	.396	.250	.746	.496	.254	0	0
Affordability mandate										
1. External funding	.15	.25	.572	.430	.150	.787	.637	.213	.064	0
2. No funding	.15	.15	.526	.286	.150	.669	.519	.331	0	0
3. Tax funding	.15	.25	.497	.410	.150	.830	.680	.270	.068	.188
Quality mandate										
4. External funding	.25	.35	.607	.535	.250	.874	.624	.126	.062	0
5. No funding	.35	.35	.573	.485	.350	.816	.466	.184	0	0
6. Tax funding	.25	.35	.524	.505	.250	.952	.702	.048	.070	.249

Table 1: Numerical examples for $u(x, \theta) = x^{1/3}\theta^{2/3}$ and $m \sim U[0, 1]$.

5.1 Improving affordability and access

Imagine now a push to reduce the size unserved population by lowering the price of the public firm's variety to $p_l = 0.15$ (i.e., only 15% of the population remain unserved). We consider three ways to achieve this goal: (1) A fully funded price reduction without a change in quality, financed through an externally provided subsidy $B > 0$; (2) an unfunded price reduction along with a matching reduction in quality to $\theta_l = 0.15$; and

(3) an price reduction that leaves quality unchanged and requires a subsidy B, funded through a income tax rate τ on all incomes above 0.15.[11] The panel labeled "Affordability mandate" in Table 1 show these three scenarios.

As predicted by Proposition 5, relative to the baseline scenario the private firm's quality increases and its quantity decreases in case 1. As predicted by Proposition 7, the opposite is true in case 2. In both case 1 and 2, the change in p_h is in the same direction as the change in θ_h. Furthermore, while in case 1 increased public supply crowds out private supply (Q_l increases and Q_h decreases relative to baseline), a crowding-in effect happens in case 2 (both Q_l and Q_h increase relative to baseline). In case 3, we use an income tax rate of $\tau = 18.8\%$ on all incomes over 0.25. Given the after-tax income distribution and a public variety $(p_l, \theta_l) = (0.15, 0.25)$, the private firm raises quality, cuts price, and reduces quantity relative to baseline. In the resulting market outcome the public firm supplies quantity $Q_l = 0.680$ and makes a loss of $B = Q_l(\theta_l - p_l) = 0.068$, which is exactly the revenue raised from the income tax. Relative to case 1, which has the same (p_l, θ_l) but without the tax, the private firm sets a lower price and quality. This is as predicted by Proposition 8.

Figure 3 plots the indirect utility of individuals in the baseline scenario as well as the three affordability scenarios. Note that an externally funded price reduction (the green line) makes all individuals better off, compared to baseline. This may seem unsurprising, given that the scenario involves an injection of resources into the economy. However, there is no guarantee in our model that such an injection always leads to a Pareto improvement. The reason is the following: While the utility gains from the policy change are largest for poorer individuals who could not afford good Y before the change but can now buy it, the gains become smaller as incomes grow, due to the diminishing marginal utility of income. They are smallest for individuals with incomes just above the new public-private income thresholds (who benefit from the private firm's quality response but are hurt by its price response), before beginning to grow again. It is possible to construct examples in which these middle-income individuals are actually hurt by an externally funded price reduction of the public firm.[12]

An unfunded price reduction (the red line) makes those individuals who could not afford good Y before better off, as well as a small proportion of individuals who were just barely able to afford good Y, but hurts all others. Interestingly, compared to this

[11] That is, the tax paid by an income-m individual is $T(m) = \max\{0, \tau(m - 0.15)\}$ and total tax revenue raised is $R = \tau \int_{0.15}^{1} (m - 0.15) dm = .36125\tau$.

[12] For example, suppose $F(m) = 1 - e^{-\frac{1}{2}m}$, $u(x, \theta) = \sqrt{x\theta}$, and $(p_l, \theta_l) = (1, 1)$. Consider a reduction of the public firm's price to $\hat{p}_l = 0.5$, leaving quality unchanged at $\theta_l = 1$. One can show that a fraction of middle-income individuals are worse off after the price reduction. These consumers previously purchased from the private firm and switch to the public provider afte the change. Even though the public firm's price-quality pair is now unambiguously better than it was before, it is not as good for middle-income individuals than the private firm's price-quality pair was before the policy change.

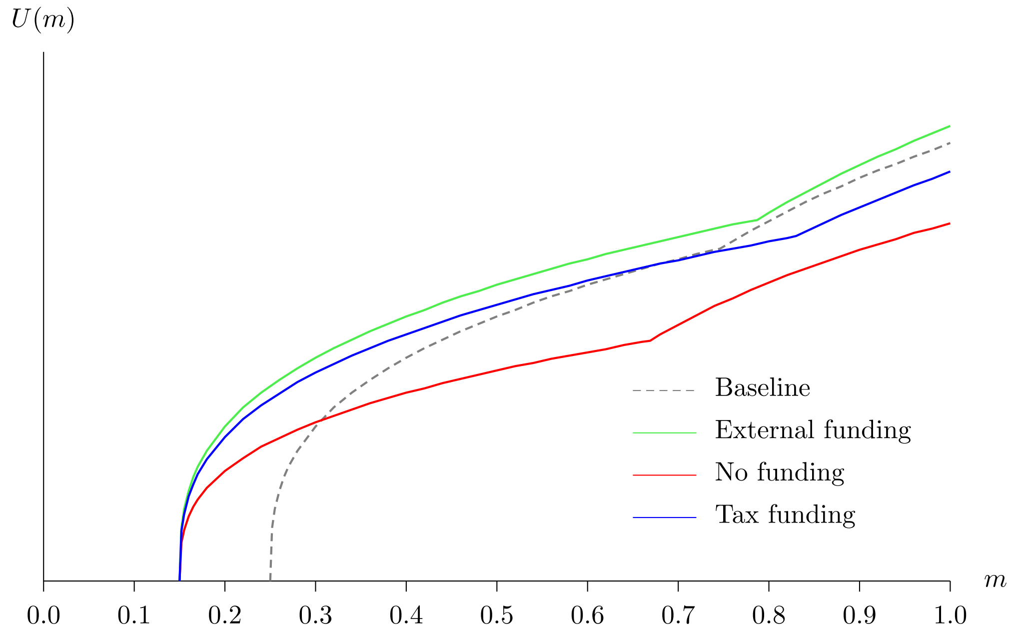

Figure 3: Distributional effects of affordability improvements.

unfunded price reduction, one that is funded by an income tax (the blue line) is preferred by all individuals, including those with high incomes who purchase from the private firm before and after the change. The reason is that these individuals, even though they are taxed to pay for the price reduction of a variety they do not consume, benefit from the quality response of the private firm. There is, again, no guarantee that these two scenarios can always be Pareto-ranked in this manner. However, the example shows that in a mixed-duopoly model with endogenous quality choice, raising taxes to make the public variety more "accessible" while maintaining its quality can have appealing welfare properties, compared to the alternative of reducing quality to pay for accessibility.

5.2 Improving quality

Next, consider an initiative to increase the quality of the publicly provided good to $\theta_l = 0.35$. Again, we consider three ways to achieve this objective: (4) A fully funded quality increase without a change in price, financed through an externally provided subsidy $B > 0$; (5) an unfunded quality increase along with a matching increase in price to $p_l = 0.35$; and (6) an internally funded quality increase that leaves price unchanged and requires a subsidy B, funded through a income tax rate τ on all incomes above p_l. The panel labeled "Quality mandate" in Table 1 show these three scenarios.

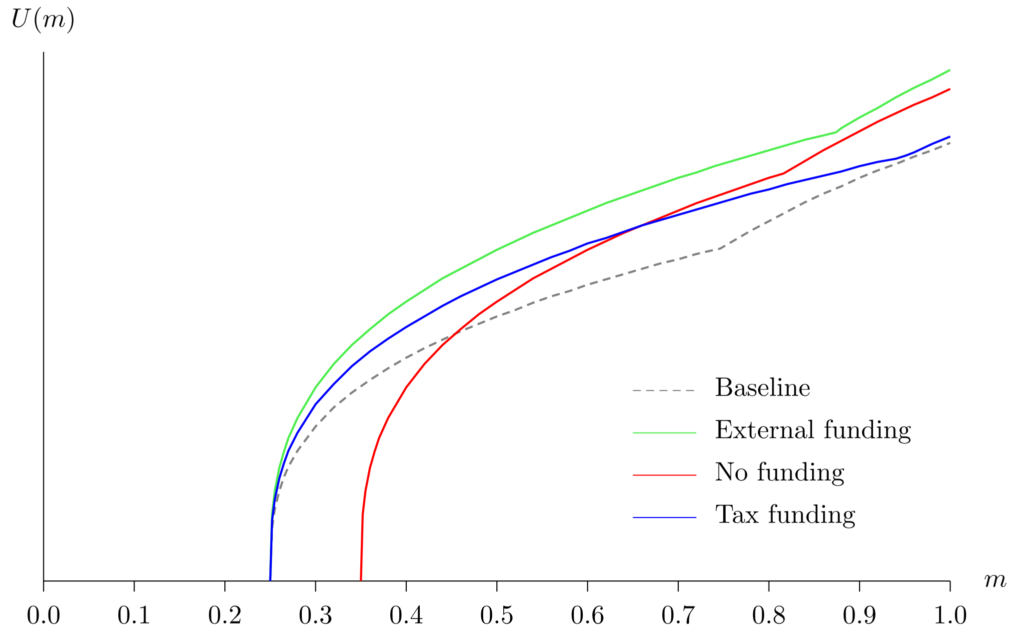

Figure 4: Distributional effects of quality improvements.

As predicted by Proposition 6, relative to the baseline scenario the private firm's quality increases and its quantity decreases in case 4; and, as predicted by Proposition 7, the same is true in case 5. Once again, the change in p_h is in the same direction as the change in θ_h. Furthermore, while in case 4 increased public supply crowds out private supply, in case 5 both Q_l and Q_h move in the same direction (they both decrease). In case 6, we use an income tax rate of $\tau = 24.9\%$ on all incomes above 0.30. Given the after-tax income distribution and a public variety $(p_l, \theta_l) = (0.25, 0.35)$, the private firm raises quality, cuts price, and reduces quantity relative to baseline. In the resulting market outcome the public firm supplies quantity $Q_l = 0.702$ and makes a loss of $B = Q_l(\theta_l - p_l) = 0.070$, which is exactly the revenue raised from the income tax. Relative to case 4, which has the same (p_l, θ_l) but no tax, the private firm sets a lower price and quality, as predicted by Proposition 8.

Figure 4 plots the indirect utility of individuals in the baseline scenario as well as the three quality scenarios. There are a number of noteworthy differences when comparing these scenarios with the affordability scenarios. First, while an injection of external funds to increase quality makes all individuals better off (as was the case before), the utility gains are now especially large for individuals in the middle and the upper end of the income distribution. This is a consequence of the complementarity of income and quality in the consumers' preferences. An unfunded quality increase has the opposite effects of an unfunded price reduction—poor individuals (some of whom are now priced out of the market) are made worse off, and richer individuals are made better off.

Lastly, in this example a tax-funded quality increase results in a Pareto-improvement over the baseline-scenario, which means that the baseline scenario is inefficient given the market structure and the policy instruments we consider. Note that the size of the private firm is drastically reduced in this case, moving the market outcome close to a tax-financed state monopoly. Provision of the good through a tax-funded public monopoly can, indeed, Pareto-dominate a mixed duopoly in our model. While there may be long-run disadvantages of restricting competition in this manner, the example does show that that maintaining high-quality public institutions—and, if necessary, funding them through progressive taxation—can benefit even those individuals who are are wealthy enough to consume the higher-end private services. The reason is that their taxes help make the public provider more attractive, and the public provider competes with the private provider for the marginal individual at the public-private threshold $\overline{m}$. Even supra-marginal consumers with income far in excess of $\overline{m}$ benefit benefit from the increased competitive pressure faced by the private provider. In the example, this benefit offsets the tax burden paid by high-income consumers.

5.3 Comparison: Affordability vs. quality

Suppose funds are available to invest in publicly provided healthcare, education, or the like. Should these funds be spent on improving affordability, or on improving quality, of the publicly provided good? The preceding analysis allows us to compare both alternatives with respect to their distributional implications.

Consider again the example where incomes are uniform and $u(x,t) = x^{1/3}\theta^{2/3}$, and $p_l = \theta_l = 0.25$. Table 1 shows that the funds required to pay for reducing price by 0.1 or increasing quality by 0.10 are roughly the same (0.064 and 0.062, respectively). Figure 3 shows that most of the benefits of an investment in price accrue at the low-end of the income spectrum. This is true even though the high-end of the spectrum benefits from both a lower price and higher quality of the privately provided good. Individuals who benefit the least are those with middle incomes. In contrast, Figure 4 shows that most of the benefits of an investment in quality accrue at the middle and the high-end of the income distribution. Depending on the social welfare function used to evaluate these gains, either policy may be preferable. However, if policy makers are concerned with distributing the gains somewhat evenly, they may prefer to allocate some share of the funds to improve affordability and the remainder to improve quality, instead of pursuing either one of these objectives exclusively.

6 Conclusion

We developed a mixed duopoly model in which firms choose both price and quality, and consumers differ in their incomes. In this model, we analyzed the private response to changes in the public firm's price and quality. Our variational analysis did not require us to specify a social welfare function. Instead, the starting point of the analysis were *changes* in the public firm's price and quality that arose from *changes* in either the government's objective or budget constraint. Utilizing the private firm's best response, the model produced a rich set of potential distributional effects caused by these changes.

Throughout the paper, we emphasized primarily changes toward greater affordability and higher quality of the publicly provided good. Most developed countries experienced prolonged phases in which public provision of certain goods expanded to lower-income segments of the population, resulting in a growing public sector along with an increasingly redistributive role of government.

However, our results apply equally to changes in the opposite direction. For example, mainstream conservative thought—at least in the United States—envisions a reduced role of government in the education, healthcare, housing, cultural, and other sectors, relative to the status quo. This view of the state is generally reflected in budget proposals that reduce funding to public schools and universities, Medicaid, the Department of Housing and Urban Development, and so on. Assuming that these changes are not too drastic, so that public providers are not entirely removed from the market, the private provider's response to such budget cuts will be of the opposite direction as the responses associated with an expansion of the public sector, and the same is true for the distributional consequences of these changes.

Appendix

Proof of Lemma 1

We will show the following single-crossing property: If an individual with income m weakly prefers (p, θ) to $(p', \theta') \ll (p, \theta)$, then every individual with income $m' > m$ strictly prefers variety (p, θ) to (p', θ'). Suppose $u(m - p, \theta) \geq u(m - p', \theta')$. Since $u_{xx} \leq 0$ we have $u_x(m-p, \theta) \geq u_x(m-p', \theta)$, and since $u_{xy} > 0$ we have $u_x(m-p', \theta) > u_x(m-p', \theta')$. Thus, $u_x(m - p, \theta) > u_x(m - p', \theta')$, and therefore $u(m' - p, \theta) > u(m' - p', \theta')$ for all $m' > m$. The result now follows immediately. □

Proof of Proposition 3

Define $P(m) = p_h(m|p_l, \theta_l) - \theta_h(m|p_l, \theta_l)$, where $\big(p_h(m|p_l, \theta_l), \theta_h(m|p_l, \theta_l)\big)$ is the point on the private firm's price-quality locus associated with marginal individual m, given (p_l, θ_l).

Define $\hat{P}(m)$ in the same way, but given $(\hat{p}_l, \hat{\theta}_l)$. The firm's profit if it serves $1 - F(m)$ individuals is $\pi(m) = P(m)(1 - F(m))$ before the change, and $\hat{\pi}(m) = \hat{P}(m)(1 - F(m))$ after the change. Let $\overline{m} = \arg\max \pi(m)$ and let $\hat{m} = \arg\max \hat{\pi}(m)$.

By Topkis' Theorem, if $\hat{\pi}(m) - \pi(m)$ increases at $m = \overline{m}$ then $\hat{m} > \overline{m}$ and thus $1 - F(\hat{m}) < 1 - F(\overline{m})$. This condition can be expressed as

$$\begin{aligned} &- f(\overline{m})\big(\hat{P}(\overline{m}) - P(\overline{m})\big) + (1 - F(\overline{m}))\big(\hat{P}'(\overline{m}) - P'(\overline{m})\big) > 0 \\ &\qquad\qquad \Leftrightarrow \quad \hat{P}'(\overline{m}) - P'(\overline{m}) > \lambda(\overline{m})\big(\hat{P}(\overline{m}) - P(\overline{m})\big). \qquad (12) \end{aligned}$$

Observe that condition (12) is equivalent to an increase in the elasticity of demand: Under the original policy (p_l, θ_l), when m increases marginally the relative change in P is $P'(m)/P(m)$ and relative change in quantity is $-f(m)/(1 - F(m)) = -\lambda(m)$. At the profit maximum this elasticity equals one, so that $P'(\overline{m})/P(\overline{m}) = \lambda(\overline{m})$. Substituting this in (12) gives

$$\hat{P}'(\overline{m}) - P'(\overline{m}) > \frac{P'(\overline{m})}{P(\overline{m})}\Big(\hat{P}(\overline{m}) - P(\overline{m})\Big) \quad \Leftrightarrow \quad \frac{\hat{P}'(\overline{m})}{\hat{P}(\overline{m})} > \frac{P'(\overline{m})}{P(\overline{m})}.$$

Thus, under the new policy, if m is increased marginally above $\overline{m}$, the relative change in P is larger than what it was under the old policy, while the relative change in $1 - F(\overline{m})$ is still $-\lambda(\overline{m})$. It follows that, at $\overline{m}$, demand elasticity is now below one in absolute value.

Fix an individual with arbitrary income m. We make three geometric observations. First, if preferences are homothetic the iso-MRS curves in (p_h, θ_h)-space are rays through $(m, 0)$. Second, if an h.o.d.-1 utility representation is chosen, then as one moves along any iso-MRS ray the change in utility is proportional to the distance travelled along the ray. Third, regardless of preferences, as one moves along any straight line in (p_h, θ_h)-space the change in the profit margin $p_h - \theta_h$ is proportional to the distance travelled along the line.

Now apply these observations to an individual with income $\overline{m}$ and h.o.d.-1 utility. Note that the profit margin $P(\overline{m})$ is determined by the intersection of the iso-MRS ray $u_x(\overline{m}-p_h, \theta_h)/u_{\theta_h}(\overline{m}-p_h, \theta_h) = 1$ with the indifference curve $u(\overline{m}-p_h, \theta_h) - u(\overline{m}-p_l, \theta_l) = 0$. Similarly, the profit margin $\hat{P}(\overline{m})$ is determined by the intersection of the same iso-MRS ray with the indifference curve $u(\overline{m} - p_h, \theta_h) - u(\overline{m} - \hat{p}_l, \hat{\theta}_l) = 0$. It follows that

$$\hat{P}(\overline{m}) - P(\overline{m}) \;=\; k \cdot \big[u(\overline{m} - \hat{p}_l, \hat{\theta}_l) - u(\overline{m} - p_l, \theta_l)\big] \;=\; k\Delta(\overline{m})$$

for some $k < 0$, and therefore $\hat{P}'(\overline{m}) - P'(\overline{m}) \;=\; k\Delta'(\overline{m})$. Plugging these expressions into (12) gives us

$$\Delta'(\overline{m}) \;<\; \lambda(\overline{m})\Delta(\overline{m}).$$

This establishes part (b) of the result; the proof of part (a) is symmetric. $\square$

Proof of Proposition 4

It will be convenient to view the private firm as choosing its profit margin $P = p_h - \theta_h$, instead of a marginal individual $\overline{m}$. For given P, let $(p_h(P), \theta_h(P))$ be the unique point on the price-quality locus where $p_h - \theta_h = P$. Let $m(P)$ be the associated income of the individual who is indifferent between the two firms, defined by $u(m(P) - p_l, \theta_l) = u(m(P) - p_h(P), \theta_h(P))$. Note that $p_h(P)$, $\theta_h(P)$, and $m(P)$ do not depend on F and are all strictly increasing.

Now, for a given income distribution F, let $D(P) = 1 - F(m(P))$ be the private firm's demand if it sets profit margin P. Its profit function is then $\pi(P) = PD(P)$, and at the profit maximum we have $\pi'(P) = D(P) + P(D'(P)) = 0$, or equivalently,

$$\frac{D'(P)}{D(P)}P \;=\; -\frac{f\big(m(P)\big)}{1 - F\big(m(P)\big)}m'(P)P \;=\; -1. \tag{13}$$

Let P^* is the solution to (13) and not that $\overline{m} = m(P^*)$.

Similarly, given income distribution $\hat{F}$, let $\hat{D}(P) = 1 - \hat{F}(m(P))$ be the private firm's demand associated with profit margin P. Since $\hat{F}(m) = F(m + \sigma(m))$, we have

$$\frac{\hat{D}'(P^*)}{\hat{D}(P^*)}P^* \;=\; -\frac{f\big(\overline{m} + \sigma(\overline{m})\big)}{1 - F\big(\overline{m} + \sigma(\overline{m})\big)}\big(1 + \sigma'(\overline{m})\big)m'(P^*)P^*. \tag{14}$$

Because $\lambda'(m) \geq 0$ and $m'(P) > 0$, (14) is strictly less than -1 if either $\lambda'(\overline{m}) > 0$ and $\sigma(\overline{m}) > 0$, or if $\sigma'(\overline{m}) > 0$. In this case, the private firm will decrease its profit margin P to below P^* and thereby increase its profit. Because $p_h(P)$ and $\theta_h(P)$ are strictly increasing in P, the private firm's price and quality decrease when the income distribution becomes less dispersed. $\square$

Proof of Proposition 6 and Proposition 7

We begin with a technical result to be used later:

Lemma 9. *Suppose $u(x, \theta) = x^\alpha \theta^{1-\alpha}$ and $m \sim U[0, M]$. At the private firm's interior profit maximum, the following holds:*

$$\overline{m} \;>\; K \equiv \frac{1}{2}\left(M + p_l + \frac{\alpha}{1-\alpha}\theta_l\right) \;>\; L \equiv \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_l.$$

Proof. Recall that the private firm sets price and quality to satisfy conditions $I(p_h, \theta_h|\overline{m}) = 0$ and $H(p_h, \theta_h|\overline{m}) = 1$. With Cobb-Douglas preferences, these can be written as follows:

$$(\overline{m} - p_h)^\alpha \theta_h^{1-\alpha} - (\overline{m} - p_l)^\alpha \theta_l^{1-\alpha} \quad = \quad 0, \tag{15}$$

$$\frac{\alpha}{1-\alpha} \frac{\theta_h}{\overline{m} - p_h} \quad = \quad 1. \tag{16}$$

Since we are only considering interior profit maxima where $(p_h, \theta_h) \gg (p_l, \theta_l)$, (16) implies

$$\frac{\alpha}{1-\alpha} \frac{\theta_l}{\overline{m} - p_l} < 1 \quad \Leftrightarrow \quad \frac{\theta_l}{\overline{m} - p_l} < \frac{1-\alpha}{\alpha}, \tag{17}$$

and because we assume that $\theta_l \geq p_l$, (17) implies

$$\frac{p_l}{\overline{m} - p_l} < \frac{1-\alpha}{\alpha} \quad \Leftrightarrow \quad \overline{m} > \frac{p_l}{1-\alpha} \quad \Leftrightarrow \quad p_l < (1-\alpha)\overline{m}. \tag{18}$$

Solving (15)–(16) for p_h and θ_h as functions of $\overline{m}$, we get

$$p_h(\overline{m}) \quad = \quad \overline{m} - \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} (\overline{m} - p_l)^\alpha \theta_l^{1-\alpha},$$

$$\theta_h(\overline{m}) \quad = \quad \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} (\overline{m} - p_l)^\alpha \theta_l^{1-\alpha}.$$

If incomes are uniformly distributed on $[0, M]$, the firm's profit, expressed as a function of a marginal individual $\overline{m}$, is given by

$$\pi(\overline{m}) \;=\; \left(1 - \frac{\overline{m}}{M}\right)\big(p_h(\overline{m}) - \theta_h(\overline{m})\big) \;=\; \left(1 - \frac{\overline{m}}{M}\right)\left(\overline{m} - A(\overline{m} - p_l)^\alpha \theta_l^{1-\alpha}\right),$$

where $A = \big[\alpha/(1-\alpha)\big]^{1-\alpha} + \big[(1-\alpha)/\alpha\big]^\alpha$. The first and second derivative of the firm's profit function are as follows:

$$\pi'(\overline{m}) \quad = \quad 1 - 2\frac{\overline{m}}{M} + \frac{A}{M}\left(\frac{\theta_l}{\overline{m} - p_l}\right)^{1-\alpha} \big[(1+\alpha)\overline{m} - \alpha M - p_l\big], \tag{19}$$

$$\begin{aligned}
\pi''(\overline{m}) \quad &= \quad -\frac{2}{M} + \frac{A}{M}\left(\frac{\theta_l}{\overline{m}-p_l}\right)^{1-\alpha}\left(\frac{-(1-\alpha)}{\overline{m}-p_l}\big[(1+\alpha)\overline{m} - \alpha M - p_l\big] + 1 + \alpha\right) \\
&= \quad -\frac{2}{M} + \frac{A}{M}\left(\frac{\theta_l}{\overline{m}-p_l}\right)^{1-\alpha}\left(2\alpha + (1-\alpha)\alpha\frac{M - \overline{m}}{\overline{m} - p_l}\right).
\end{aligned} \tag{20}$$

We now prove the result in a series of steps.

Step 1. We show that, at the private firm's profit maximum, $\overline{m} > L$. Suppose, to the contrary, that $\overline{m} \leq L$. Then

$$(1+\alpha)\overline{m} - \alpha M - p_l \ \leq \ (1+\alpha)\left[\frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_l\right] - \alpha M - p_l \ = \ 0. \tag{21}$$

From (17) it follows that

$$A\left(\frac{\theta_l}{\overline{m}-p_l}\right)^{1-\alpha} \ < \ \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \ = \ \frac{1}{\alpha}. \tag{22}$$

Together, (19), (21), and (22) imply that

$$\pi'(\overline{m}) \ \geq \ B(\overline{m}) \ \equiv \ 1 - 2\frac{\overline{m}}{M} + \frac{1}{M}\frac{1}{\alpha}\big[(1+\alpha)\overline{m} - \alpha M - p_l\big].$$

Note that

$$B'(\overline{m}) \ = \ -\frac{2}{M} + \frac{1}{M}\frac{1+\alpha}{\alpha} \ = \ \frac{1}{M}\frac{1-\alpha}{\alpha} \ > \ 0,$$

which, using (18), implies that

$$B(\overline{m}) \ > \ B\left(\frac{p_l}{1-\alpha}\right) \ = \ 1 - 2\frac{p_l}{(1-\alpha)M} + \frac{1}{\alpha M}\left[\frac{1+\alpha}{1-\alpha}p_l - \alpha M - p_l\right] \ = \ 0.$$

Thus, for $p_l/[1-\alpha] < \overline{m} \leq L$, $\pi'(\overline{m}) > 0$, and it follows that $\overline{m} > L$ at the private firm's profit maximum.

Step 2. Next, we show that $K > L$. Since K increases in θ_l but L does not, and $\theta_l \geq p_l$, it is sufficient to show that $K > L$ when $\theta_l = p_l$. Assuming $\theta_l = p_l$, suppose that $K \leq L$:

$$\frac{1}{2}\left(M + p_l + \frac{\alpha}{1-\alpha}p_l\right) \ = \ \frac{1}{2}\left(M + \frac{1}{1-\alpha}p_l\right) \ \leq \ \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_l$$

$$\Leftrightarrow \ (1-3\alpha)p_l \ \geq \ (1-\alpha)^2 M. \tag{23}$$

If $\alpha > 1/3$, the inequality in (23) would clearly be violated. If $\alpha < 1/3$, (23) implies

$$p_l \ \geq \ \frac{(1-\alpha)^2}{1-3\alpha}M \ = \ \frac{1-\alpha}{1-3\alpha}(1-\alpha)M \ > \ (1-\alpha)M \ \geq \ (1-\alpha)\overline{m}.$$

But by (18) we know this to be false, since $p_l < (1-\alpha)\overline{m}$ (otherwise, no interior equilibrium exists in which $(p_h, \theta_h) \gg (p_l, \theta_l)$). It follows that $K > L$.

Step 3. Finally, we show that $\overline{m} > K$. Since $\overline{m}$ decreases in p_l by Proposition 5, and because K increases in p_l, it is sufficient to show $\overline{m} > K$ when $p_l = \theta_l$. Furthermore, in Step 1 we showed that $\pi'(\overline{m}) > 0$ for $\overline{m} \in \big(p_l/[1-\alpha], L\big]$. (20) shows that $\pi''(\overline{m})$ is strictly decreasing in $\overline{m}$. Thus, if $\pi'(K|\theta_l = p_l) > 0$, then $\pi(\overline{m})$ is maximized at $\overline{m} > K$. Define

$$t \equiv \frac{p_l}{(1-\alpha)M}, \quad k(t) \equiv \frac{1}{2}(1+t).$$

Note that $p_l > 0$ implies $t > 0$; furthermore (18) implies that in an interior equilibrium in which $\overline{m} < M$ we have $p_l < (1-\alpha)M$ and, thus, $t < 1$. Using $p_l = tM$ and $K = k(t)M$, we can write

$$\begin{aligned}
\pi'(K|\theta_l = p_l) &= 1 - 2\frac{k(t)M}{M} + \frac{A}{M}\left(\frac{tM}{k(t)M - tM}\right)^{1-\alpha}\big[(1+\alpha)k(t)M - \alpha M - tM\big] \\
&= -t + \frac{1}{2}A\left(\frac{2(1-\alpha)t}{1+(2\alpha-1)t}\right)^{1-\alpha}[1-\alpha+(3\alpha-1)t] \\
&= -t + \frac{1}{2}\frac{1}{\alpha^\alpha}\left(\frac{2t}{1+(2\alpha-1)t}\right)^{1-\alpha}[1-\alpha+(3\alpha-1)t] \\
&> 0 \;\Leftrightarrow\; \big[1-\alpha+(3\alpha-1)t\big] \;>\; (2\alpha t)^\alpha\big(1+(2\alpha-1)t\big)^{1-\alpha}. \qquad (24)
\end{aligned}$$

Denote the difference between the left and right side of the inequality in (24) by

$$D(t) \;\equiv\; 1-\alpha+(3\alpha-1)t - (2\alpha t)^\alpha\big(1+(2\alpha-1)t\big)^{1-\alpha}.$$

Note that $D(1) = 0$; thus, in order to show that $D(t) > 0$ for $t < 1$ we will show that $D'(t) < 0$ for all $t < 1$. We have

$$\begin{aligned}
D'(t) &= 3\alpha - 1 - (2\alpha)^\alpha \alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{1-\alpha} - (2\alpha t)^\alpha(1-\alpha)\big(1+(2\alpha-1)t\big)^{-\alpha}(2\alpha-1) \\
&= 3\alpha - 1 - (2\alpha)^\alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{-\alpha}\Big[\alpha\big(1+(2\alpha-1)t\big) + (1-\alpha)(2\alpha-1)t\Big] \\
&= 3\alpha - 1 - (2\alpha)^\alpha t^{\alpha-1}\big(1+(2\alpha-1)t\big)^{-\alpha}\Big[\alpha + (2\alpha-1)t\Big].
\end{aligned}$$

Note that $D'(1) = 0$; thus, in order to show that $D'(t) < 0$ for $t < 1$ we will show that $D''(t) > 0$ for all $t < 1$. Let $w = 2\alpha - 1$ and write

$$\begin{aligned}
D''(t) &= -(2\alpha)^\alpha \frac{t^{1-\alpha}(1+wt)^\alpha - (\alpha+wt)\left[(1-\alpha)t^{-\alpha}(1+wt)^\alpha + \alpha t^{1-\alpha}w(1+wt)^{\alpha-1}\right]}{\left[t^{1-\alpha}(1+wt)^\alpha\right]^2} \\
&= -\frac{(2\alpha)^\alpha t^{-\alpha}(1+wt)^{\alpha-1}}{\left[t^{1-\alpha}(1+wt)^\alpha\right]^2}\Big[wt(1+wt) - (\alpha+wt)\big[(1-\alpha)(1+wt) + \alpha wt\big]\Big] \\
&= -(2\alpha)^\alpha t^{\alpha-2}(1+wt)^{-\alpha-1}\Big[\alpha^2 - \alpha\Big] \;>\; 0.
\end{aligned}$$

This establishes (24), and hence that $\overline{m} > K$. $\square$

We now proceed with the proof of both Proposition 6 and Proposition 7. Consider a change in the public firm's price and quality from (p_l, θ_l) to $(p_l + dp, \theta_l + d\theta)$. With Cobb-Douglas preferences, we can write

$$\begin{aligned}
\Delta(\overline{m}) &= (\overline{m} - p_l - dp)^{\alpha}(\theta_l + d\theta)^{1-\alpha} - (\overline{m} - p_l)^{\alpha}\theta_l^{1-\alpha}, \\
\Delta'(\overline{m}) &= \alpha(\overline{m} - p_l - dp)^{\alpha-1}(\theta_l + d\theta)^{1-\alpha} - \alpha(\overline{m} - p_l)^{\alpha-1}\theta_l^{1-\alpha}.
\end{aligned}$$

If $|dp|$ and $|d\theta|$ are close to zero these expressions are approximated by

$$\begin{aligned}
\Delta(\overline{m}) &= -dp \cdot \alpha(\overline{m} - p_l)^{\alpha-1}\theta_l^{1-\alpha} + d\theta \cdot (1-\alpha)(\overline{m} - p_l)^{\alpha}\theta_l^{-\alpha}, \\
\Delta'(\overline{m}) &= dp \cdot \alpha(1-\alpha)(\overline{m} - p_l)^{\alpha-2}\theta_l^{1-\alpha} + d\theta \cdot \alpha(1-\alpha)(\overline{m} - p_l)^{\alpha-1}\theta_l^{-\alpha},
\end{aligned}$$

and we have

$$\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} = \frac{\alpha(1-\alpha)}{\overline{m} - p_l}\frac{dp \cdot \theta_l + d\theta \cdot (\overline{m} - p_l)}{-dp \cdot \alpha\theta_l + d\theta \cdot (1-\alpha)(\overline{m} - p_l)}. \tag{25}$$

Furthermore, with uniformly distributed incomes on $[0, M]$, the hazard rate at $\overline{m}$ is given by

$$\lambda(\overline{m}) = \frac{1/M}{1 - \overline{m}/M} = \frac{1}{M - \overline{m}}. \tag{26}$$

Now consider the following cases. First, suppose $dp = 0$ (the case considered in Proposition 6). (25)–(26) imply that

$$\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} < \lambda(\overline{m}) \quad \Leftrightarrow \frac{\alpha}{\overline{m} - p_l} < \frac{1}{M - \overline{m}} \quad \Leftrightarrow \quad \overline{m} > \frac{\alpha}{1+\alpha}M + \frac{1}{1+\alpha}p_l = L.$$

By Lemma 9, we know this to be true. It follows that $\Delta'(\overline{m}) >/< \lambda(\overline{m})\Delta(\overline{m})$ if and only if $\Delta(\overline{m}) </> 0$, which is the case if and only if $d\theta </> 0$. Proposition 3 then implies that the private firm increases/decreases its quantity if and only if $d\theta </> 0$. Piecing together many such small changes $d\theta$, Proposition 6 follows.

Next, suppose $dp = d\theta$ (the case considered in Proposition 7). In this case, (25) becomes

$$\begin{aligned}
\frac{\Delta'(\overline{m})}{\Delta(\overline{m})} &= \frac{\alpha(1-\alpha)}{\overline{m} - p_l}\frac{\theta_l + \overline{m} - p_l}{-\alpha\theta_l + (1-\alpha)(\overline{m} - p_l)} = \frac{\alpha(1-\alpha)\left(\dfrac{\theta_l}{\overline{m} - p_l} + 1\right)}{-\alpha\theta_l + (1-\alpha)(\overline{m} - p_l)} \\
&< \frac{\alpha(1-\alpha)\left(\dfrac{1-\alpha}{\alpha} + 1\right)}{-\alpha\theta_l + (1-\alpha)(\overline{m} - p_l)} = \frac{1-\alpha}{-\alpha\theta_l + (1-\alpha)(\overline{m} - p_l)},
\end{aligned}$$

where the inequality is due to (17). Thus, to show that $\Delta'(\overline{m})/\Delta(\overline{m}) < \lambda(\overline{m})$ it is sufficient that

$$\frac{1-\alpha}{-\alpha\theta_l + (1-\alpha)(\overline{m} - p_l)} < \frac{1}{M - \overline{m}} \quad \Leftrightarrow \quad \overline{m} > \frac{1}{2}\left(M + p_l + \frac{\alpha}{1-\alpha}\theta_l\right) = K.$$

By Lemma 9, we know this to be true. It follows that $\Delta'(\overline{m}) > / < \lambda(\overline{m})\Delta(\overline{m})$ if and only if $\Delta(\overline{m}) < / > 0$. (17) implies that the marginal rate of substitution for an income-$\overline{m}$ individual who purchases from the public firm is less than one. Thus, $\Delta(\overline{m}) < / > 0$ if and only if $dp = d\theta < / > 0$. Proposition 3 then implies that the private firm increases/decreases its quantity if and only if $dp = d\theta < / > 0$. Piecing together many such small changes $dp = d\theta$, Proposition 7 follows. □

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