STAT 679 - Assignment 1 - Due date is on the course outline

1. Let $\hat{\gamma}_{XY}(m)$ be the usual sample cross-covariance function for jointly stationary series $\{X_t, Y_t\}$, with data $\{x_t, y_t\}_{t=1}^n$. Show that

$$E\left[\hat{\gamma}_{XY}(m)\right] = \left(\frac{n-m}{n}\right)\left\{\gamma_{XY}(m) - \text{COV}\left[\bar{y}, \bar{x}^*\right] - \text{COV}\left[\bar{x}, \bar{y}_*\right] + \text{COV}\left[\bar{x}, \bar{y}\right]\right\},$$

   where $\bar{x}^*$ is the average of $\{x_t\}_{t=m+1}^n$ and $\bar{y}_*$ is the average of $\{y_t\}_{t=1}^{n-m}$.

2. Assume $\mu_X = 0$; consider the problem of minimizing the function

$$f_m(\alpha_{1,m}, ..., \alpha_{m,m}) = E\left[\{X_t - \alpha_{1,m}X_{t-1} - ... - \alpha_{m,m}X_{t-m}\}^2\right],$$

   which is the MSE when X_t is forecast by $\alpha_{1,m}X_{t-1}+...+\alpha_{m,m}X_{t-m}$. Let the minimizers be $\alpha_{1,m}^*, ..., \alpha_{m,m}^*$. The **lag-m PACF value**, written ϕ_{mm}, is defined to be $\alpha_{m,m}^*$. Show that

$$\phi_{mm} = \text{CORR}\left[X_t - \hat{X}_t, X_{t-m} - \hat{X}_{t-m}\right],$$

   where each $\hat{X}$ denotes the best (i.e. minimum MSE) predictor which is a linear function of $X_{t-1}, ..., X_{t-m+1}$.

3. The obvious estimate of

$$f(\nu_k) = \sum_{m=-\infty}^{\infty} e^{-2\pi i \nu_k m}\gamma(m),$$

   using data $\{x_t\}_{t=1}^n$ is

$$\hat{f}(\nu_k) = \sum_{m=-(n-1)}^{n-1} e^{-2\pi i \nu_k m}\hat{\gamma}(m).$$

   Show that this reduces to the periodogram:

$$\hat{f}(\nu_k) = I(\nu_k) = |X(k)|^2.$$

4. Suppose that $\{X_t\}$ is AR(2), mean zero and stationary.

   (a) In the representation

$$X_t = w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + ...$$

   show that

$$\psi_k = \phi_1\psi_{k-1} + \phi_2\psi_{k-2}, \text{where } \psi_1 = \phi_1, \psi_0 = 1.$$

(b) Obtain a closed form expression for ψ_k. (*Hint*: write $\psi_k - b\psi_{k-1} = a\left(\psi_{k-1} - b\psi_{k-2}\right)$ for suitable constants a and b; then iterate.)

5. Suppose that filter coefficients are approximated by

$$a_s^M = \frac{1}{M}\sum_{k=0}^{M-1} A(\omega_k) e^{2\pi i \omega_k s}$$

and then $Y_t^M = \sum_{|s|<M/2} a_s^M X_{t-s}$ for $t = M/2 - 1, ..., n - M/2$.

(a) Show that these coefficients are real and symmetric ($a_s^M = a_{-s}^M$), if A is real and symmetric: $A(\omega) = A(-\omega)$.

(b) Show that

$$a_s^M = \sum_{t=-\infty}^{\infty} a_t \left[I(t - s \text{ is a multiple of } M)\right].$$

(c) Establish the bound in Problem 4.32:

$$E\left[\left(Y_t^M - Y_t\right)^2\right] \leq 4\gamma_X(0)\left(\sum_{|k|\geq M/2} |a_k|\right)^2.$$