

## Note on the Independence of Certain Statistics

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Let  $x_1, x_2, \dots, x_n$  be a sample from an  $n$ -variate normal population with the probability density function:

$$(2\pi)^{-\frac{n}{2}} |R|^{-\frac{1}{2}} e^{-\frac{1}{2} \sum_{i,j=1}^n r_{ij} x_i x_j},$$

where  $R$  denotes the matrix  $(r_{ij})^{-1}$ , which we assume to be symmetric and positive-definite, and let  $\varphi = \sum a_{ij} x_i x_j$  and  $\psi = \sum b_{ij} x_i x_j$  be real Hermitian forms of  $x_1, x_2, \dots, x_n$ . Then the following theorem holds, as is known<sup>(1)</sup>, with respect to the independence of  $\varphi$  and  $\psi$ .<sup>(2)</sup>

**Theorem.** *A necessary and sufficient condition that the statistics  $\varphi = \sum_{i,j=1}^n a_{ij} x_i x_j$  and  $\psi = \sum_{i,j=1}^n b_{ij} x_i x_j$  be mutually independent, is that the product of the matrices  $A=(a_{ij})$ ,  $R$  and  $B=(b_{ij})$  is the zero-matrix. i. e.,  $ARB=0$ .*

As the known proofs of this theorem seem to me not simple enough, I shall give my proof in the following lines.<sup>(3)</sup>

*Proof.* As  $R$  is a positive-definite and symmetric matrix, there exist a proper-orthogonal matrix  $T$  and a real diagonal matrix  $P$  with non-negative components so that  $T'R T = P^2$ .<sup>(4)</sup> Then simple calculations give for the moment generating function  $f(z)$  of a Hermitian form  $\sum_{i,j=1}^n c_{ij} x_i x_j$  the expression  $f(z) = |E - 2z\tilde{C}|^{-\frac{1}{2}}$ , where  $E$  and  $\tilde{C}$  denote the unit matrix and the matrix  $P'T'(c_{ij})TP$  respectively, which is also symmetric. Therefore the mutual independence of  $\varphi$  and  $\psi$  is equivalent to the equation:

$$|E - 2(z_1\tilde{A} + z_2\tilde{B})|^{-\frac{1}{2}} = |E - 2z_1\tilde{A}|^{-\frac{1}{2}} |E - 2z_2\tilde{B}|^{-\frac{1}{2}},$$

where  $z_1, z_2$  are mutually independent variables, and  $A$  and  $B$  represent the matrices  $P'T'ATP$  and  $P'T'BTTP$  respectively. From this equation it follows at once, that  $ARB=0$  is a sufficient condition. To prove that this condition is also necessity, assume that  $\varphi$  and  $\psi$  are mutually independent. Then we see that the equation

$$(*) \quad |E - (\lambda \tilde{A} + \mu \tilde{B})| = |E - \lambda \tilde{A}| |E - \mu \tilde{B}|$$

holds identically in  $\lambda$  and  $\mu$ . Now, let  $\alpha_1, \alpha_2, \dots, \alpha_r$  and  $\beta_1, \beta_2, \dots, \beta_s$  be the non-vanishing eigenvalues of  $\tilde{A}$  and  $\tilde{B}$  respectively, and let  $U$  be an orthogonal matrix such that  $U' \tilde{A} U$  has a diagonal form

$$\begin{pmatrix} \alpha_1 & & & & & \\ & \alpha_2 & & & & \\ & & \ddots & & & \\ & & & \alpha_r & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & & 0 \end{pmatrix}.$$

The left-hand side of (\*) can then be rewritten

$$\begin{vmatrix} 1 - \lambda \alpha_1 - \mu b_{11}^{**} & -\mu b_{12}^{**} & \dots & \dots & \dots & -\mu b_{1n}^{**} \\ -\mu b_{21}^{**} & 1 - \lambda \alpha_2 - \mu b_{22}^{**} & \dots & \dots & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 - \lambda \alpha_r - \mu b_{rr}^{**} & \dots & \dots \\ & & & & 1 - \mu b_{r+1, r+1}^{**} & \dots \\ -\mu b_{n1}^{**} & \dots & \dots & \dots & \dots & 1 - \mu b_{nn}^{**} \end{vmatrix}$$

where  $(b_{ij}^{**})$  denotes the matrix  $U' \tilde{B} U$ . This is a polynomial in  $\lambda$ , which has

$$\alpha_1, \dots, \alpha_r \left| \begin{array}{ccc} 1 - \mu b_{r+1, r+1}^{**} & \dots & -\mu b_{r+1, n}^{**} \\ \vdots & \ddots & \vdots \\ -\mu b_{n, r+1}^{**} & \dots & 1 - \mu b_{nn}^{**} \end{array} \right|$$

as the coefficient of  $\lambda^r$ , and this must be equal to  $\alpha_1 \dots \alpha_r |E - \mu \tilde{B}|$  (identically in  $\mu$ ), as can be immediately seen from (\*). This fact implies that the non-vanishing eigenvalues of the matrix

$$\tilde{B}_1 = \begin{pmatrix} b_{r+1, r+1}^{**} & \dots & b_{r+1, n}^{**} \\ \dots & \dots & \dots \\ b_{n, r+1}^{**} & \dots & b_{nn}^{**} \end{pmatrix} \text{ are } \beta_1, \beta_2, \dots, \beta_s. \text{ Consequently,}$$

there exists such an orthogonal matrix  $V_1$  of degree  $n-r$  that

$$V_1' \tilde{B}_1 V_1 \text{ has a diagonal form } \begin{pmatrix} \beta_1 & & & & \\ & \beta_2 & & & \\ & & \ddots & & \\ & & & \beta_s & \\ & 0 & & & 0 \end{pmatrix}, \text{ for the matrix } U' \tilde{B} U =$$



consequently

$$\tilde{A}\tilde{B}=0,$$

and finally we have

$$ARB=0.$$

**Corollary.** *A necessary and sufficient condition that the statistics  $\sum_{i,j=1}^n a_{ij}x_ix_j$ , and  $\sum_{i=1}^n a_ix_i$ , or  $\sum_{i=1}^n a_ix_i$  and  $\sum_{i=1}^n b_ix_i$  be mutually independent, is respectively,  $AR \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = 0$  or  $(a_1, a_2, \dots, a_n) R \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = 0$ .*

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1) A. T. Craig. Note on the independence of certain quadratic forms. *Annals of Mathematical Statistics*. Vol. 14. 1943.  
proof at this stage.

H. Hotelling. Note on a matrix theorem of A. T. Craig. *Annals of Mathematical Statistics*, Vol. 15 (1944).

2) I had this result in 1943, independently of A. T. Craig when H. Sakamoto asked me about the independence of quadratic forms, and informed it to him. He and some other colleagues of mine have searched for other proofs of this theorem (esp. M. Sugawara and S. Nabeya have found new proofs) or applied it to various problems. On seeing these recent investigations and those of some others, it seems to me of some use to publish my proof at this stage.

3) In this note we denote the transposed matrix of a matrix  $T$ , as usual, by  $T'$ .