STAT 512 - Sample Midterm Exam

Instructions: Answer all six equally weighted questions.
Time: 80 minutes.

1. (a) Suppose that V is a vector space in $\mathbb{R}^n$ and that W is a subset of V. What properties must W possess in order that it be a *subspace* of V?

   (b) Define the *orthogonal complement* of V, and show that it is also a vector space.

2. Let $\mathbf{A}$ be a real, symmetric, positive semi-definite matrix.

   (a) Show that there is a matrix $\mathbf{B}$ such $\mathbf{A} = \mathbf{B}'\mathbf{B}$.

   (b) Show that for any vector $\mathbf{v}$, $\mathbf{v}'\mathbf{A}\mathbf{v} = 0$ iff $\mathbf{A}\mathbf{v} = \mathbf{0}$.

3. Consider vectors of the form $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{y}$ is $n \times 1$, $\mathbf{X}$ is $n \times p$ with rank p, and $\boldsymbol{\beta}$ is $p \times 1$. Such vectors occur in the theory of linear regression.

   (a) Show that this vector can be expressed as the sum of two vectors: $\mathbf{y} - \mathbf{X}\boldsymbol{\beta} = \mathbf{u} + \mathbf{v}$, in such a way that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other and $\mathbf{v}$ lies in the column space of $\mathbf{X}$.

   (b) Use (a) to show that the vector $\boldsymbol{\beta}$ which minimizes the norm of $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$ is the solution $\hat{\boldsymbol{\beta}}$ to the equation $\mathbf{v} = \mathbf{0}$. Solve this equation.

4. (a) Define what it means for a function $f : D \to \mathbb{R}$ to be (i) continuous on D, (ii) uniformly continuous on D.

   (b) Show that if a function f is defined and differentiable on $\mathbb{R}$ and has a bounded derivative there, then the function is uniformly continuous on $\mathbb{R}$.

5. (a) Define what it means for a function $f(x)$ to approach a (finite) limit L as x approaches a number a.

   (b) Suppose that $f(x) \to L_1$ as $x \to 1$, and also $f(x) \to L_2$ as $x \to 1$. Both L_1 and L_2 are finite. Show that L_1 and L_2 must be equal.

6. Suppose that U is a random variable uniformly distributed over the interval $(0, \pi)$. Define a new random variable by $X = \cos U$.

   (a) What is the distribution function of X?

   (b) What is the density function of X?