STAT 512 - Assignment 2 - due date is on course outline

1. Suppose that $f(x)$ is differentiable on $(0,\infty)$ and that $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x+1) - f(x)$. Show that $g(x) \to 0$ as $x \to \infty$.

2. Let f be convex function on a closed interval $[a,b]$.

   (a) Show that f is bounded on $[a,b]$.

   (b) Show that f is continuous on (a,b).

3. The moment generating function of a random variable X is the function of t defined by $M_X(t) = E\left[e^{tX}\right]$, provided the expectation exists for all t in a neighbourhood of zero. Assuming that it does, and that X has a mean μ_X, show that $M_X(t) \geq e^{t\mu_X}$, so that
$$\mu_X \leq \frac{\log M_X(t)}{t} \text{ for } t > 0.$$
Show that this inequality becomes an equality as $t \to 0$. Point out where you need to perform a certain interchange of operations in your derivation.

4. Let $\mathbf{A}$ be a $p \times p$ positive definite matrix. Show that, if $\mathbf{a}$ is any vector with $||\mathbf{a}|| = 1$, then
$$(\mathbf{a}'\mathbf{A}\mathbf{a})\left(\mathbf{a}'\mathbf{A}^{-1}\mathbf{a}\right) \geq 1.$$
(*Hints*: 1. A symmetric matrix is 'almost' diagonal. 2. Suppose first that $\mathbf{A}$ *is* diagonal. Write out the inequality as a statement about expectations, interpreting the squares of the elements of $\mathbf{a}$ as probabilities. (Are they? Why?) 3. Think about Jensen's Inequality.)

5. Prove the following special case of Slutsky's Theorem: If $X_n \xrightarrow{pr} c$ as $n \to \infty$, and that $a_n \to a$ (finite) as $n \to \infty$, where $\{a_n\}$ is a sequence of constants, then $a_n X_n \xrightarrow{pr} ac$.

6. Let X be a r.v. denoting the age of failure of an electrical component, and assume that X has a d.f. F with density f. The *failure rate* is defined as the probability of failure in a finite interval of time, given the age of the component, say x. This is therefore given by
$$P\left(x \leq X \leq x + h | X \geq x\right).$$
The *hazard rate* is defined as the instantaneous failure rate:
$$h(x) = \lim_{h \to 0} \frac{P\left(x \leq X \leq x + h | X \geq x\right)}{h}.$$

...over

(a) Show that
$$h(x) = \frac{f(x)}{1 - F(x)}.$$

(b) Show that X has a constant hazard rate iff it has an exponential distribution (i.e. $f(x) = \lambda e^{-\lambda x}$ for some λ).

7. Prove: If $f(x)$ is strictly increasing, twice differentiable, and convex on $[a, \infty)$ then it is unbounded.

8. In the theory of robust estimation in Statistics, one encounters the differential equation $2\psi'(x) - \psi^2(x) = -\lambda^2$, where λ is a positive constant. There are several solutions to this equation – $\psi(x) = \lambda$ is an obvious one. Without actually solving the equation, show that if $\psi(x)$ is any *bounded* solution, then $\psi(x) \leq \lambda$ for all x.

9. Show that, if an angle θ is uniformly distributed over $(-\pi/2, \pi/2)$, then $Y = \tan\theta$ has the Cauchy distribution, with density $f_Y(y) = 1/\left[\pi\left(1 + y^2\right)\right], -\infty < y < \infty$. (This is also known as "Student's" t-distribution on 1 degree of freedom.)

10. (a) Prove: If $h(x) > 0$ for all x, then $h(x)$ is maximized at x_0 if and only if $\log h(x)$ is maximized at x_0.

(b) Suppose that a random sample $\mathbf{x} = (x_1, ..., x_n)$ has a density, depending on an unknown parameter σ^2, of the form

$$\begin{aligned} f\left(\mathbf{x}; \sigma^2\right) &= h(t; \sigma^2) g\left(\mathbf{x}\right), \text{ for} \\ h(t; \sigma^2) &= \frac{\frac{n-1}{\sigma^2}\left(\frac{(n-1)t}{2\sigma^2}\right)^{\frac{n-1}{2}-1} \exp\{-\frac{(n-1)t}{2\sigma^2}\}}{\Gamma\left(\frac{n-1}{2}\right)}, \ 0 < t < \infty. \end{aligned}$$

Here t is the value of a random variable $T = T\left(\mathbf{x}\right)$ computed from the sample, and $g\left(\cdot\right)$ does not depend upon the parameter. If we observe the value of T then we can compute an estimate of σ^2. This estimate, called the Maximum Likelihood Estimate (MLE), is the value of σ^2 which maximizes $h(t; \sigma^2)$, hence $f\left(\mathbf{x}; \sigma^2\right)$. Obtain the MLE. That is, if t is observed, what is the estimate of σ^2? [Note: In the sampling situation in which this problem arises, T is a 'sufficient statistic' for σ^2 and contains all the information in the sample about that parameter, through its density $h(t; \sigma^2)$.]