Outline of theory you are expected to know for Exam III.

- Any zero-mean, weakly stationary time series may be approximated arbitrarily closely by a series $\{X_{t,N}\}_{t=-\infty}^{\infty}$ of the form

$$X_{t,N} = \sum_{k=0}^{N} \{A_k \cos(\lambda_k t) + B_k \sin(\lambda_k t)\}$$

for Fourier frequencies $\lambda_k \in [-\pi, \pi]$. Here $A_0, ..., A_N$, $B_0, ..., B_N$ are uncorrelated, zero-mean r.v.s with $VAR[A_k] = VAR[B_k] = \sigma_k^2$.

- The ACF of $\{X_{t,N}\}$ was derived and expressed as

$$\rho(m) = E\left[\cos(\Lambda m)\right]$$

for some discrete, symmetric r.v. Λ with possible values $\lambda_0 = 0, \pm\lambda_1, ..., \pm\lambda_N$.

- As we improve the approximation by letting $N \to \infty$, the distribution of Λ tends to one with a density. Now we write $\lambda = 2\pi\nu$, $-1/2 \leq \nu \leq 1/2$, and write the density of ν in the form $f(\nu)/\sigma^2$. The relationship above becomes (how?)

$$\begin{aligned}\frac{\gamma(m)}{\sigma^2} &= \rho(m) = \int_{-1/2}^{1/2} \cos(2\pi\nu m)\frac{f(\nu)}{\sigma^2}d\nu \\ &= \frac{1}{\sigma^2}\int_{-1/2}^{1/2} e^{2\pi i\nu m} f(\nu)\, d\nu.\end{aligned}$$

- These transforms are all invertible:

$$\begin{aligned}\gamma(m) &= \int_{-1/2}^{1/2} e^{2\pi i\nu m} f(\nu)\, d\nu \\ &\Longleftrightarrow \\ f(\nu) &= \sum_{m=-\infty}^{\infty} e^{-2\pi i\nu m}\gamma(m).\end{aligned}$$

A consequence is (how?)

$$f(\nu) = \gamma(0) + 2\sum_{m=1}^{\infty}\left\{\cos(2\pi\nu m)\cdot\gamma(m)\right\}.$$

- Similarly the CCF and cross-spectrum are related by

$$\begin{aligned}\gamma_{XY}(m) &= \int_{-1/2}^{1/2} e^{2\pi i \nu m} f_{XY}(\nu) d\nu, \\ f_{XY}(\nu) &= \sum_{m=-\infty}^{\infty} e^{-2\pi i \nu m} \gamma_{XY}(m).\end{aligned}$$

- The squared coherence is defined by

$$\rho^2_{Y \cdot X}(\nu) = \frac{|f_{YX}(\nu)|^2}{f_Y(\nu) f_X(\nu)} \in [0, 1].$$

The 1 is attained (why? - you should be able to give the derivation) if

$$Y_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s}$$

for constants $\{a_s\}_{s=-\infty}^{\infty}$. In this latter case we say $\{Y_t\}$ is a linear filter of $\{X_t\}$.

- If $\{Y_t\}$ is a filter of $\{X_t\}$ then

$$f_Y(\nu) = |A(\nu)|^2 f_X(\nu)$$

and

$$f_{YX}(\nu) = f_X(\nu)A(\nu)$$

(recall you were asked to derive this latter equality) where

$$A(\nu) = \sum_{s=-\infty}^{\infty} a_s e^{-2\pi i \nu s}$$

is the IFT. One can invert this to get the a_s's from $A(\nu)$ - how?

  - One consequence - If $\{X_t\}$ is MA(q) it can be viewed as a filter of $\{w_t\}$, leading to the result

$$f_X(\nu) = |\theta(e^{-2\pi i \nu})|^2 \sigma_w^2.$$

    Similarly if $\{X_t\}$ is AR(p) then $\{w_t\}$ can be viewed as a filter of $\{X_t\}$, leading to an expression for the power of $\{X_t\}$ - what is it? Derive the power of an ARMA(1,1) series.

- Estimating the power. Since $f(\nu)$ is the IFT of the ACF, a natural estimate is the corresponding IFT of the estimated ACF:

$$\hat{f}(\nu_k) = \sum_{m=-(n-1)}^{n-1} e^{-2\pi i \nu_k m} \hat{\gamma}(m) = |X(k)|^2,$$

(the "periodogram") where

$$X(k) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-2\pi i \nu_k t} x_t$$

is the DFT of the data. The data can be recovered from this:

$$x_t = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{2\pi i \nu_k t} X(k);$$

you should know the derivation of this result.

- For various reasons (what are they?) one generally computes a smoothed version of the periodogram:

$$\hat{f}(\nu_k) = \frac{1}{L} \sum_{l=-\frac{L-1}{2}}^{\frac{L-1}{2}} |X(k+l)|^2.$$

The approximate distribution is

$$\hat{f}(\nu_k) \overset{d}{\approx} \frac{f(\nu_k)}{df} \chi^2_{df}$$

with $df = 2Ln/n'$. What then are the approximate mean and variance? Derive the form of a confidence interval on $f(\nu_k)$.

- One application is "lagged regression" or "impulse-response" problems: If an examination of the estimated coherence indicates that series $\{X_t\}$, $\{Y_t\}$ are strongly coherent at some frequencies, then we might try to fit a model of the form

$$Y_t = \sum_{s=-\infty}^{\infty} \beta_s X_{t-s} + v_t.$$

You should be able to write down the MSE and differentiate it w.r.t. each β_r so as to obtain the equations

$$\gamma_{YX}(r) = \sum_{s=-\infty}^{\infty} \beta_s \gamma_X(r-s),\ r = 0, \pm 1, \pm 2, \ldots .$$

From this, derive the equation

$$B(\nu) = \frac{f_{YX}(\nu)}{f_X(\nu)},$$

where $B(\nu)$ is the IFT of $\{\beta_s\}$. How then is $\hat{\beta}_s$ obtained?

- Another application is the construction of filters. If an examination of the estimated spectrum of a series reveals certain ranges of interesting frequencies, we might choose a frequency response function $A(\nu)$ to be large at these frequencies, small otherwise. In theory, the original and filtered spectra are related through

  $$f_Y(\nu) = |A(\nu)|^2 f_X(\nu).$$

  On R we take the desired $A(\nu)$, and then do an approximate integration (how?) so as to obtain M filter coefficients a_s^M approximating those of $A(\nu)$. Then the filtered series can be computed - what will it be? - and has spectrum $|A^M(\nu)|^2 f_X(\nu)$, where $A^M(\nu) =$ (what?).