STAT 312 Lab 9

1. (a) Recall the discussion of improper Riemann integrals. Show that, as a consequence of these definitions, the expectation $E[X]$ of a r.v. with density $f(x)$ $(-\infty < x < \infty)$ is defined (i.e., the integral converges to a finite value) if and only if $E[|X|]$ is finite.

   (b) Show that, if an angle θ is uniformly distributed over $(-\pi/2, \pi/2)$, then $Y = \tan\theta$ has the Cauchy distribution, with density $f_Y(y) = 1/\left[\pi\left(1+y^2\right)\right], -\infty < y < \infty$. This is also known as "Student's" t-distribution on 1 degree of freedom. Does $E[Y]$ exist? If so, what is its value?

2. The Gamma function:

   (a) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint*: convert the integral to one involving the normal density.]

   (b) Show that the function $f_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!}\lambda e^{-\lambda s}$ $(s, \lambda > 0)$ is a probability density.

3. Customers arrive for service; the times between successive arrivals are independently and exponentially distributed with mean $1/\lambda$. Let N_x denote the number of arrivals before time x. Show that $N_x \sim \mathbb{P}(\lambda x)$. [*Hint*: Recall Example 2 from Lecture 27; show that that $N_x \leq n-1$ iff $S_n > x$, where S_n has the density $f_n(s)$ exhibited above. This will lead to

$$p_{n-1} \stackrel{def}{=} P\left(N_x \leq n-1\right) = \int_x^{\infty} f_n(s)\, ds;$$

an integration by parts now leads to the relationship $p_n - p_{n-1} = e^{-\lambda x}\frac{(\lambda x)^n}{n!}$.]

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4. Customers enter a store according to a Poisson process with a mean of λ arrivals per day, i.e. if N is the number of arrivals per day then

$$P(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots \, .$$

Each customer, independently of the others, makes a purchase with probability p. Let S be the total number of purchases during the day.

   (a) Show that the probability generating function $G(z)$ of S is $G(z) = e^{-\lambda p(1-z)}$. [*Hint*: Note that S depends on N. What is the distribution of S when $N = n$? Now recall the Double Expectation Theorem, by which the expected value of one r.v., say Y, can be evaluated in stages by first conditioning on another r.v., say X, on which Y depends: $E[Y] = E\{E[Y|X]\}$. More precisely, $E[Y|X = x]$ will be a function of x, say $h(x)$, and then $E\{E[Y|X]\} = E\{h(X)\}$.]

   (b) From (a), what is the probability distribution of S?

5. Show that the m.g.f. of $X \sim N(\mu, \sigma^2)$ is $E[e^{tX}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$. [*Hint*: Express $\int_{-\infty}^{\infty} e^{tz}\phi(z; \mu, \sigma^2)dz$ as $\psi(t) \cdot \int_{-\infty}^{\infty} \phi(z; \alpha, \sigma^2)dz$ for some α.]