STAT 312 Lab 6

1. (a) Let X be a random variable mapping from a sample space Ω into the real line. Define what we mean by the 'distribution function $F(x)$ of X'.

   (b) Show that the distribution function $F(x)$ is weakly increasing: $x < y \Rightarrow F(x) \leq F(y)$.

2. Suppose that $X_n \xrightarrow{pr} c$ and $Y_n \xrightarrow{pr} d$ as $n \to \infty$. Show that then $X_n + Y_n \xrightarrow{pr} c + d$.

3. Let Y be a continuous random variable with a strictly increasing distribution function F. Let G be another strictly increasing, continuous distribution function. Show that the distribution function of the random variable $G^{-1}(F(Y))$ is G.

4. Suppose that X is a $N(0,1)$ r.v. Obtain the density of X^2 – this is the χ^2_1 density.

5. A r.v. X has the *exponential distribution* with mean $1/\lambda$ if the d.f. is $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Suppose that such a r.v. represents the length of time, in minutes, that a randomly chosen person waits for his bus, and that $X_1, ..., X_n$ is a sample of such waiting times. If each person has a mean waiting time of 3 (minutes), then:

   (a) What is the d.f. of the shortest of these n waiting times? What is the expected value of this r.v. "shortest of n waiting times"?

   (b) Among 6 such people, what is the probability that the shortest waiting time exceeds 1 minute?

6. Let $\mathbf{A}$ be a $p \times p$ positive definite matrix. Show that, if $\mathbf{a}$ is any vector with $||\mathbf{a}|| = 1$, then

$$(\mathbf{a}'\mathbf{A}\mathbf{a})\left(\mathbf{a}'\mathbf{A}^{-1}\mathbf{a}\right) \geq 1.$$

   (*Hints*: 1. A symmetric matrix is 'almost' diagonal. 2. Suppose first that $\mathbf{A}$ *is* diagonal. Write out the inequality as a statement about expectations, interpreting the squares of the elements of $\mathbf{a}$ as probabilities. (Are they? Why?) 3. Think about Jensen's Inequality.)