STAT 312 Lab 5

1. (a) Define what is means for a set to be *open*.

   (b) Show that if a set A is open then the complement A^c is closed.

2. (a) Show that if $A_1, ..., A_n$ are open sets, then $\cap_{i=1}^n A_i$ is open.

   (b) Use (a), together with problem #1 and its converse (shown in class) to formulate and prove statements regarding unions and intersections of closed sets.

3. (a) Define what we mean by the 'supremum' of a set of numbers.

   (b) Prove: If $\{x_n\}$ is an increasing and bounded sequence of numbers then $S = \sup x_n$ is finite and $x_n \to S$ as $n \to \infty$.

4. State and prove the triangle inequality, as it applies to the absolute value of the sum of two numbers.

5. Let $f : (0,1) \to \mathbb{R}$ be continuous and such that $f(x) = 0$ for every <u>rational</u> $x \in (0,1)$. Show that $f(x) = 0$ for <u>every</u> $x \in (0,1)$. [*Hint* - a real number in $(0,1)$ can be represented as a decimal $.a_1 a_2 \cdots a_n \cdots$, where each $a_i \in \{0, 1, 2, ..., 9\}$. Put $b_n = .a_1 a_2 \cdots a_n$. What is the relationship between b_n and x? What is $f(b_n)$? (why?)]

6. Prove: If a function f is differentiable on (a,b) and attains a maximum (or minimum) at $c \in (a,b)$ then $f'(c) = 0$.

7. Suppose that a random variable X has mean μ_X and variance σ_X^2. Show that, if we can assume that X will be very close to μ_X, then the transformed random variable $Y = \psi(X)$ will have an approximate mean and variance of

$$\mu_Y \approx \psi(\mu_X) \text{ and } \sigma_Y^2 \approx [\sigma_X \psi'(\mu_X)]^2 .$$

   (The assumption used here will be seen to be quite reasonable, if X is the average of a large sample – this is essentially the statement of the Weak Law of Large Numbers.)

8. Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.