STAT 312 Lab 4

1. Let
$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$
Let $\mathbf{Y}$ consist of the independent columns of $\mathbf{X}$, and let $\mathbf{z}$ be a 4×1 vector lying anywhere on the 'unit sphere' $||\mathbf{z}|| = 1$. For such vectors $\mathbf{z}$, what is the *maximum* value of $||\mathbf{Y}'\mathbf{z}||^2$? State the relevant results from this course which you are applying in determining your answer.

2. Consider the $n \times n$ 'equicorrelation' matrix
$$\mathbf{P} = \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ \rho & \rho & \cdots & \rho & 1 \end{pmatrix},$$
where ρ is a correlation, so that $|\rho| \leq 1$.

   (a) Show that the determinant is
$$|\mathbf{P}| = (1-\rho)^{n-1}\left(1+(n-1)\rho\right).$$

   (b) Assuming that $|\mathbf{P}| \neq 0$ (what values of ρ are being excluded here?), determine the inverse.

3. Let $\mathbf{x}$ be an $n \times 1$ vector of random variables, with mean vector $\boldsymbol{\mu}_{n\times 1}$ and covariance matrix $\boldsymbol{\Sigma}_{n\times n}$. Consider a linear combination $\mathbf{a}'\mathbf{x} = \sum_{i=1}^{n} a_i X_i$ of the elements of $\mathbf{x}$.

   (a) Show that the variance of this linear combination is $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$.

   (b) Among all such linear combinations with $\sum_{i=1}^{n} a_i^2 = 1$, which one has the largest variance?

4. Show that a matrix which is (symmetric and) idempotent is necessarily positive semi-definite.

...over

5. Recall that if $\mathbf{A}$ is an $n \times m$ matrix, then $vec(\mathbf{A})$ is the $mn \times 1$ vector formed by stringing the columns of $\mathbf{A}$ out, one after the other.

   (a) Show that $tr\left(\mathbf{A}'\mathbf{A}\right) = \|vec(\mathbf{A})\|^2$. The square root of this quantity is called the *Frobenius norm* of $\mathbf{A}$, written $\|\mathbf{A}\|$.

   (b) Show that $\|\mathbf{Ax}\| \leq \|\mathbf{A}\| \, \|\mathbf{x}\|$. (*Hint*: $\|\mathbf{Ax}\|^2 = \frac{\mathbf{x}'\mathbf{A}'\mathbf{Ax}}{\mathbf{x}'\mathbf{x}} \cdot \|\mathbf{x}\|^2$. Bound the first term by an eigenvalue.)

   (c) An equation $\mathbf{Ax} = \mathbf{w}$ might be solved for $\mathbf{x}$ by writing $\mathbf{B} = \mathbf{I} - \mathbf{A}$ and solving $\left(\mathbf{I} - \mathbf{B}\right)\mathbf{x} = \mathbf{w}$ by iteratively calculating

   $$\mathbf{x}_n = \mathbf{B}\mathbf{x}_{n-1} + \mathbf{w}.$$

   Show that $\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq \|\mathbf{B}\| \, \|\mathbf{x}_n - \mathbf{x}_{n-1}\|$. If the Frobenius norm $\|\mathbf{B}\| < 1$ then the iterative scheme is called a *contraction*, and is guaranteed to converge to a solution.

6. Complete the proof that $vec\left(\mathbf{ABC}\right) = \left(\mathbf{C}' \otimes \mathbf{A}\right) vec\mathbf{B}$. (*Hint*: we established this in class, in the case that $\mathbf{B}$ is of the form $\mathbf{B} = \mathbf{xy}'$. So show that *any* matrix can be written as a sum of such 'rank one' matrices.)