STAT 312 Lab 3

1. This question is meant to reinforce the maxim that *a symmetric matrix is 'almost' diagonal.* Let $\mathbf{S}$ be an $n \times n$ symmetric matrix. Suppose that $\mathbf{S}$ <u>is</u> diagonal.

   (a) Show that the diagonal elements are the eigenvalues.

   (b) Show that $tr(\mathbf{S}^3) = \sum_{i=1}^{n} \lambda_i^3$, where the λ_i are the eigenvalues of $\mathbf{S}$.
   Now drop the assumption that $\mathbf{S}$ is diagonal and show that (b) continues to hold (although <u>of course</u> (a) does not), using only the fact that $\mathbf{S}$ is symmetric.

2. (a) Define what we mean by the 'characteristic polynomial' of a matrix $\mathbf{A}$, and use this definition to define what we mean by 'the eigenvalues of $\mathbf{A}$'.

   (b) Let $\mathbf{A}$ be an $n \times n$ matrix, and $\mathbf{C}$ a nonsingular $n \times n$ matrix. Show that $\mathbf{A}$ and $\mathbf{C}^{-1}\mathbf{AC}$ have the same eigenvalues.

3. Let $\mathbf{A}$ be any $n \times p$ matrix.

   (a) Show that $tr\,(\mathbf{A}'\mathbf{A}) = 0$ iff $\mathbf{A} = \mathbf{0}$.

   (b) Show that if $\mathbf{A}$ has rank p, then $\mathbf{A}'\mathbf{A}$ is a positive definite matrix.

4. Let $\mathbf{A}$ be a non-negative definite, $n \times n$ matrix with rank $p \leq n$.

   (a) Show that we may represent $\mathbf{A}$ as $\mathbf{A} = \mathbf{BB}'$, where $\mathbf{B}$ is $n \times p$ and $\mathbf{B}'\mathbf{B}$ is a diagonal matrix whose diagonal elements are the non-zero eigenvalues of $\mathbf{A}$.

   (b) Show that for any $n \times 1$ vector $\mathbf{v}$, $\mathbf{v}'\mathbf{A}\mathbf{v} = 0$ iff $\mathbf{A}\mathbf{v} = \mathbf{0}$.

5. (a) What do we mean when we say that a matrix $\mathbf{A}$ is 'positive semidefinite'? Answer in terms of (i) possible values of $\mathbf{x}'\mathbf{A}\mathbf{x}$, and again in terms of (ii) the eigenvalues of $\mathbf{A}$.

   (b) Prove: Any symmetric matrix can be represented as the difference between two p.s.d. matrices which are mutually orthogonal and the sum of whose ranks equals the rank of the original matrix.

6. Show that the diagonal elements of a symmetric matrix are bounded above and below by the maximum and minimum eigenvalues, respectively. Conclude that a matrix for which any diagonal element is negative cannot be positive definite.