STAT 312 Lab 2

1. Let
$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix},$$
as in Lab 1 Problem 1. Exhibit an orthonormal basis for $\text{col}(\mathbf{X})$, with appropriate verifications.

2. Let $\mathbf{x}$ be a random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Show that
$$E[\mathbf{x}'\mathbf{A}\mathbf{x}] = tr(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}.$$
Note: I will take a dim view of a solution which starts by expanding $\mathbf{x}'\mathbf{A}\mathbf{x}$ as a double sum, and taking the sum of expectations. I want you to use properties of the trace, and of expectations, starting with the observation that the trace of a number - such as $\mathbf{x}'\mathbf{A}\mathbf{x}$ - is the number itself, then continuing by applying other properties of the trace. For instance if $\mathbf{M}$ is a square random matrix then
$$E[tr\{\mathbf{M}\}] = E\left[\sum_i M_{ii}\right] = \sum_i E[M_{ii}] = tr\{E[\mathbf{M}]\}.$$

3. (a) What do we mean when we say that vectors $\mathbf{u}_1, ..., \mathbf{u}_n$ are 'linearly independent'?
   (b) Show that if $\mathbf{u}_1, ..., \mathbf{u}_n$ are mutually orthogonal non-zero vectors, then they are linearly independent.

4. (a) Define what we mean by the 'transition matrix for a Markov chain with s states'.
   (b) Suppose that $\mathbf{P}$ is the transition matrix for a Markov chain with s states. Show that $\mathbf{1}_s$ is an eigenvector of $\mathbf{P}$. What is the eigenvalue?

...over

5. Consider a regression model in which one makes n observations on a variable Y, which varies with regressors $X_1, ..., X_{p-1}$ according to

$$Y = \beta_0 + X_1\beta_1 + \cdots + X_{p-1}\beta_{p-1} + \varepsilon.$$

In matrix terms,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $\mathbf{X}$ is the $n \times p$ matrix with columns $\mathbf{1}_n, \mathbf{z}_1, ..., \mathbf{z}_{p-1}$ and $\mathbf{z}_i$ contains all n values of the variable X_i. Assume that the rank of $\mathbf{X}$ is p, so that the 'hat' matrix is $\mathbf{H} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$.

(a) Show that the sum of the elements in any row of $\mathbf{H}$ is one. (*Hint*: $\mathbf{HX} = \mathbf{X}$; what is the first column?)

(b) Show that the average of these diagonal elements is $\bar{h} = p/n$.

(c) Let $\hat{\boldsymbol{\beta}}$ be the vector of LSEs, $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$ the vector of 'fitted values', and $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$ the vector of residuals. Assuming that the errors ε_i are i.i.d., with mean 0 and variance σ^2, show that the covariance matrices of $\hat{\mathbf{y}}$ and $\mathbf{e}$ are $\sigma^2\mathbf{H}$ and $\sigma^2\left(\mathbf{I} - \mathbf{H}\right)$ respectively. [Note: This result implies that the variance of the i^{th} residual is $\text{VAR}[e_i] = \sigma^2\left(1 - h_{ii}\right)$, so that if h_{ii} is near 1, the corresponding residual must be near its expected value of 0 and $\hat{y}_i$ must be near $\mathbf{x}_i'\hat{\boldsymbol{\beta}}$, regardless of the observed value y_i. When this happens $\mathbf{x}_i$ is called a "highly influential" value, and the h_{ii} are called 'influence measures' - they are important tools in regression diagnostics.]

(d) Show that the i^{th} diagonal element of $\mathbf{H}$ is $h_{ii} = \mathbf{x}_i'\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{x}_i$, where $\mathbf{x}_i'$ is the i^{th} row of $\mathbf{X}$, and that $0 \leq h_{ii} \leq 1$.

6. (a) Define what we mean when we say that a matrix is symmetric and idempotent.

(b) Let $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$ be symmetric idempotent matrices. Show that $\mathbf{A} - \mathbf{B}$ is idempotent if $\mathbf{AB} = \mathbf{B}$.