# Lab 1

1. The rank of $\mathbf{X}$ is the dimension of the column space, and is also the number of independent columns of $\mathbf{X}$. This is '2' - which 2 columns of $\mathbf{X}$ are independent? (First note that one of them is clearly a linear combination of the other two.) They will form a basis - so write them down and verify that they span $\mathrm{col}(\mathbf{X})$ and are independent. Finally, relate the rank of $\mathbf{X}'\mathbf{X}$ to that of $\mathbf{X}$ – we covered exactly this in class.

2. *The $n$-step transition matrix, for a Markov chain with $s$ states, is given by $P^{(n)} = P^n$.* We did this in class for $n = 2$. To go from $n-1$ to $n$ (for any $n$) note that $\left[\mathbf{P}^{(n)}\right]_{ij}$ is by definition the probability of going from state $i$ to state $j$ in $n$ steps. So ask where you are after $n-1$ steps - you have to be somewhere. As in class:

$$\left[\mathbf{P}^{(n)}\right]_{ij} = \sum_{k=1}^{s} P\left(\text{go from } i \text{ to } k \text{ in } n-1 \text{ steps, then to } j \text{ in one step}\right)$$

   and exactly as in class – review this point now if you need to – you should conclude that this is the $(i,j)^{th}$ element of $\mathbf{P}^{(n-1)}\mathbf{P}$. So $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1)}\mathbf{P}$ – how do you finish off now? Fill in the steps of the inductive argument. For part (b) phrase the problem in terms of $\mathbf{P}^2$.

3. (a) See the list of axioms in Lecture 3. (b) *Prove: In any vector space the identity element* 0 *is unique.*
   You want to show that if $\tilde{\mathbf{0}}$ has the same properties as $\mathbf{0}$, i.e. if $\mathbf{x}+\tilde{\mathbf{0}} = \mathbf{x}$ for any $\mathbf{x}$ (call this statement (*)), then necessarily $\tilde{\mathbf{0}} = \mathbf{0}$. Look at the list of axioms: we have

$$\begin{aligned} \tilde{\mathbf{0}} &= \tilde{\mathbf{0}} + \mathbf{0} \text{ (by \#3)} \\ &= \mathbf{0} + \tilde{\mathbf{0}} \text{ (by\#2)} \\ &= \mathbf{0} \text{ (by statement (*) with } \mathbf{x} = \mathbf{0}\text{).} \end{aligned}$$

4. (a) Lecture 4. (b) *Prove: If $W$ is a vector subspace of a vector space $V$ then $\dim(W) \le \dim(V)$.*
   Suppose, for contradiction, that $\dim(W) = s > r = \dim(V)$. Then there is a basis $\mathbf{w}_1, ..., \mathbf{w}_s$ of $W$, and these vectors are also in $V$. Look now at Fact 2 from Lecture 3, to get the desired contradiction.

5. *Here you will investigate some properties of covariance matrices...*

Part (a) starts with expanding the product in the definition of $\mathbf{\Sigma}$ to get

$$\begin{aligned} E\left[\mathbf{x}\mathbf{x}' - \boldsymbol{\mu}\mathbf{x}' - \mathbf{x}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}'\right] &= E\left[\mathbf{x}\mathbf{x}'\right] - \boldsymbol{\mu}E\left[\mathbf{x}'\right] - E\left[\mathbf{x}\right]\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}' \\ &= E\left[\mathbf{x}\mathbf{x}'\right] - \boldsymbol{\mu}\boldsymbol{\mu}' - \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}' \\ &= E\left[\mathbf{x}\mathbf{x}'\right] - \boldsymbol{\mu}\boldsymbol{\mu}'. \end{aligned}$$

What BASIC fact – stressed in the lectures – is used in the first equality of this argument?

Part (b) involves the definition (Lecture 2) of the expected value of a vector or matrix as the vector or matrix of expected values. That independent random variable are uncorrelated starts with

$$\operatorname{COV}\left[X, Y\right] = E\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right]$$

and then uses the suggested characterization. For (c) start with the definition:

$$\operatorname{COV}\left[\mathbf{y}\right] = E\left[\left(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}\right)\left(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}\right)'\right],$$

where $\boldsymbol{\mu}_{\mathbf{y}}$ is the mean vector of $\mathbf{y}$. Now replace $\mathbf{y}$ by its expression in terms of $\mathbf{x}$, and $\boldsymbol{\mu}_{\mathbf{y}}$ with its expression in terms of $\boldsymbol{\mu}_{\mathbf{x}}$. Then let the linearity properties guide you. After a few steps you will be able to just insert your answer to (a).

6. *Suppose we gather data $X_1, ..., X_n$ and compute the sample average and variance. Show that ...*

Start by writing

$$\begin{aligned} S^2 &= \frac{1}{n-1}\left(X_1 - \bar{X}, ..., X_n - \bar{X}\right)\begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix} \\ &= \frac{1}{n-1}\left\{\left(X_1, ..., X_n\right) - \left(\bar{X}, ..., \bar{X}\right)\right\}\left\{\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} - \begin{pmatrix} \bar{X} \\ \vdots \\ \bar{X} \end{pmatrix}\right\}. \end{aligned}$$

Write the second term in braces in terms of $\mathbf{x}$ and $\mathbf{1}_n$ (hint: $\sum X_i = \mathbf{1}_n'\mathbf{x}$), and continue.