STAT 312 Lab 12

1. Suppose that $X_1, ..., X_n$ are i.i.d. r.v.s with density $p(x;\lambda) = \lambda e^{-\lambda x} / \left(1 - e^{-\lambda}\right)$ for $0 \le x \le 1$. This is the density of an $\mathbb{E}(\lambda)$ (exponential with parameter λ) r.v., subject to the restriction that it lie in $[0,1]$:

$$\begin{aligned} p(x;\lambda) &= \frac{d}{dx} P\left(X \le x | 0 \le X \le 1\right) = \frac{d}{dx} \frac{P\left(X \le x \text{ and } 0 \le X \le 1\right)}{P\left(0 \le X \le 1\right)} \\ &= \frac{d}{dx} \frac{\int_0^x \lambda e^{-\lambda t} dt}{1 - e^{-\lambda}} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda}}. \end{aligned}$$

   (a) Derive the Newton-Raphson iteration scheme for the computation of the maximum likelihood estimator $\hat{\lambda}$. Suggest a starting value. (As regards starting values there is no 'right' answer - just use your intuition to suggest something reasonable.)

   (b) Describe the asymptotic distribution of $\hat{\lambda}$, including an explicit expression for the variance.

2. Discuss the computation of the least squares estimates $\left(\hat{\alpha}, \hat{\beta}\right)$, obtained from the Gauss-Newton method, in the nonlinear regression model with exponential response

$$Y_i = \alpha e^{-\beta x_i} + \varepsilon_i.$$

   Write down the scheme explicitly. Suggest a method of obtaining starting values, based on the observation that, if the random error is ignored, then $\log Y$ is a linear function of the parameters.

3. Suggest a method of obtaining approximate 95% confidence intervals on each of the two parameters in the previous question.

4. Complete the example of Lecture 32, in order to obtain the 'Student's' t_{n-1} density

$$g_{n-1}(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}} \left(1 + \frac{t^2}{n-1}\right)^{-\frac{n}{2}}, -\infty < t < \infty.$$

   [*Hint*: You can differentiate $G_{n-1}(t)$, obtained in class, under the integral sign. Then make a change of variables in the resulting integral so as to recognize it as one for which you know the value.]

...over

5. Suppose that $X \sim N(\mu, \sigma^2)$. A 'normalized mean' is

$$\text{NM} = \mu/\sigma,$$

and expresses the mean as a multiple of the standard deviation (and hence has no units associated with it). Find an unbiased estimator of NM, based on a sample $X_1, ..., X_n \sim N(\mu, \sigma^2)$. [*Hint*: Start by looking at the mle, or at $\bar{X}/S$; evaluate its expected value by exploiting what you have learned about the distribution of $(\bar{X}, S)$ in Normal samples.]

6. Let $X_1, ..., X_n$ be a sample from a $N(\mu, \sigma^2)$ population, and let $\widehat{cv}$ be the *mle* of the coefficient of variation $\text{CV} = \sigma/\mu$. Thus the parameter vector is $\boldsymbol{\theta} = (\mu, \sigma^2)'$ and CV is a function $\tau(\boldsymbol{\theta})$. Obtain the approximate normal distribution of $\widehat{cv}$, including an explicit expression for the variance (which should turn out to be $\text{CV}^2\left(\frac{1}{2} + \text{CV}^2\right)/n$).