STAT 312 Lab 11

1. Define a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ by $f(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a symmetric matrix. Calculate:

   (a) the Jacobian $J_f(\mathbf{x})$,

   (b) the gradient $\nabla_f(\mathbf{x})$,

   (c) the Hessian $H_f(\mathbf{x})$.

2. Consider the problem of estimating a parameter vector $\boldsymbol{\beta}$, by Least Squares, in the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} +$ *random error.* Here $\mathbf{y} : n \times 1$ and $\mathbf{X} : n \times p$ are constants and $\mathbf{X}$ has rank $p < n$. Suppose that the parameters are required to satisfy q independent linear constraints of the form $\mathbf{A}\boldsymbol{\beta} = \mathbf{0}_{q\times 1}$, where $\mathbf{A}_{q\times p}$ has rank q. Thus the mathematical problem is

   $$\textit{Minimize } S(\boldsymbol{\beta}) = ||\mathbf{y} - \mathbf{X}\boldsymbol{\beta}||^2 \textit{ over } \boldsymbol{\beta} \in R^p\textit{, subject to constraints } \mathbf{A}\boldsymbol{\beta} = \mathbf{0}_{q\times 1}.$$

   Show that the solution to this problem is

   $$\hat{\boldsymbol{\beta}} = \left[\mathbf{I} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'\left(\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\right]\hat{\boldsymbol{\beta}}_{LS},$$

   where $\hat{\boldsymbol{\beta}}_{LS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is the unconstrained LS estimator.

3. The following problem is of interest in Mathematical Finance. A portfolio consists of n investments, whose possible returns can be viewed as random variables. The i^{th} investment has a mean return of μ_i, a variance of σ_{ii}, and the covariance between investments i and j is σ_{ij}. Define $\boldsymbol{\mu}_{n\times 1} = (\mu_1, ..., \mu_n)'$ and $\boldsymbol{\Sigma}_{n\times n} = (\sigma_{ij})$. Suppose that one invests a fraction w_i of the total (which we take as $= 1$, in appropriate units) into investment i. Define $\boldsymbol{w}_{n\times 1} = (w_1, ..., w_n)'$.

   (a) Show that the expected return on this portfolio is $\boldsymbol{\mu}'\boldsymbol{w}$, and that the variance is $\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$.

   (b) Suppose that one requires an expected return of μ_0. Derive the weights which will accomplish this, with a minimum possible variance. [*Hint*: this is a version of Problem (P) in Lecture 31.]

   (c) Explain why (b) is only a reasonable question if μ_0 lies between the smallest and largest of the μ_i.

...over

4. Recall from Lab 6 that the density of $X = Z^2$, where $Z \sim N(0,1)$, is $f_1(x) = \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}-1}}{2\Gamma\left(\frac{1}{2}\right)} e^{-x/2}$, the χ^2_1 density. Show that the χ^2_n density is $f_n(x) = \frac{\left(\frac{x}{2}\right)^{\frac{n}{2}-1}}{2\Gamma\left(\frac{n}{2}\right)} e^{-x/2}$. [*Hint*: What r.v. has this density? Use m.g.f.s.]

5. Suppose that $X_1, ..., X_n$ are i.i.d. $N(\mu, \sigma^2)$ r.v.s.

   (a) Obtain the mles of μ and σ^2.

   (b) Derive the information matrix.

   (c) What is the joint asymptotic distribution of the two mles? [*Note*: Expression (34.2) in the notes is usually the easier of the two ways to calculate the Information matrix.]

   (d) Obtain the mle of σ^2 if μ is <u>known</u>.

   (e) Is either mle of σ^2 unbiased?

6. Let $X_1, ..., X_n$ be a sample from the Uniform density

$$\begin{aligned} p(x;\theta) &= \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta, \\ 0, & \text{otherwise} \end{cases} \\ &= \frac{1}{\theta} I(0 \le x \le \theta). \end{aligned}$$

   Determine the MLE of θ. [*Hint*: This is a problem in which the 'regularity conditions' – being able to differentiate the likelihood, etc. – don't hold. Just plot the likelihood function to see where it is maximized.]