STAT 312 Lab 1

1. Let
$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

   (a) What is the rank of $\mathbf{X}$?

   (b) What is the dimension of the column space of $\mathbf{X}$?

   (c) Exhibit any basis for the column space of $\mathbf{X}$. Include a verification that the basis is a basis.

   (d) What is the rank of $\mathbf{X}'\mathbf{X}$? State (proof not required) a result which allows you to answer this without calculating $\mathbf{X}'\mathbf{X}$, and use this result in presenting your answer.

2. Recall the discussion of Markov chains, from class. Part (a) of this question was outlined there, for $n = 2$.

   (a) Show that the n-step transition matrix, for a Markov chain with s states and transition matrix $\mathbf{P}$, is given by $\mathbf{P}^{(n)} = \mathbf{P}^n$.

   (b) Consider a 2-state Markov chain. The two states (of the economy) are 'booming' and 'in recession'. Suppose that, if the economy is booming in one year, it remains in that state for one more year with probability .8, and otherwise goes into recession. If in recession, it recovers and booms the next year with probability .4. If the economy is booming this year, what is the probability that it will still be booming two years from now?

3. (a) State the definition of an identity element ('$\mathbf{0}$') in a vector space.

   (b) Prove: In any vector space the identity element $\mathbf{0}$ is unique.

...over

4. (a) Define what we mean by the 'dimension of a vector space'.

   (b) Prove: If W is a vector subspace of a vector space V, then $\dim(W) \leq \dim(V)$.

5. Here you will investigate some properties of covariance matrices. The major tool will be the linearity property of expectations. Suppose that $\mathbf{x} = (X_1, .., X_n)'$ is a random vector. Denote by $\boldsymbol{\mu}$ the mean vector $E[\mathbf{x}]$, and by $\boldsymbol{\Sigma}$ the covariance matrix $E\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'\right]$.

   (a) Show that
$$\boldsymbol{\Sigma} = E[\mathbf{x}\mathbf{x}'] - \boldsymbol{\mu}\boldsymbol{\mu}'.$$

   (b) Show that, if the X_i are i.i.d. ('independently and identically distributed') with mean μ and variance σ^2, then
$$\boldsymbol{\mu} = \mu \mathbf{1}_n, \;\; \boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_n.$$

   Your derivation should include a proof that independent random variables are uncorrelated, starting with the characterization that if X, Y are independent then $E[f(X)g(Y)] = E[f(X)]\,E[g(Y)]$ for all functions f, g for which $f(X)$ and $g(Y)$ are also random variables.

   (c) Show that if $\mathbf{A}$ is a matrix of constants (i.e. is non-random) and $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}$ as in (a), then
$$\textsc{cov}[\mathbf{y}] = \mathbf{A}\textsc{cov}[\mathbf{x}]\,\mathbf{A}' = \sigma^2 \mathbf{A}\mathbf{A}'.$$

6. Suppose we gather data $X_1, ..., X_n$ and compute the sample average and variance. Show that, if the data are placed into a vector: $\mathbf{x} = (X_1, ..., X_n)'$, and if $\mathbf{1}_n$ is the vector of n ones, then the sample average can be represented as $\bar{X} = \mathbf{1}_n'\mathbf{x}/n$, and the sample variance $S^2 = (n-1)^{-1} \sum_{i=1}^n \left(X_i - \bar{X}\right)^2$ can be represented as
$$S^2 = \frac{1}{n-1}\mathbf{x}'(\mathbf{I}_n - \mathbf{J})\,\mathbf{x},$$
   where $\mathbf{J} = \mathbf{1}_n\mathbf{1}_n'/n$ is an $n \times n$ matrix, each of whose elements equals $1/n$. The quantity S^2 is of course the usual (unbiased) estimator of the variance σ_X^2, in the case that the X_i form a random sample, i.e. are i.i.d.