Tentative list of possible midterm exam problems

The 'in class' midterm exam questions will be chosen from those on this list, which will not be 'official' until the word 'tentative' is removed from the title.

1. [**Lecture 3**] Let V be an n-dimensional vector space.

   (a) What does it mean to say that a set of elements of V spans the space?

   (b) Suppose that $\{\mathbf{v}_1, .., \mathbf{v}_n\}$ is a basis. Show that no proper subset of this basis can span the entire space.

2. [**Lecture 4**] Let $\mathbf{X}$ be an $n \times p$ matrix.

   (a) Define the 'column space of $\mathbf{X}$', and show that it is a vector space.

   (b) Define 'rank of $\mathbf{X}$' in terms of this column space.

3. [**Lecture 4**] Prove: If $\mathbf{A}_{n\times n}$ is of full rank, so that there is a right inverse $\mathbf{B}$ satisfying $\mathbf{AB} = \mathbf{I}_n$, then also $\mathbf{BA} = \mathbf{I}$.

4. [**Lecture 5**] Prove: if $\mathbf{x}, \mathbf{y}$ are nonzero $n \times 1$ vectors then there is an angle θ for which
$$\cos\theta = \frac{\mathbf{x}'\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

5. [**Lecture 6**] Define the 'trace' of a matrix. Show that if $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times m$ then $tr(\mathbf{AB}) = tr(\mathbf{BA})$.

6. [**Lecture 6**] Suppose that $\mathbf{v}_1, ..., \mathbf{v}_{m+1}$ are linearly independent vectors, and that one has constructed mutually orthogonal vectors $\mathbf{q}_1, ...\mathbf{q}_m$, each with a norm of 1, and with each $\mathbf{q}_i$ a linear combination of $\mathbf{v}_1, ...\mathbf{v}_i$. Show how one can construct a further vector $\mathbf{q}_{m+1}$ which is a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_{m+1}$, orthogonal to each of $\mathbf{q}_1, ...\mathbf{q}_m$, and with a norm of 1.

7. [**Lecture 6**] Suppose that $\mathbf{Q}$ is an $n \times n$ matrix. In terms of the columns of $\mathbf{Q}$, define what it means for $\mathbf{Q}$ to be *orthogonal*. Show that a consequence of your definition is that $\mathbf{QQ}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$. [Added note: on an exam you could state and then use, without proof, the result of question 3 above.]

8. [**Lectures 6-7**] Let $\mathbf{X}$ be an $n \times p$ matrix with full column rank.

(a) Suppose that one has carried out the Gram-Schmidt process, so as to obtain an orthogonal basis for the column space of $\mathbf{X}$. Explain how this process also yields the QR-decomposition of $\mathbf{X}$ into the product of a matrix with orthogonal columns, and an upper triangular, nonsingular matrix.

(b) In the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, the least squares estimates are $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. Explain how the QR-decomposition of $\mathbf{X}$ is employed, in order to compute $\hat{\boldsymbol{\beta}}$ without any matrix inversions – one need only solve a number of linear equations, each in only one unknown.

9. [**Lecture 7**] Consider vectors of the form $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{y}$ is $n \times 1$, $\mathbf{X}$ is $n \times p$ with rank p, and $\boldsymbol{\beta}$ is $p \times 1$. Such vectors occur in the theory of linear regression.

   (a) Show that this vector can be expressed as the sum of two vectors: $\mathbf{y} - \mathbf{X}\boldsymbol{\beta} = \mathbf{u} + \mathbf{v}$, in such a way that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other and $\mathbf{v}$ lies in the column space of $\mathbf{X}$.

   (b) Use (a) to show that the vector $\boldsymbol{\beta}$ which minimizes the norm of $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$ is the solution $\hat{\boldsymbol{\beta}}$ to the equation $\mathbf{v} = \mathbf{0}$. Solve this equation.

10. [**Lecture 7**] In the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, with independent errors having variance σ_ε^2, assume that $\mathbf{X}_{n\times p}$ has full rank, so that the least squares estimates are $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. Let $\hat{\mathbf{y}}$ be the vector $\mathbf{X}\hat{\boldsymbol{\beta}}$ of 'fitted values', and $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$ the vector of residuals.

    (a) Express $\mathbf{e}$ in terms of an appropriate 'hat' matrix.

    (b) Using properties of the matrix in (a) show that the sum of squares of the residuals, divided by $n - p$, is an unbiased estimator of σ_ε^2.

11. [**Lecture 8**] Let $\mathbf{M}$ be an $n \times n$ matrix, λ an eigenvalue. Show that the set of all eigenvectors which have λ as their eigenvalue is a vector space.

12. [**Lecture 8**] Prove: eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal to each other.

13. [**Lecture 9**] Prove: if $\mathbf{M}$ is a real, symmetric $n \times n$ matrix then the minimum value of $\mathbf{x}'\mathbf{M}\mathbf{x}$, over all vectors $\mathbf{x}$ with unit norm, is given by the smallest of the eigenvalues of $\mathbf{M}$.

14. [**Lecture 9**] Let $\mathbf{M}$ be a real, symmetric $n \times n$ matrix, and suppose that $\mathbf{M}$ is positive semidefinite. Show how one can construct a symmetric square root of $\mathbf{M}$.

15. [**Lecture 9**] Prove: if $\mathbf{H}$ is an $n \times n$ idempotent matrix, and λ is an eigenvalue, then $\lambda \in \{0, 1\}$.

16. [**Lecture 9**] Let $\mathbf{H}$ be an $n \times n$ idempotent matrix of rank r. Show that one can represent $\mathbf{H}$ as $\mathbf{H} = \mathbf{V}\mathbf{V}'$, where $\mathbf{V}$ is $n \times r$ and $\mathbf{V}'\mathbf{V} = \mathbf{I}_r$.

17. [**Lecture 9**] Let $\mathbf{A}_{p\times p}$ be a (symmetric and) positive definite matrix and let $\mathbf{B}_{p\times q}$ have rank $q \leq p$. Consider the matrix $\mathbf{M} = \mathbf{B}'\mathbf{A}\mathbf{B}$.

    (a) In terms of quadratic forms $\mathbf{x}'\mathbf{M}\mathbf{x}$, what does it mean to say that $\mathbf{M}$ is <u>positive definite</u>?

    (b) Show that $\mathbf{M}$ is positive definite.

18. [**Lecture 9**] Prove: if $\mathbf{S}$ is a $p \times p$ (symmetric and) positive semidefinite matrix of rank $q \leq p$, then one can find $\mathbf{M}_{p\times q}$ such that $\mathbf{S} = \mathbf{M}\mathbf{M}'$ and $\mathbf{M}'\mathbf{M}$ is the $q \times q$ diagonal matrix of the positive eigenvalues of $\mathbf{S}$.

19. [**Lectures 10-11**] Calculate (a) the determinant, and (b) the inverse, of the matrix $\mathbf{I}_n + \mathbf{1}_n\mathbf{1}_n'$.

20. [**Lecture 10**] In the 'classification of skulls' example discussed in class, we reached a point at which we were to take linear functions $\boldsymbol{\alpha}'\bar{\mathbf{x}}$ and $\boldsymbol{\alpha}'\bar{\mathbf{y}}$ of the vectors of averages of the two 'training samples', and then classify a new observation $\mathbf{z}$ on the basis of the value of $\boldsymbol{\alpha}'\mathbf{z}$. The 'best' $\boldsymbol{\alpha}$ was to be the maximizer of

$$\frac{\boldsymbol{\alpha}'\left(\bar{\mathbf{x}} - \bar{\mathbf{y}}\right)\left(\bar{\mathbf{x}} - \bar{\mathbf{y}}\right)'\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\mathbf{S}\boldsymbol{\alpha}},$$

    where $\mathbf{S}$ is the pooled covariance matrix of the training samples, assumed positive definite. Derive this best $\boldsymbol{\alpha}$.

21. [**Lecture 11**] Suppose that $\mathbf{x}$ is an $n \times 1$ nonzero vector and $\mathbf{M}_{n\times n} = \mathbf{x}\mathbf{x}'$.

    (a) Show (state any results from the lectures used in your derivation) that $\mathbf{M}$ has exactly one nonzero eigenvalue.

    (b) What is the eigenvalue in (a)?

22. [**Lecture 12**] Consider the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, with uncorrelated, equally varied errors ε and with $\mathbf{X}_{n\times p}$ having full column rank. Suppose that we seek to estimate a linear combination $\alpha = \mathbf{a}'\boldsymbol{\beta}$ by an *unbiased, linear* estimate $\hat{\alpha} = \mathbf{c}'\mathbf{Y}$. Show that the minimum variance estimate in this class, i.e. the 'Best Linear Unbiased Estimate' (BLUE), is $\hat{\alpha}_{BLUE} = \mathbf{a}'\hat{\boldsymbol{\beta}}_{OLS} = \mathbf{a}'\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$.

23. [**Lectures 13-14**] Prove: Unions of (arbitrarily many) open sets in $\mathbb{R}^n$ are themselves open.

24. [**Lectures 13-14**] Prove: If $f(x) \to 0$ and $g(x) \to L$ (finite) as $x \to 0$, then $f(x)g(x) \to 0$.

25. [**Lectures 13-14**] Prove: If a function $f(x)$ is non-negative for all x, and $f(x) \to L$ as $x \to a$, then $L \geq 0$.

26. [**Lecture 14**] Suppose that f is a function defined on an open interval $D \subset \mathbb{R}$, and that $f'(x_0)$ exists for some $x_0 \in D$. Show that then f is continuous at x_0.

27. [**Lecture 15**] Prove the Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b) then $\exists c \in (a, b)$ with $f(b) = f(a) + f'(c)(b - a)$.

28. [**Lecture 16**] Let $(\Omega, \mathbb{B}, P)$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a random variable. This requires the inverse image, under X, of any open set in $\mathbb{R}$ to be an event. Show that then the inverse image of any closed set is an event. State any results from the lectures which you use in your derivation.

29. [**Lecture 16**] Let $(\Omega, \mathbb{B}, P)$ be a probability space. Prove, from the three basic axioms which must be satisfied by $\mathbb{B}$, that if E and F are events, then the difference $F \backslash E$ is an event.