Addendum: Derivation of H_3

From (5) and (6) we have

$$H_3(v;\mu,\sigma) = \int_{-\infty}^{v/3} H_1(u;\mu,\sigma) \sum_{j_1<j_2<j_3} \sum_{l=1}^{3} \psi_l\left(u;j_1,j_2,j_3\right) du,$$

where

$$\psi_l\left(u;j_1,j_2,j_3\right) = \frac{\phi\left(u+\frac{\mu_{j_l}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_l}}{\sigma}\right)} \int_u^{v-(k-1)u} \frac{\phi\left(u_k+\frac{\mu_{j_k}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_k}}{\sigma}\right)} \cdots \int_u^{v-lu-\sum_{i>l+1}u_i} \frac{\phi\left(u_{l+1}+\frac{\mu_{j_{l+1}}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_{l+1}}}{\sigma}\right)} \cdot$$

$$\cdot \int_u^{v-(l-1)u-\sum_{i>l-1,i\neq l}u_i} \frac{\phi\left(u_{l-1}+\frac{\mu_{j_{l-1}}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_{l-1}}}{\sigma}\right)} \cdots \int_u^{v-u-\sum_{i>1,i\neq l}u_i} \frac{\phi\left(u_1+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}$$

$$du_1\cdots du_{l-1}du_{l+1}\cdots du_k.$$

So

$$\psi_1\left(u;j_1,j_2,j_3\right) = \frac{\phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)} \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \int_u^{v-u-t} \frac{\phi\left(s+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} dsdt,$$

$$\psi_2\left(u;j_1,j_2,j_3\right) = \frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \int_u^{v-u-t} \frac{\phi\left(s+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)} dsdt,$$

$$\psi_3\left(u;j_1,j_2,j_3\right) = \frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \int_u^{v-u-t} \frac{\phi\left(s+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)} dsdt.$$

Put, in general,

$$\begin{aligned}\psi\left(u;j_1,j_2,j_3\right) &= \frac{\phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)} \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \int_u^{v-u-t} \frac{\phi\left(s+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dsdt \\ &= \frac{\phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)} \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - 1\right] dt,\end{aligned}$$

then

$$\begin{aligned}\sum_{j_1<j_2<j_3} \sum_{l=1}^{3} \psi_l\left(u;j_1,j_2,j_3\right) &= \sum_{j_1<j_2<j_3} \left\{\psi\left(u;j_1,j_3,j_2\right) + \psi\left(u;j_2,j_3,j_1\right) + \psi\left(u;j_3,j_2,j_1\right)\right\} \\ &= \left\{\sum_{j_1<j_3<j_2} + \sum_{j_2<j_3<j_1} + \sum_{j_3<j_2<j_1}\right\} \psi\left(u;j_1,j_2,j_3\right).\end{aligned}$$

An integration by parts shows that $\psi\left(u;j_1,j_2,j_3\right) = \psi\left(u;j_1,j_3,j_2\right)$, so that the above is one-half the sum over *all* $n(n-1)(n-2)$ triples (j_1,j_2,j_3) of distinct indices. Thus

$$\begin{aligned}H_3(v;\mu,\sigma) &= \frac{1}{2}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma) \sum_{j_2\neq j_3} \sum_{j_1\notin\{j_2,j_3\}} \psi\left(u;j_1,j_2,j_3\right) du \\ &= \frac{1}{2} \sum_{j_2\neq j_3} \int_{-\infty}^{v/3} \left\{H_1(u;\mu,\sigma) \sum_{j_1\notin\{j_2,j_3\}} \frac{\phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_1}}{\sigma}\right)}\right\} \cdot \\ &\cdot \int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - 1\right] dtdu. \qquad \text{(B.1)}\end{aligned}$$

In terms of the density h_1 of H_1, this is

$$
\begin{aligned}
&H_3(v;\mu,\sigma)\\
&=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}\left\{h_1(u;\mu,\sigma)-H_1(u;\mu,\sigma)\left[\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}+\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right]\right\}\cdot\\
&\cdot\int_{u}^{v-2u}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dtdu\\
&=I_1-I_2,
\end{aligned}
\tag{B.2}
$$

where

$$
I_1=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}\left\{\int_{u}^{v-2u}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\right\}dH_1(u;\mu,\sigma),
$$

$$
I_2=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left[\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}+\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right]\left\{\int_{u}^{v-2u}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\right\}du.
$$

Evaluating I_1 by parts (noting that the integral in braces vanishes at both $u=v/3$ and $u=-\infty$) gives

$$
\begin{aligned}
I_1&=-\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{d}{du}\left\{\int_{u}^{v-2u}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\right\}du\\
&=-\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left(\begin{array}{c}-\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]\\+\int_{u}^{v-2u}\frac{d}{du}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\end{array}\right)du\\
&=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]du\\
&-\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\int_{u}^{v-2u}\frac{d}{du}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dtdu.
\end{aligned}
\tag{B.3}
$$

Substituting (B.3) into (B.2) gives

$$
\begin{aligned}
&H_3(v;\mu,\sigma)\\
&=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]du\\
&-\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left\{\begin{array}{c}\int_{u}^{v-2u}\frac{d}{du}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]\\+\left[\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}+\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right]\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\end{array}\right\}du\\
&=J_1-J_2,
\end{aligned}
\tag{B.4}
$$

where

$$
J_1=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]du,
$$

$$
J_2=\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left\{\begin{array}{c}\int_{u}^{v-2u}\frac{d}{du}\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]\\+\left[\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}+\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right]\frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dt\end{array}\right\}du.
$$

We evaluate J_1 by noting again that the integrand involves the density of H_1:

$$\begin{aligned}
J_1 &= \frac{1}{2}\sum_{j_3=1}^{n}\int_{-\infty}^{v/3}\left\{h_1(u;\mu,\sigma) - H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right\}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]du \\
&= \frac{1}{2}\sum_{j_3=1}^{n}\int_{-\infty}^{v/3}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]dH_1(u;\mu,\sigma) \\
&- \frac{1}{2}\sum_{j_3=1}^{n}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]du \\
&= \frac{1}{2}\sum_{j_3=1}^{n}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\left\{\begin{array}{c}\frac{2\phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}+\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \\ -\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\left[\frac{\Phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right]\end{array}\right\}du \\
&= \sum_{j_3=1}^{n}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}du + \frac{1}{2}\sum_{j_3=1}^{n}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}du \\
&= \sum_{j_3=1}^{n}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\frac{\phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}du + \frac{1}{2}H_1(v/3;\mu,\sigma). \qquad \text{(B.5)}
\end{aligned}$$

To evaluate J_2 we substitute

$$\begin{aligned}
&\frac{d}{du}\frac{1}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right] \\
&= \frac{-\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-1\right] \\
&+ \frac{1}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}\left[\frac{-\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right] \\
&= -\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}+\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \\
&- \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}-\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi^2\left(u+\frac{\mu_{j_3}}{\sigma}\right)}
\end{aligned}$$

into the definition of J_2, obtaining

$$J_2 = \frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3}H_1(u;\mu,\sigma)\Omega(u)du, \qquad \text{(B.6)}$$

where

$$\begin{aligned}
\Omega(u) &= \int_u^{v-2u} \left\{ \begin{array}{c} \phi\left(t+\frac{\mu_{j_2}}{\sigma}\right) \left[\begin{array}{c} -\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} + \frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \\ -\frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - \frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi^2\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \end{array} \right] \\ + \left[\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} + \frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}\right] \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - 1\right] \end{array} \right\} dt \\
&= \int_u^{v-2u} \left\{ \begin{array}{c} -\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} + \frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \\ -\frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - \frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi^2\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \\ +\frac{\phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi^2\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - 1\right] + \frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left[\frac{\Phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - 1\right] \end{array} \right\} dt \\
&= \int_u^{v-2u} \left\{ -\frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} - \frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \right\} dt \\
&= -\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left(\frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right) + \phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \right) dt. \qquad \text{(B.7)}
\end{aligned}$$

Now (B.7) in (B.6) gives

$$\begin{aligned}
J_2 &= -\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \left(\frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right) + \phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \right) dtdu \\
&= -\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dtdu \\
&\quad -\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\frac{\phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} \left[\frac{\Phi\left(v-2u+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} - 1\right] du \\
&= -\frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dtdu - J_1. \qquad \text{(B.8)}
\end{aligned}$$

Finally, (B.5) and (B.8) in (B.4) give

$$\begin{aligned}
H_3(v;\mu,\sigma) &= 2J_1 + \frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dtdu \\
&= 2\sum_{j_3=1}^{n}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\frac{\phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} du + H_1(v/3;\mu,\sigma) \\
&\quad + \frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dtdu \\
&= H_1(v/3;\mu,\sigma) + 2\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\sum_{j_3=1}^{n} \frac{\phi\left(v-2u+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} du \\
&\quad + \frac{1}{2}\sum_{j_2\neq j_3}\int_{-\infty}^{v/3} H_1(u;\mu,\sigma)\int_u^{v-2u} \frac{\phi\left(t+\frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v-u-t+\frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u+\frac{\mu_{j_3}}{\sigma}\right)} dtdu.
\end{aligned}$$

Since (as is seen by making the change of variables $s = v - u - t$)

$$\int_u^{v-2u} \frac{\phi\left(t + \frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v - u - t + \frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_3}}{\sigma}\right)} dt = \int_u^{v-2u} \frac{\phi\left(t + \frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_3}}{\sigma}\right)} \frac{\phi\left(v - u - t + \frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_2}}{\sigma}\right)} dt,$$

the last sum is

$$\sum_{j_2 < j_3} \int_{-\infty}^{v/3} H_1(u; \mu, \sigma) \int_u^{v-2u} \frac{\phi\left(t + \frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v - u - t + \frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_3}}{\sigma}\right)} dt du,$$

giving

$$\begin{aligned} H_3(v; \mu, \sigma) = H_1(v/3; \mu, \sigma) + 2 \int_{-\infty}^{v/3} H_1(u; \mu, \sigma) \sum_{j_3=1}^{n} \frac{\phi\left(v - 2u + \frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_3}}{\sigma}\right)} du \\ + \sum_{j_2 < j_3} \int_{-\infty}^{v/3} H_1(u; \mu, \sigma) \int_u^{v-2u} \frac{\phi\left(t + \frac{\mu_{j_2}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_2}}{\sigma}\right)} \frac{\phi\left(v - u - t + \frac{\mu_{j_3}}{\sigma}\right)}{\Phi\left(u + \frac{\mu_{j_3}}{\sigma}\right)} dt du, \end{aligned}$$

which is (8).