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DISTRIBUTIONS MINIMIZING FISHER INFORMATION FOR SCALE IN KOLMOGOROV NEIGHBOURHOODS

by

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DISTRIBUTIONS MINIMIZING FISHER INFORMATION FOR SCALE IN KOLMOGOROV NEIGHBOURHOODS

(Abbreviated title: MINIMUM INFORMATION DISTRIBUTIONS)

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ABSTRACT

We construct those distributions minimizing Fisher information for scale in Kolmogorov neighbourhoods $\mathcal{K}_\varepsilon(G) = \{F | \sup_x |F(x) - G(x)| \leq \varepsilon\}$ of d.f.s. G satisfying certain mild conditions. The theory is sufficiently general to include those cases in which G is normal, Laplace, logistic, Student's t, etc. As well, we consider $G(x) = 1 - e^{-x}$, $x \geq 0$, and correct some errors in the literature concerning this case.

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1. Introduction and Summary. In the theory of robust, minimax variance estimation as developed by Huber (1964, 1981), a frequently occuring problem is that of determining that member of a certain class of distributions, representing all "reasonable" departures from a "target" distribution, which minimizes Fisher information for the quantity being estimated. Such departures are often modelled by Kolmogorov neighbourhoods:

$$\mathcal{K}_\varepsilon(G) = \{F | \sup_{-\infty<x<\infty} |F(x) - G(x)| \leq \varepsilon\}$$

in which ε and G are known and fixed.

Huber (1964) minimized information for *location* in $\mathcal{K}_\varepsilon(\Phi)$, $\varepsilon \leq .0303$. Here, Φ is the standard normal d.f. Sacks and Ylvisaker (1972) extended this to the range $.0303 \leq \varepsilon \leq .5$. Wiens (1986) considered this problem for general, symmetric G. Collins and Wiens (1989) extended these results to Lévy neighbourhoods of d.f.s G satisfying conditions similar to those imposed in Wiens (1986).

The problem of minimizing information for *scale* in Kolmogorov neighbourhoods has hitherto not received a systematic treatment in the literature. Note that if σ is a scale parameter for a r.v. X, then $\log \sigma$ is a location parameter for $\log |X|$. By this device, certain results for location estimation may be transfered to the problem of scale estimation. This approach was taken by Huber (1981) in minimizing information for scale in the gross errors neighbourhood

$$\mathcal{G}_\varepsilon(\Phi) = \{F = (1-\varepsilon)\Phi + \varepsilon H, \; H \;\; \text{arbitrary}\}.$$

The log tranformation was useful here since, under it, the gross errors structure is maintained *and* there already existed a theory of minimum information for location for $\mathcal{G}_\varepsilon(G)$, with G asymmetric.

In contrast, although the Kolmogorov neighbourhood structure is maintained under the log tranformation, a common assumption in all of the aforementioned papers on minimizing information for location in $\mathcal{K}_\varepsilon(G)$ is that G be symmetric. This symmetry is typically destroyed by the taking of logs. Thus, the problem of minimizing information for scale requires either a direct approach, or the derivation of a location estimation theory which is valid in $\mathcal{K}_\varepsilon(G)$ for asymmetric G.

We here adopt the former approach. We shall present a theory, valid for scale estimation in $\mathcal{K}_\varepsilon(G)$, with G satisfying certain mild assumptions. The results are general enough to include the cases $G = \Phi$, G the Laplace d.f., and more generally $G = G_\ell$, with density proportional to $\exp(-|x|^\ell/\ell)$, $1 \le \ell \le 2$. The logistic and Student's t distributions are covered as well. We also consider the case in which $G(x) = 1 - e^{-x}$, $x \ge 0$; in so doing we correct some errors made in Thall (1979).

The proof of Theorem 1 below is completely analogous to that of Theorem 4.4.2 of Huber (1981), and so is omitted.

DEFINITION: Fisher information for scale of a distribution F on the real line is

$$I(F;1) = \sup_{x} \frac{(\int_{-\infty}^{\infty} x\chi'(x)dF(x))^2}{\int_{-\infty}^{\infty} \chi^2(x)dF(x)},$$

where the sup is taken over all continuously differentiable functions χ with compact support, satisfying $\int_{-\infty}^{\infty} \chi^2(x)dF(x) > 0$.

THEOREM 1. *The following two assertions are equivalent:*

(1) $I(F;1) < \infty$.

(2) F *has a density, absolutely continuous on* $\mathbb{R}\backslash\{0\}$, *satisfying:*

(i) $xf(x) \to 0$ *as* $x \to 0, \pm\infty$;

(ii) $\int_{-\infty}^{\infty}(-x\,\frac{f'}{f}\,(x) - 1)^2 f(x)dx < \infty$.

In either case, we have

$$I(F;1) = \int (-x\,\frac{f'}{f}\,(x) - 1)^2 f(x)dx < \infty.$$

REMARK 1: Define $F_\sigma(x) = F(\frac{x}{\sigma})$ for $\sigma > 0$. Then if $I(F;\sigma)$ is defined as $I(F_\sigma;1)$, we have that the value of this functional is $\frac{1}{\sigma^2}\, I(F;1)$.

REMARK 2: An M-estimate of scale is defined as $S(F_n)$, where F_n is the empirical distribution function based on a sample $X_1, \ldots, X_n \sim F$, and the functional $S(F)$ is defined implicitly by

$$\int_{-\infty}^{\infty} \chi\left(\frac{x}{S(F)}\right) dF(x) = 0. \tag{1.1}$$

Under appropriate regularity conditions (see for example Boos and Serfling (1980) and Serfling (1981))

$$\sqrt{n}\left(\frac{S(F_n)}{S(F)} - 1\right) \xrightarrow{w} N(0, V(\chi, F)), \tag{1.2}$$

where

$$V(\chi, F) = \frac{\int_{-\infty}^{\infty} \chi^2\left(\frac{x}{S(F)}\right) dF(x)}{\left[\int_{-\infty}^{\infty} \chi'\left(\frac{x}{S(F)}\right) \frac{x}{S(F)}\, dF(x)\right]^2}\,.$$

Now let $\mathcal{F}$ be a given convex class of distributions, and suppose that F_0 minimizes $I(F;1)$ in $\mathcal{F}$. Define $\chi_0(x) = -x\, \frac{f_0'}{f_0}\,(x) - 1$, corresponding to maximum likelihood estimation of σ if $X_1, \ldots, X_n \sim F_{0,\sigma}$. Define $S_0(F)$ by (1.1), with $\chi = \chi_0$. Then $S_0(F_0) = 1$. We have

$$V(\chi_0, F) \le V(\chi_0, F_0) = \frac{1}{I(F_0;1)} \le V(\chi, F_0) \tag{1.3}$$

for all $F \in \mathcal{F}_1 = \{F \in \mathcal{F} | S_0(F) = 1\}$ and all χ such that (1.1) holds for $F \in \mathcal{F}_1$. The second inequality in (1.3) is essentially the Cramèr-Rao Inequality; the first is established by variational arguments, as in Huber (1964, 1981). It follows from (1.3) that

$$\sup_{\mathcal{F}_1} V(\chi_0, F) = \inf_{\chi} \sup_{\mathcal{F}_1} V(\chi, F),$$

so that χ_0 yields a minimax variance estimate of scale for $F \in \mathcal{F}_1$.

The question of whether or not the saddlepoint property (1.3) extends to all of $\mathcal{F}$ will be considered, for $\mathcal{F} = \mathcal{K}_\varepsilon(G)$ and $\mathcal{F} = \mathcal{G}_\varepsilon(G)$, in a forthcoming paper. We note that this question has been answered, in the affirmative, by Huber (1981) in the class $\mathcal{G}_\varepsilon(\Phi)$, $\varepsilon \leq .04$.

2. General Theory. Define $\mathcal{K}'_\varepsilon(G) = \{F \in \mathcal{K}_\varepsilon(G) | I(F;1) < \infty\}$ and assume that $G \in \mathcal{K}'_\varepsilon(G)$. Then as at Vandelinde (1979), $\mathcal{K}'_\varepsilon(G)$ is weakly dense in $\mathcal{K}_\varepsilon(G)$. The methods of Huber (1964, 1981) may now be employed to show the existence of an information minimizing $F_0 \in \mathcal{K}'_\varepsilon(G)$. By Lemma 4.4.4 of Huber (1981), $I(F;1)$ is a convex functional of $F \in \mathcal{K}'_\varepsilon(G)$. It follows that $I(F_0;1) = \min\limits_{\mathcal{K}_\varepsilon(G)} I(F;1)$ iff, for all $F_1 \in \mathcal{K}'_\varepsilon(G)$

$$\begin{aligned} 0 &\leq \frac{d}{dt} I((1-t)F_0 + tF_1;1)_{|t=0} \\ &= \int_{-\infty}^{\infty} \{2x[x \frac{f_0'}{f_0}(x) + 1](f_1' - f_0')(x) \\ &\quad - [x^2(\frac{f_0'}{f_0})^2(x) - 1](f_1 - f_0)(x)\}dx. \end{aligned} \tag{2.1}$$

This condition becomes more useful if an integration by parts is possible. First, define

$$\chi_0(x) = -x \frac{f_0'}{f_0}(x) - 1,$$

and define an operator J on the class of absolutely continuous functions on $\mathbb{R}$ by

$$J(\chi)(x) = 2x\chi'(x) - \chi^2(x).$$

Extend J by left continuity where χ' is discontinuous.

LEMMA 2. *If F_0 is such that χ_0 is absolutely continuous and bounded, then in order that F_0 minimize $I(F;1)$ in $\mathcal{K}_\varepsilon(G)$ it is necessary and sufficient that*

(1) $F_0 \in \mathcal{K}_\varepsilon(G)$;

(2) $0 \leq \int_{-\infty}^{\infty} J(\chi_0)(x) d(F_1 - F_0)(x)$, *for all* $F_1 \in \mathcal{K}'_\varepsilon(G)$.

PROOF: Integrate by parts in (2.1), using condition 2(i) of Theorem 1.

□

REMARK 3: As at Section 5.6 of Huber (1981), if for each $F \in \mathcal{K}'_\varepsilon(G)$ we have also $\bar{F} \in \mathcal{K}'_\varepsilon(G)$, where $\bar{F}(x) = 1 - F(-x)$, then $\tilde{F} = \frac{1}{2}(F + \bar{F}) \in \mathcal{K}'_\varepsilon(G)$ and $I(\tilde{F}; 1) \leq I(F; 1)$. We need then consider only symmetric distributions. This condition on $\mathcal{K}_\varepsilon(G)$ is satisfied if G is symmetric.

Motivated by the above considerations, we make the following assumptions on G :

G1) G is symmetric and strictly increasing on $(-\infty, \infty)$, with $G(\infty) = 1$.

G2) $0 < I(G; 1) < \infty$.

G3) The denisty g of G, absolutely continuous by G2), is strictly decreasing on $(0, \infty)$.

G4) $\xi(x) := -x \frac{g'}{g}(x) - 1$ is absolutely continuous on $(-\infty, \infty)$, with an absolutely continuous derivative $\xi'(x)$; $\xi(0) = -1$ and $\lim_{x \to 0} x\xi''(x) = 0$.

In Section 3 below we exhibit the minimum information members F_0 of $\mathcal{K}_\varepsilon(G)$ for a variety of distributions G satisfying G1)-G4). In each case, that these F_0 do minimize $I(F; 1)$ follows from the following theorem.

THEOREM 3. *If* F_0 *possesses the following properties, then it is the unique member of* $\mathcal{K}'_\varepsilon(G)$ *minimizing* $I(F; 1)$ *over* $\mathcal{K}_\varepsilon(G)$:

S1) $F_0 \in \mathcal{K}_\varepsilon(G)$, F_0 *symmetric,* $F_0(\infty) = 1$.

S2) χ_0 *is absolutely continuous and bounded.*

S3) *There exists a sequence* $0 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_{n-1} < b_{n-1} \leq a_n$, *and constants* $\lambda_1, \ldots, \lambda_n$, *such that:*

$$(i)\quad J(\chi_0)(x) = \begin{cases} \lambda_1, & 0 < x \le b_1; \\ \lambda_i, & a_i < x \le b_i,\ i = 2, \ldots, n-1; \\ \lambda_n < 0, & a_n < x; \\ J(\xi)(x), & b_i < x \le a_{i+1}. \end{cases}$$

(ii) *With* $B_L := \{x | F_0(x) = G(x) - \varepsilon\}$ *and* $B_U := \{x | F_0(x) = G(x) + \varepsilon\}$,

$$B_U \cup B_L = \{\bigcup_{i=1}^{n-1} [b_i, a_{i+1}]\} \cup \{\bigcup_{i=1}^{n-1} [-a_{i-1}, -b_i]\}.$$

(iii) *If* $a_i \in B_L[B_U]$ *then* $J(\chi_0)(a_i^+) = \lambda_i \le [\ge] J(\chi_0)(a_i^-)$. *If* $b_i \in B_L[B_U]$ *then* $J(\chi_0)(b_i^-) = \lambda_i \ge [\le] J(\chi_0)(b_i^+)$.

(iv) *If* (b_i, a_{i+1}) *is non-empty and contained in* $B_L[B_U]$ *then* $J(\xi)$ *is weakly decreasing [increasing] there.*

PROOF: Conditions S1) and S2) allow one to apply Lemma 2, and to satisfy condition (1) of the Lemma. Condition S3) guarantees that condition (2) of Lemma 2 is satisfied. To see this, split the range of integration up into intervals, and then integrate by parts over those non-empty intervals (b_i, a_{i+1}), using G4). The integral can then be re-arranged as a sum of non-negative terms, using (iii) and (iv) of S3).

The uniqueness of F_0 now follows from Proposition 4.4.5 of Huber (1981), if

(i) $f_0(x) > 0, \quad x \in (-\infty, \infty)$;

(ii) $0 < I(F_0; 1) = \int_{-\infty}^{\infty} \chi_0^2(x) dF_0(x)$.

These follow from G3), and the observation that no solutions to the equation $J(\chi_0) = constant$ can remain bounded as $f_0(x) \to 0$ — see Remark 4 below.

□

REMARK 4: The possible solutions to the equation $J(\chi_0)(x) = constant$ are given by:

(1) $J(\chi_0)(x) = \lambda^2,\ \lambda > 0:$

$$\chi_0(x) = \lambda \tan(\frac{\lambda}{2} \log x + c);$$
$$f_0(x) = \frac{c_1 \cos^2(\frac{\lambda}{2} \log x + c)}{x}; \quad c \in \mathbb{R},\ c_1 > 0.$$

(2) $J(\chi_0)(x) = -\lambda^2, \quad \lambda > 0:$

$$\chi_0(x) = \lambda, -\lambda, \quad \text{or} \quad \frac{\lambda(1 + c_2 x^\lambda)}{1 - c_2 x^\lambda};$$

correspondingly

$$f_0(x) = c_3 x^{-(\lambda+1)}, \quad c_4 x^{\lambda-1}, \quad \textit{or} \quad \frac{(1 - c_2 x^\lambda)^2}{x^{\lambda+1}}; \quad c_2 \in \mathbb{R}, \quad c_3, c_4 > 0.$$

3. Some Classes of Solutions. In this section we present the minimum information members $F_0 \in \mathcal{K}_\varepsilon(G)$ for those G satisfying assumptions G1) - G4) of Section 2, and for which the following additional assumptions are satisfied:

J1) $J(\xi)(0) = -1$; $\frac{d^2}{dx^2} J(\xi)(x)_{|x=0} > 0$.

J2) There exists $k \in [1, \infty]$ for which

$$J(\xi)(x) \to -k \quad \text{as} \quad x \to \pm\infty.$$

J3) $J(\xi)(x)$ has exactly one local maximum $(m, J(\xi)(m))$ for $m \in (0, \infty)$, and $J(\xi)(m) > 0$.

J4) $J(\xi)(x)$ is strictly increasing on $(0, m)$, strictly decreasing on (m, ∞).

REMARK 5: Assumptions G1) - G4), J1) - J4), henceforth referred to as G) and J), hold for the logistic distribution, all Student's t distributions, and those G_ℓ defined in

Section 1, for $1 < \ell \leq 2$. Note that G_2 is the normal d.f., G_1 the Laplace. For G_1, see Remark 6 below.

Lemma 4. *If G and $J(\xi)$ satisfy assumptions G) and J), then $\xi(x)$ is strictly increasing on $[0,\infty)$, with $\xi(0) = -1$, $\xi(\infty) > 0$.*

Proof: The proof is lengthy and rather technical, and so the reader is referred to the Ph.D. thesis of Wu (1990) for details.

□

Remark 6: The remaining results of this section are stated under the assumptions G) and J). In all of them, J) may be replaced by "J2), J3), J4) and the conclusions of Lemma 4." In particular, the solutions below then apply to the Laplace distribution G_1.

Theorem 5. *(Small ε) If G) and J) hold, then there exists $\varepsilon_0(G)$ such that for $0 < \varepsilon < \varepsilon_0(G)$, Fisher information for scale is minimized in $\mathcal{K}_\varepsilon(G)$ by that F_0 with:*

$$\chi_0(x) = \chi_0(-x) = \begin{cases} \xi(a), & 0 \leq x \leq a; \\ \xi(x), & a \leq x \leq b; \\ \delta \ \tan(\frac{\delta}{2} \ \log \ x + w) & b \leq x \leq c; \\ \xi(x), & c \leq x \leq d; \\ \xi(d), & d \leq x; \end{cases}$$

$$f_0(x) = f_0(-x) = \begin{cases} g(a)(a/x)^{1+\xi(a)}, & 0 < x \leq a; \\ g(x), & a \leq x \leq b; \\ \dfrac{bg(b)}{\cos^2(\frac{\delta}{2} \ \log \ b + w)} \cdot \dfrac{\cos^2(\frac{\delta}{2} \ \log \ x + w)}{x}, & b \leq x \leq c; \\ g(x), & c \leq x \leq d; \\ g(d)(d/x)^{1+\xi(d)}, & d \leq x. \end{cases}$$

The constants δ *and* w *are given by*

$$\delta =: \left(\frac{bg(b)\xi^2(b) - cg(c)\xi^2(c)}{cg(c) - bg(b)}\right)^{1/2},$$

$$w = \tan^{-1}\left(\frac{\xi(b)}{\delta}\right) - \frac{\delta}{2}\log b.$$

The remaining constants are determined by

$$\begin{aligned} &(1)\quad \xi(c) = \delta\,\tan(\tfrac{\delta}{2}\log c + w), &&(2)\quad F_0(a) = G(a) + \varepsilon, \\ &(3)\quad F_0(c) = G(c) - \varepsilon, &&(4)\quad F_0(\infty) = 1. \end{aligned} \tag{3.1}$$

Then

$$B_U \cap [0,\infty) = [a,b], \quad B_L \cap [0,\infty) = [c,d]. \tag{3.2}$$

Minimum information is

$$\begin{aligned} I(F_0;1) = 2[&-\xi^2(a)(G(a) - \frac{1}{2} + \varepsilon) + \delta^2(G(c) - G(b) - 2\varepsilon) \\ &- \xi^2(d)(1 - G(d) + \varepsilon) + \int_{[a,b]\cup[c,d]} J(\xi)(x)dG(x)]. \end{aligned}$$

Proof: The functions χ_0, f_0 are defined in such a way that, as long as (3.1.1) holds, χ_0 and f_0 are continuous. Clearly, χ_0 is then absolutely continuous and bounded. In the notation of S3) of Theorem 3, we have

$$(\lambda_1, \lambda_2, \lambda_3; b_1, a_2, b_2, a_3) = (-\xi^2(a), \delta^2, -\xi^2(d); a, b, c, d).$$

To verify S1) and (3.2), it then suffices if

$$f_0(x) \geq g(x), \quad x \in (0,a) \cup (d,\infty); \quad f_0(x) \leq g(x), \quad x \in (b,c). \tag{3.3}$$

The remaining parts (iii), (iv), of S3) will follow if

$$\delta^2 \geq \max(J(\xi)(b), J(\xi)(c)); \quad b \leq m \leq c. \tag{3.4}$$

Note that $-\xi^2(d) \leq J(\xi)(d)$ and $-\xi^2(a) \leq J(\xi)(a)$ are also necessary to satisfy S3) (iii), since $a_3 = d \in B_L$ and $b_1 = a \in B_U$, but that these inequalities are immediate from Lemma 4.

For the details of the proof that, for sufficiently small ε, there exist a, b, c, d determined by (3.1), and that (3.3), (3.4) are then satisfied as well, see Wu (1990).

□

It is now a fairly straightforward matter to infer the form of the solution for $\varepsilon > \varepsilon_0(G)$. For the "medium ε" and "large ε" forms of F_0 given below, the conditions of Theorem 3 are easily verified, given the existence of the constants. See Wu (1990) for details.

Medium ε : Under G) and J), and for a range $\varepsilon_0(G) \leq \varepsilon \leq \varepsilon_1(G)$, the minimum information $F_0 \in \mathcal{K}_\varepsilon(G)$ assumes one of the two following forms. Both forms occur — see Examples 1 and 2 below.

<u>Form 1</u>:

$$\chi_0(x) = \chi_0(-x) = \begin{cases} \xi(a), & 0 \leq x \leq a; \\ \xi(x), & a \leq x \leq b; \\ \delta \tan(\frac{\delta}{2} \log x + w), & b \leq x \leq d; \\ \delta \tan(\frac{\delta}{2} \log d + w), & d \leq x; \end{cases}$$

$$f_0(x) = f_0(-x) = \begin{cases} g(a)(\frac{a}{x})^{1+\xi(a)}, & 0 \leq x \leq a; \\ g(x), & a \leq x \leq b; \\ \dfrac{bg(b)}{\cos^2(\frac{\delta}{2} \log b + w)} \dfrac{\cos^2(\frac{\delta}{2} \log x + w)}{x}, & b \leq x \leq d; \\ g(d)(\frac{d}{x})^{\delta \tan(\frac{\delta}{2} \log d + w)}, & d \leq x. \end{cases}$$

The constants a, b, d, δ, w are determined by

$$
\begin{aligned}
&(1) \quad \xi(b) = \delta \tan(\tfrac{\delta}{2} \log b + w), \\
&(2) \quad \frac{bg(b)}{dg(d)} = \frac{\cos^2(\frac{\delta}{2} \log b + w)}{\cos^2(\frac{\delta}{2} \log d + w)}, \\
&(3) \quad F_0(a) = G(a) + \varepsilon, \qquad (4) \quad F_0(d) = G(d) - \varepsilon, \qquad (5) \quad F_0(\infty) = 1.
\end{aligned}
\tag{3.5}
$$

Then $B_U \cap [0, \infty) = [a, b]$, $B_L \cap [0, \infty) = \{d\}$. Minimum information is

$$
\begin{aligned}
I(F_0; 1) = 2[&- \xi^2(a)(G(a) - \frac{1}{2} + \varepsilon) + \delta^2(G(d) - G(b) - 2\varepsilon) \\
&- \delta^2 \tan^2(\frac{\delta}{2} \log d + w)(1 - G(d) + \varepsilon) + \int_a^b J(\xi)(x) dG(x)].
\end{aligned}
$$

<u>Form 2</u>:

$$
\chi_0(x) = \chi_0(-x) = \begin{cases} \delta \tan(\frac{\delta}{2} \log a + w), & 0 \le x \le a; \\ \delta \tan(\frac{\delta}{2} \log x + w), & a \le x \le c; \\ \xi(x), & c \le x \le d; \\ \xi(d), & d \le x; \end{cases}
$$

$$
f_0(x) = f_0(-x) = \begin{cases} g(a)(\frac{a}{x})^{\delta \tan(\frac{\delta}{2} \log a + w) + 1}, & 0 < x \le a; \\ \dfrac{ag(a)}{\cos^2(\frac{\delta}{2} \log a + w)} \cdot \dfrac{\cos^2(\frac{\delta}{2} \log x + w)}{x}, & a \le x \le c; \\ g(x), & c \le x \le d; \\ g(d)(\frac{d}{x})^{1+\xi(d)}, & d \le x. \end{cases}
$$

The constants a, c, d, δ, w are determined by

$$
\begin{aligned}
&(1) \quad \xi(c) = \delta \tan(\tfrac{\delta}{2} \log c + w), \\
&(2) \quad \frac{cg(c)}{ag(a)} = \frac{\cos^2(\frac{\delta}{2} \log c + w)}{\cos^2(\frac{\delta}{2} \log a + w)}, \\
&(3) \quad F_0(a) = G(a) + \varepsilon, \quad (4) \quad F_0(c) = G(c) - \varepsilon, \quad (5) \quad F_0(\infty) = 1.
\end{aligned}
\tag{3.6}
$$

Table III. Numerical constants for $\mathcal{K}_\varepsilon(G), G =$ Exponential.

ε	a	b	c	d	δ	w	$I(F_0)$
10^{-4}	.0141	1.3910	2.729	7.360	1.690	−.052	.9940
10^{-3}	.0440	1.1000	3.216	5.421	1.629	−.015	.9631
.003	.0754	.9111	3.570	4.544	1.577	.017	.9227
.005	.0966	.8207	3.775	4.151	1.545	.037	.8896
.006	.1054	.7875	3.856	4.013	1.532	.045	.8746
.0065	.1096	.7725	3.893	3.953	1.525	.049	.8662
.00683	.1122	.7635		3.916	1.522	.051	.8635
.008	.1210	.7331		3.852	1.509	.059	.8323
.01	.1345	.6868		3.763	1.486	.072	.7983
.02	.1860	.5279		3.504	1.389	.116	.7301
.05	.2807	.3116		3.252	1.191	.170	.5131
.0546	.2976	.2920		3.237	1.168	.174	.4893
.1	.2213			3.220	.9742	.187	.3092
.2	.1330			3.535	.6648	.175	.1090
.3	.0751			4.123	.4315	.150	.03091
.4	.0312			5.145	.2341	.114	.0046
.45	.0136			6.127	.1374	.088	8×10^{-4}
.49	.0021			8.269	.0456	.046	1.9×10^{-5}