Sampling with Partial Replacement Extended To Include Growth Projections

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ABSTRACT: The original theory of Sampling with Partial Replacement (SPR) is modified to incorporate growth model projections of the unmatched plot data on the initial measurement. The Modified Sampling with Partial Replacement (MSPR) theory is illustrated by estimating growth and current volume using simulated plot data. In addition to the assumptions required by SPR, MSPR requires that the growth projection errors be random and have a mean of zero. If these assumptions are met, the modified theory improves the precision of the volume and growth estimators over the original SPR estimator. For. Sci. 42(3):328–334.

Additional key words: Continuous forest inventory, tree-growth modeling, forest stand projections, volume estimation, growth estimation.

Ware and Cunia (1962) presented a theory of sampling with partial replacement (SPR). The basic aim of the theory was to provide estimators for current stand volume and growth which improved the precision of the estimates by taking advantage of the correlation between repeated measurements. These estimators were based on the theory of weighted means (Brownlee 1965) and provided a basis for combining unmatched temporary sample plot (TSP) data from two occasions with remeasured permanent sample plot (PSP) data. The improvement in precision came first from a direct increase in sample size and second from exploiting the correlation between the matched PSP and the unmatched TSP on both occasions.

Since the original paper, several extensions of the original theory have appeared in the literature. Cunia (1965) extended the theory of SPR by using multiple regression estimates. This extension showed that using multiple linear regression to estimate the second occasion parameter was more efficient than using the simple linear estimates, as in the original SPR theory. Cunia and Chevrou (1969) extended the theory of SPR to accept measurements on three or more occasions. Scott (1984) clarified the presentation of SPR theory and resolved the issue of appropriate estimators for variance when samples are small. Van Deusen (1989) formulated the SPR estimators using matrix notation and incorporated additivity constraints and estimators for components of growth. Newton et al. (1974) developed multivariate estimators for sampling with partial replacement. This extension allowed for the simultaneous estimation in the change of several forest characteristics for two successive forest measurements. The procedure is considered more efficient because it takes advantage of the inherent correlation that exists between different forest characteristics. Hansen and Hahn (1983) described the use of growth projection simulators in forest inventory using double sampling for regression estimation Hansen (1990) combined the use of a growth model with several sampling methods including a variation of SPR. This paper, independently of Hansen's work, extends the mathematical and statistical presentation of the SPR theory by incorporating growth projections for the temporary plots. It is assumed that the growth simulation model provides unbiased estimates of future volume, subject to additive random error so that the initial and projected values are not perfectly correlated. An application of this approach is presented using simulated plot data.

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The Modified Sampling with Partial Replacement Procedure (MSPR)

Notation for the development of estimators of current volume, growth, and variances follows that of Ware and Cunia (1962). The naming conventions to derive the MSPR estimators are presented in Table 1. The sample volumes from the first occasion are algebraically represented by the letter X. The sample volumes from the second occasion are represented algebraically by the letter Y. Unmatched temporary sample plot volumes from the initial occasion are represented by X_u. New TSP on the second occasion are represented by Y_n. Remeasured (matched) PSP data are algebraically represented on the first occasion by an I_m and on the second occasion by a y_m.

The new source of data for the MSPR procedure is a prediction from a growth model and is represented algebraically by the symbol Z_u. The input data for the growth model is the same TSP data X_u representing the unmatched data on the initial occasion. Since the TSP data are assumed to be a random sample from the population, and the growth model is assumed to provide unbiased projections, the model predictions are random, unbiased estimates of the population stand volume on the second occasion. It is also assumed that the growth model used to make the predictions is an individual tree growth model composed of relationships derived from independent data. As will be seen in the development of estimators, the magnitude of unbiased random errors in the growth model is reflected in the variance of the estimators.

Current Volume Estimator

The MSPR volume estimator $\bar{Y}$ MSPR is a linear combination of the five sample averages $\bar{X}_m, \bar{X}_u, \bar{Y}_m, \bar{Y}_n$ and $\bar{Z}_u$. The coefficients are to be chosen so that the estimator has a minimum variance, subject to the requirement that it be unbiased. Let μ_1 and μ_2 be the true mean volumes on the first and second occasions respectively, so that

$$E\left(\bar{X}_m\right) = E\left(\bar{X}_u\right) = \mu_1 \text{ and } E\left(\bar{Y}_m\right) = E\left(\bar{Y}_n\right) = E\left(\bar{Z}_u\right) = \mu_2$$

Define

$$\beta_{X|Y} = \mathrm{COV}[X_m, Y_m]/\sigma^2(Y_m) \text{ and } \alpha_{X|Y} = \mu_1 - \beta_{X|Y}\mu_2$$

$$\beta_{X|Z} = COV[X_u, Z_u]/\sigma^2(Z_u) \text{ and } \alpha_{X|Z} = \mu_1 - \beta_{X|Z}\mu_2$$

These are the slope and intercept coefficients of the linear regressions of x_m (dependent) on Y_m (independent) and X_u on Z_u. The residual variances around the regression lines are:

$$\sigma^2_{X|Y} = E[\{X_m - \alpha_{X|Y} - \beta_{X|Y}Y_m\}^2] = \sigma^2(X_m)(1-\rho_m^2) \quad (1)$$

$$\sigma^2_{X|Z} = E[\{X_u - \alpha_{X|Z} - \beta_{X|Z}Z_u\}^2] = \sigma^2(X_u)(1-\rho_u^2) \quad (2)$$

where ρ_m and ρ_u are the coefficients of correlation between X_m and Y_m and between X_u and Z_u respectively.

Note that any linear combination of the five sample averages can be written as an equivalent linear combination containing $\bar{Y}_m, \bar{Y}_n, \bar{Z}_u$ and the differences (residuals) $\bar{X}_m - \beta_{X|Y}\bar{Y}_m$ and $\bar{X}_u - \beta_{X|Z}\bar{Z}_u$. We consider these differences, rather than the sample averages $\bar{X}_m$ and $\bar{X}_u$ themselves, for two reasons:

1. These differences and the averages $\bar{Y}_m, \bar{Y}_n, \bar{Z}_u$, are pairwise uncorrelated. This is mathematically convenient, since their variances are then additive. It also allows for straightforward interpretations of the contributions of the individual terms. See the discussion at the end of this section.
2. In the presence of $\bar{Y}_m$ and $\bar{Z}_u$ the averages $\bar{X}_m$ and $\bar{X}_u$ contain additional information about μ_2 only through the differences. This explains why the differences are those from regressions of X_m and X_u on Y_m and Z_u respectively, rather than from Y_m and Z_u on X_m and X_u. Recall that in Ware and Cunia (1962), the results are expressed in terms of the regression of Y_m on X_m. Although expressions analogous to the approach of Ware and Cunia (1962) can be constructed for the MSPR approach, they are much more cumbersome than those presented here.

Upon adjusting the coefficients of an arbitrary linear combination of these three averages and two differences for unbiasedness, we see that it is sufficient to consider only estimators of the form

Table 1. The sample data sources for MSPR sampling.

Initial occasion		Second occasion	
Unmatched TSP	$X_{u1}, X_{u2}, \ldots, X_{ui}, \ldots, X_{uu}$	Matched Model	$Z_{u1}, Z_{u2}, \ldots, Z_{uk}, \ldots, Z_{uu}$
Matched PSP	$X_{m1}, X_{m2}, \ldots, X_{mi}, \ldots, X_{mm}$	Matched PSP	$Y_{m1}, Y_{m2}, \ldots, Y_{mj}, \ldots, Y_{mm}$
		Unmatched TSP	$Y_{n1}, Y_{n2}, \ldots, Y_{nh}, \ldots, Y_{nn}$

$$\bar{Y}_{MSPR} = \alpha_1 \frac{\bar{Y}_m}{\sigma^2(\bar{Y}_m)} + \alpha_2 \frac{\bar{Y}_n}{\sigma^2(\bar{Y}_n)} + \alpha_3 \frac{\bar{Z}_u}{\sigma^2(\bar{Z}_u)} + \alpha_4 \left[\frac{(\bar{X}_m - \beta_{X|Y}\bar{Y}_m) - (\bar{X}_u - \beta_{X|Z}\bar{Z}_u)}{\frac{\sigma^2_{X|Y}}{m} + \frac{\sigma^2_{X|Z}}{u}} \right] \quad (3)$$

where the following restriction ensures unbiasedness:

$$\frac{\alpha_1}{\sigma^2(\bar{Y}_m)} + \frac{\alpha_2}{\sigma^2(\bar{Y}_n)} + \frac{\alpha_3}{\sigma^2(\bar{Z}_u)} + \frac{\alpha_4(\beta_{X|Z} - \beta_{X|Y})}{\frac{\sigma^2_{X|Y}}{m} + \frac{\sigma^2_{X|Z}}{u}} = 1 \quad (4)$$

Equation (3) requires a little explanation. When we first approached the minimization problem, we carried it out without dividing the means by their variances. Upon noticing that the resulting estimate was of the form given in (3), we chose to *start* from this point, in order to make the resulting expressions easier to present. There is of course no loss of generality in this approach, since the variances could have been absorbed into the alphas, at the cost of some additional complexity of presentation. Note also that, in Equations (3) and (4), we are not *assuming* that regression-based estimators are to be used. Rather, we are merely pointing out that any unbiased linear combination of the five averages ($\bar{X}_m, \bar{X}_u, \bar{Y}_m, \bar{Y}_n$ and $\bar{Z}_u$) can be written in this form.

By (4) we have $E[\bar{Y}_{MSPR}] = \mu_2$. We are then to choose $\alpha_1, \ldots, \alpha_4$ so as to minimize

$$\sigma^2(\bar{Y}_{MSPR}) = \frac{\alpha_1^2}{\sigma^2(\bar{Y}_m)} + \frac{\alpha_2^2}{\sigma^2(\bar{Y}_n)} + \frac{\alpha_3^2}{\sigma^2(\bar{Z}_u)} + \frac{\alpha_4^2}{\frac{\sigma^2_{X|Y}}{m} + \frac{\sigma^2_{X|Z}}{u}} \quad (5)$$

subject to (4).

Van Deusen (1989) presents the solution to the SPR problem using matrix notation. Our approach is analogous to theirs since the optimization goal is the same, but retains the benefit that formulas derived for the estimators provide a link to the original presentation of Ware and Cunia (1962). In addition, our notation simplifies the original presentation and allows interpretation of the components of the estimators.

This formulation (5) is a standard variational problem whose solution is obtained with the aid of Lagrange multipliers. The solution is

$$\alpha_1 = \alpha_2 = \alpha_3 = 1/A \text{ and } \alpha_4 = \{\beta_{X|Z} - \beta_{X|Y}\}/A \quad (6)$$

where

$$A = \frac{1}{\sigma^2(\bar{Y}_m)} + \frac{1}{\sigma^2(\bar{Y}_n)} + \frac{1}{\sigma^2(\bar{Z}_u)} + \frac{(\beta_{X|Z} - \beta_{X|Y})^2}{\frac{\sigma^2_{X|Y}}{m} + \frac{\sigma^2_{X|Z}}{u}} \quad (7)$$

The minimum-variance volume estimator is then given by (3), (6), and (7). The minimum variance is obtained by substituting (6) into (5) and simplifying, and is

$$\min \sigma^2(\bar{Y}_{MSPR}) = 1/A \quad (8)$$

The estimate in (3) now admits the following interpretation. The four summands are uncorrelated estimates of μ_2 weighted by terms proportional to the inverses of their variances. The final summand is weighted by the difference between the regression slopes and so has a more dominant influence on the estimate when this difference is large.

Of course $\bar{Y}_{MSPR}$ as presented above is not yet a statistic, the coefficients must be estimated. For this, let $S^2(Y_m), S^2(Y_n)$, and $S^2(Z_u)$ be the sample variances. Let $\hat{\beta}_{X|Y}, \hat{\alpha}_{X|Y}$ and $S^2_{X|Y}$ be the slope, intercept, and residual mean square resulting from a regression of $X_{m1}, \ldots, X_{mm}$ on $Y_{m1}, \ldots, Y_{mm}$ Define $\hat{\beta}_{X|Z}, \hat{\alpha}_{X|Z}$ and $S^2_{X|Z}$ analogously. Then

$$\bar{X}_u - \hat{\beta}_{X|Z}\bar{Z}_u = \hat{\alpha}_{X|Z},$$

$$\bar{X}_m - \hat{\beta}_{X|Y}\bar{Y}_m = \hat{\alpha}_{X|Y}$$

and $\bar{Y}_{MSPR}$ is consistently estimated by

$$\hat{Y}_{MSPR} = \frac{1}{\hat{A}} \left\{ \frac{m\bar{Y}_m}{S^2(Y_m)} + \frac{n\bar{Y}_n}{S^2(Y_n)} + \frac{u\bar{Z}_u}{S^2(Z_u)} + \frac{(\hat{\beta}_{X|Z} - \hat{\beta}_{X|Y})(\hat{\alpha}_{X|Y} - \hat{\alpha}_{X|Z})}{\frac{S^2_{X|Y}}{m} + \frac{S^2_{X|Z}}{u}} \right\} \quad (9)$$

where

$$\hat{A} = \frac{m}{S^2(Y_m)} + \frac{n}{S^2(Y_n)} + \frac{u}{S^2(Z_u)} + \frac{(\hat{\beta}_{X|Z} - \hat{\beta}_{X|Y})^2}{\frac{S^2_{X|Y}}{m} + \frac{S^2_{X|Z}}{u}} \quad (10)$$

Three observations can be made about this formulation of the current volume estimator:

1. The equations above may also be applied to the original SPR problem. They then yield expressions different from, but equivalent to those in Ware and Cunia (1962). For this, note that if $Z_{u1}, \ldots, Z_{uu}$ are not available, then $\hat{\beta}_{X|Z} = 0$,

$\hat{\alpha}_{X|Z} = \bar{X}_u$ and $S^2{}_{X|Z} = S^2(X_u)$. Equations (9) and (10) above then allow formulation of the original SPR ($\hat{Y}_{SPR}$) estimate as

$$\hat{Y}_{SPR} = \frac{1}{\hat{A}_{SPR}}\left\{\frac{m\bar{Y}_m}{S^2(Y_m)} + \frac{n\bar{Y}_n}{S^2(Y_n)} - \frac{\hat{\beta}_{X|Y}(\hat{\alpha}_{X|Y} - \bar{X}_u)}{\frac{S^2{}_{X|Y}}{m} + \frac{S^2(X_u)}{u}}\right\} \quad (11)$$

$$\hat{A}_{SPR} = \frac{m}{S^2(Y_m)} + \frac{n}{S^2(Y_n)} + \frac{\hat{\beta}^2_{X|Y}}{\frac{S^2_{X|Y}}{m} + \frac{S^2(X_u)}{u}} \quad (12)$$

Note that $1/\hat{A}_{SPR}$ is the estimated variance of current volume. For small sample sizes, a correction factor may also be applied to the estimate for variance of the current volume as described by Scott (1984). As sample sizes increase, the effect of the correction factor becomes negligible.

2 If it may be assumed that $\sigma^2(Y_m) = \sigma^2(Y_n)$, then $S^2(Y_m)$ and $S^2(Y_n)$ may each be replaced by the pooled estimate

$$S^2(Y) = \{(m-1)S^2(Y_m) + (n-1)S^2(Y_n)\}/(m+n-2)$$

A similar remark applies to $\sigma^2(X_m)$ and $\sigma^2(X_u)$.

3 From (7) and (8) we see that incorporating information from $Z_{u1}, \ldots, Z_{uu}$ will result in a decrease in the variance of the estimator, relative to that of $\bar{Y}_{SPR}$, insofar as $\sigma^2(\bar{Z}_u)$ is small, $|\beta_{X|Z} - \beta_{X|Y}|$ is large, or $\sigma^2_{X|Z}$ is small relative to $\sigma^2(X_u)$. This latter requirement holds if X_u and Z_u are highly correlated.

Growth Estimator

The derivation of the MSPR growth estimator $\bar{G}_{MSPR}$ parallels that of the volume estimator. We write the estimator in the form

$$\bar{G}_{MSPR} = \alpha_1 \frac{\bar{Y}_m}{\sigma^2(\bar{Y}_m)} + \alpha_2 \frac{\bar{Y}_n}{\sigma^2(\bar{Y}_n)} + \alpha_3 \frac{\bar{Z}_u}{\sigma^2(\bar{Z}_u)} - \alpha_4 \frac{(\bar{X}_m - \beta_{X|Y}\bar{Y}_m)}{\frac{\sigma^2_{X|Y}}{m}} - \alpha_5 \frac{(\bar{X}_u - \beta_{X|Z}\bar{Z}_u)}{\frac{\sigma^2_{X|Z}}{u}}$$

where, in order that $E[\bar{G}_{MSPR}] = \mu_2 - \mu_1$, the coefficients must satisfy the conditions

$$\frac{\alpha_1}{\sigma^2(\bar{Y}_m)} + \frac{\alpha_2}{\sigma^2(\bar{Y}_n)} + \frac{\alpha_3}{\sigma^2(\bar{Z}_u)} + \frac{\alpha_4\beta_{X|Y}}{\frac{\sigma^2_{X|Y}}{m}} + \frac{\alpha_5\beta_{X|Z}}{\frac{\sigma^2_{X|Z}}{u}} = 1$$

and

$$\frac{\alpha_4}{\frac{\sigma^2_{X|Y}}{m}} + \frac{\alpha_5}{\frac{\sigma^2_{X|Z}}{u}} = 1$$

Subject to these conditions we are to minimize

$$\sigma^2(\bar{G}_{MSPR}) = \frac{\alpha_1^2}{\sigma^2(\bar{Y}_m)} + \frac{\alpha_2^2}{\sigma^2(\bar{Y}_n)} + \frac{\alpha_3^2}{\sigma^2(\bar{Z}_u)} + \frac{\alpha_4^2}{\frac{\sigma^2_{X|Y}}{m}} + \frac{\alpha_5^2}{\frac{\sigma^2_{X|Z}}{u}}$$

The solution is most conveniently described in terms of the two quantities

$$B = \frac{1}{\sigma^2(\bar{Y}_m)} + \frac{1}{\sigma^2(\bar{Y}_n)} + \frac{1}{\sigma^2(\bar{Z}_u)}$$

$$C = B\left\{\frac{1}{\frac{\sigma^2_{X|Y}}{m}} + \frac{1}{\frac{\sigma^2_{X|Z}}{u}}\right\} + \frac{\{\beta_{X|Y} - \beta_{X|Z}\}^2}{\left(\frac{\sigma^2_{X|Y}}{m}\right)\left(\frac{\sigma^2_{X|Z}}{u}\right)}$$

and is given by

$$\alpha_1 = \alpha_2 = \alpha_3 = \left\{\frac{1-\beta_{X|Y}}{\frac{\sigma^2_{X|Y}}{m}} + \frac{1-\beta_{X|Z}}{\frac{\sigma^2_{X|Z}}{u}}\right\}/C$$

$$\alpha_4 = \left\{B + \frac{(1-\beta_{X|Z})(\beta_{X|Y} - \beta_{X|Z})}{\frac{\sigma^2_{X|Z}}{u}}\right\}/C$$

$$\alpha_5 = \left\{B + \frac{(1-\beta_{X|Y})(\beta_{X|Z} - \beta_{X|Y})}{\frac{\sigma^2_{X|Y}}{m}}\right\}/C$$

The minimum variance is

$$\min \sigma^2(\overline{G}_{MSPR}) = \left\{ B + \frac{(1-\beta_{X|Y})^2}{\frac{\sigma^2_{X|Y}}{m}} + \frac{(1-\beta_{X|Z})^2}{\frac{\sigma^2_{X|Z}}{u}} \right\} / C$$

The parameters which appear in these expressions may be estimated in the same manner as was described for the volume estimator, by replacing parameters by sample estimates. This results in a consistent estimate $\hat{G}_{MSPR}$ of $\overline{G}_{MSPR}$.

An Illustrative Example

An application of the modified MSPR theory is presented using simulated plot data. Additional estimates using traditional approaches are also included for comparison. The estimators selected for the comparison along with a description of the data sources they utilize is presented in Table 2.

A complete listing of the data and summary statistics used in the illustration are presented in Table 3. The unmatched plot data consisted of 15 TSP representing the initial occasion (X_u) and 16 TSP for the second occasion (Y_n). The matched plots were represented by eight PSP on both occasions (X_m and Y_m). The growth model data (Z_u) consisted of 15 plots obtained by projecting the unmatched TSP (X_u) 5 yr to the second occasion. All of the TSP and PSP used in the illustration were simulated to represent the typical relationships that would exist under normal conditions.

Current Volume

The current mean volume estimates and their estimated variances for the five procedures [with both pooled and separate estimates of $\sigma^2(Y_m)$ and $\sigma^2(Y_n)$] are presented in Table 4. The MSPR sampling method had the lowest estimated variance (35.13). The three more traditional methods, TSP alone, PSP alone, and a combination of TSP and PSP, yielded larger estimated variances, 106.44, 104.60, and 57.67 respectively. The original SPR model performed considerably better than the traditional methods with a variance of 35.59, but not as well as the MSPR model. The mean volume estimates for TSP alone, PSP alone, both TSP and PSP, the original SPR, and the MSPR were 254.70, 263.70, 257.70, 257.61, and 257.18 m^3/ha, respectively. All mean volume estimates are very close with the exception of the PSP, which are slightly higher than the other methods.

Growth

The mean growth estimates and their estimated variances for the five procedures are presented in Table 5. The MSPR sampling method had the lowest estimated variance (0.806). The PSP alone and the original SPR model yielded larger estimated variances, 1.20 and 1.15 respectively. The 5 yr growth estimates varied considerably with the MSPR method estimating growth at 6.15 m^3/ha, PSP at 5.45 m^3/ha, and the original SPR at 5.78 m^3/ha.

Discussion

The new MSPR approach improves the precision of current mean volume estimates and growth estimates over the original SPR procedure. It also resolves the difficulties in estimating the correlation coefficient, a topic that has been the focus of several papers, Scott (1984), Titus (1981), after the original paper was published by Ware and Cunia (1962). The new approach removes the assumption that the variances from the different data sources, representing a given point in time, are equal. However, the new MSPR requires the assumption that growth projections are unbiased.

Increased Precision

The precision will differ between the different estimators because of the variation in the different sample data sets, the correlation of the matched PSP data, the correlation between X_u and Z_u, and the proportion of matched and unmatched plots. In this example, the traditional methods (TSP and PSP) utilize only samples taken from the second occasion, therefore the number of samples is relatively small compared to the MSPR procedures. In other applications, the size of samples would likely be much larger. Furthermore, these samples are independent, and there is no covariance term to reduce the variance. The original SPR has a larger sample size because data is incorporated from both occasions. In addition, the matched PSP data is highly correlated, and exploiting this fact results in a decrease in the variance of the estimator. The MSPR approach increases the sample size by adding more samples (due to growth projections of the TSP) on the second occasion. Also, the TSP projected by a growth model allow for an additional reduction in the variance, again by exploiting the correlation between the initial and projected volumes.

If all assumptions are met, then since the MSPR model contains the other models considered here (SPR, PSP, etc.) the minimum-variance MSPR estimates must necessarily have smaller variances than the estimates constructed using these other methods. This occurs as a consequence of the mathematical minimization process and the addition of the growth estimates as another estimate of current volume (μ_2). The reduction in variance may, or may not, be large, as can be seen from examining the terms that make up the variance estimator.

Table 2. Number of plots and sampling methods for the example data.

Volume estimation sampling method	Number of plots X_u	X_m	Y_m	Y_n	Z_u	Total	Growth estimation sampling method	Number of plots X_u	X_m	Y_m	Y_n	Z_u	Total
TSP	—	—	—	16	—	16	—	—	—	—	—	—	—
PSP	—	—	8	—	—	8	—	—	—	—	—	—	—
TSP and PSP	—	—	8	16	—	24	PSP	—	8	8	—	—	16
Original SPR	15	8	8	16	—	47	Original SPR	15	8	8	16	—	47
Modified SPR	15	8	8	16	15	62	Modified SPR	15	8	8	16	15	62

Table 3. Data and associated summary statistics for the example data

TSP time 1 (X_u)		PSP time 1 (X_m)		PSP time 2 (Y_m)		Model time 2 (Z_u)		TSP time 2 (Y_n)	
Plot no.	Volume (m³/ha)	Plot no.	Volume (m³/ha)	Plot no.	Volume (m³/ha)	Plot no.	Volume (m³/ha)	Plot no.	Volume (m³/ha)
1	228.33	1	213.00	1	219.00	1	232.10	1	235.85
2	172.32	2	278.40	2	281.00	2	175.70	2	326.57
3	322.79	3	286.70	3	290.60	3	326.40	3	183.65
4	248.97	4	266.00	4	270.00	4	276.70	4	240.45
5	294.52	5	230.00	5	240.00	5	297.80	5	304.92
6	195.02	6	225.89	6	234.00	6	203.10	6	268.73
7	282.92	7	276.00	7	277.00	7	287.50	7	246.14
8	242.88	8	290.00	8	298.00	8	247.10	8	210.66
9	308.79					9	315.60	9	231.68
10	244.20					10	252.50	10	297.44
11	262.82					11	272.50	11	242.24
12	252.47					12	259.80	12	309.44
13	227.72					13	230.90	13	224.84
14	195.90					14	200.60	14	291.31
15	245.19					15	251.70	15	206.72
								16	254.54
$\bar{X}_u$	248.32	$\bar{X}_m$	258.25	$\bar{Y}_m$	263.70	$\bar{Z}_u$	255.32	$\bar{Y}_n$	254.70
S^2_{Xu}	1786.21	S^2_{Xm}	927.34	S^2_{Ym}	836.83	S^2_{Zu}	1826.92	S^2_{Yn}	1703.06
$S^2_{\bar{X}u}$	119.08	$S^2_{\bar{X}m}$	115.92	$S^2_{\bar{Y}m}$	104.60	$S^2_{\bar{Z}u}$	121.80	$S^2_{\bar{Y}n}$	106.44
				$\text{cov}(X_m Y_m)$	877.30	$\text{cov}(X_u Z_u)$	1794.55		
				r	0.9959	r	0.9897		

The estimated variances may of course not be ordered in this way, due to sampling variation in the estimates of the coefficients.

If the assumptions are not met, then the estimates may be biased. In the illustrative example, the PSP showed higher volume (263.70) and lower growth (5.45) compared to other methods. This occurrence could be a result of sampling bias, such as intentionally placing PSP in more productive, fully stocked stands. If these nonrandom PSP alone are used to estimate volume or growth, the precision may be high, but bias is present. The MSPR procedure lowers the estimated volume (256.03) and increases the growth estimates (6.18), while further increasing the precision. The MSPR model achieves this because it incorporates the potentially nonrandom PSP data, the random TSP data (from both occasions) and growth model data, thus increasing the sample size and decreasing the overall effect of the potential bias introduced by the PSP data set.

Table 4. Comparison of current mean volume estimates and their variances for different sampling methods.

Sampling method	Volume estimate (m³/ha)	Estimated variance of the estimate
Only TSP	254.70	106.44
PSP	263.70	104.60
TSP and PSP	257.70	57.67
Original SPR (separate)	257.61	35.59
Original SPR (pooled)	256.51	39.02
Modified SPR (separate)	257.18	35.13
Modified SPR (pooled)	256.03	37.98

Correlation Coefficients

Note that in estimating $\sigma^2_{X|Y}$, $\sigma^2_{X|Z}$, $\sigma^2 X_m$, and $\sigma^2 X_u$ we have used the classical estimates—the residual mean squares and the sample variances. Thus if Equations (1) and (2) are to remain valid with all parameters replaced by sample estimates, the only possible estimators of the correlation coefficients ρ_m and ρ_u are the sample correlation coefficients r_m and r_u based on the relevant paired data.

Growth Model Projections

Because most empirical growth models are deterministic in nature, if two different TSP having exactly the same stand characteristics are projected to the second occasion, they will have the same second occasion stand characteristics. If the projection Z_u is a deterministic, linear function of X_u, then X_u and Z_u are perfectly correlated and there is no gain in precision; Z_u is merely a surrogate for X_u. Therefore, the use of growth projections is advantageous only insofar as the projections utilize additional information beyond a linear extrapola-

Table 5. Comparison of growth estimates and their variances for different sampling designs.

Estimation procedure	Growth estimate (m³/ha)	Variance of the estimate
PSP	5.45	1.20
Original SPR (separate)	5.78	1.15
Original SPR (pooled)	5.82	1.16
Modified SPR (separate)	6.15	0.806
Modified SPR (pooled)	6.18	0.809

tion of current volume while retaining the property that model projections are unbiased.

Applicability of the MSPR estimator

From the applied perspective, if the original SPR estimator is already being used, the application of the modified estimator does not require any changes with the exception of using the new estimators and obtaining a suitable growth model. If appropriate growth models are available the decision to change from traditional estimator to the MSPR estimator is appealing because of the potential for gains in precision without the need for additional field data collection.

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