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ON THE BUSY PERIOD DISTRIBUTION OF THE *M/G/2* QUEUEING SYSTEM

DOUGLAS P. WIENS,* *University of Alberta*

Abstract

Equations are derived for the distribution of the busy period of the *GI/G/2* queue. The equations are analyzed for the *M/G/2* queue, assuming that the service times have a density which is an arbitrary linear combination, with respect to both the number of stages and the rate parameter, of Erlang densities. The coefficients may be negative. Special cases and examples are studied.

GI/G/2 QUEUE; LAPLACE TRANSFORMS; MIXTURES OF ERLANG DENSITIES

1. Introduction

In this paper, we study the busy period distribution for the *GI/G/2* and *M/G/2* queueing systems. The queue discipline is that customers receive service in order of their arrival, with each server drawing upon the pool of customers as soon as the server becomes available. The first arrival, at time $t = 0$, finds the queue empty; the busy period ends at the next instant at which both servers become idle.

In Section 2, we derive a system of equations, the solution to which gives the density of the first busy period of the *GI/G/2* queue. Our equations are not as general of those of De Smit (1973a) who generalizes the methods of Pollaczek (1961) to derive integral equations for waiting time and queue length, as well as busy period, distributions for the *GI/G/s* queue. The equations of De Smit have, however, only been solved for the *GI/M/s* (De Smit 1973b) and $GI/H_m/s$ (De Smit 1983) systems. In Section 3, we analyze our equations in the case that the interarrival times are exponentially distributed, and the service times have density

$$(1.1) \qquad g(t)=\sum_{i=1}^{I}\sum_{j=1}^{J}A_{ij}g_{ij}(t); \qquad -\infty<A_{ij}<\infty, \quad \sum_{i,j}A_{ij}=1,$$

where $g_{ij}(t)=(v_i t)^{j-1}v_i\exp(-v_i t)/(j-1)!$. We give a method for obtaining the joint Laplace transform of the busy period length, and generating function of the number of

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* Postal address: Department of Statistics and Applied Probability, University of Alberta, Edmonton, Canada T6G 2G1.
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customers served during the busy period. In Section 4, special cases and examples are analyzed.

Note that (1.1) contains as special cases the Erlang density $E_J(I=1, A_{1j}=\delta_{jJ})$ and the hyperexponential density ($J=1$). Cohen (1982) analyses waiting time distributions for the $M/G/2$ queue, assuming that $g(t)$ is a *convex* combination of negative exponential distributions. Erlang service times are used in Heffer (1969), who analyzes waiting time distributions for the $M/E_k/s$ queue. There have been few exact results published for the *busy period* of the two-server queue. Karlin and McGregor (1958) obtained the busy period distribution for the $M/M/s$ queue, Bhat (1966) for the $GI/M/2$ queue and De Smit (1973b) for the $GI/M/s$ queue. The results of these papers are in decreasing order of explicitness. Hokstad (1978) proposed an *approximation* for the $M/G/s$ busy period distribution, defining the busy period as the period during which *all* servers are busy. For other approximations for waiting time and queue length distributions in the $M/G/s$ queue see Tijms et al. (1981) and references cited therein.

2. Equations for the $GI/G/2$ system

The interarrival times to the system are assumed to be i.i.d., with d.f. $F(t)$, complementary d.f. $\bar{F}=1-F$, p.d.f. $f(t)$, mean $\lambda^{-1}<\infty$. The service times are assumed to be i.i.d., with d.f. $G(t)$, p.d.f. $g(t)$, mean $E[S]<\infty$. We assume that the traffic intensity $\rho=\lambda E[S]/2<1$.

We consider the following random variables:

B = length of the busy period,

N = number of customers served during B,

U_n = time of the nth service completion,

V_n = time of commencement of service of the $(n+2)$th customer,

T_n = time of completion of that service which is in progress at time U_n,

X_n = time of arrival of the $(n+1)$th customer.

Define

$$h_n(t)=\frac{d}{dt}P(B\leqq t, N=n+1). \tag{2.1}$$

We derive a recursion for

$$l_n(t,u,x)=\frac{\delta}{\delta t}\frac{\delta}{\delta u}\frac{\delta}{\delta x}P(T_n\leqq t, U_n\leqq u, X_n\leqq x, N\geqq n+1), \qquad (0\leqq x\leqq u\leqq t<\infty),$$

and then

$$h_n(t)=\int_0^t\int_0^u l_n(t,u,x)\bar{F}(t-x)dxdu. \tag{2.2}$$

By definition, $U_0\equiv 0$, $X_0\equiv 0$ and $T_0\sim G(t)$, so that

$$l_0(t,u,x)=g(t)\delta_0(u)\delta_0(x) \tag{2.3}$$

where $\delta_0(x)$ is Dirac's delta, with all mass at 0.

We obtain l_{n+1} from l_n by considering separately four cases. The events at times V_n, U_{n+1} may be realized by (1) the same server or (2) distinct servers, and the $(n+2)$th customer may (a) be already waiting at time U_n or (b) have not yet arrived at U_n. Considering these cases in the order (1a), (1b), (2a), (2b) gives, for $n \geqq 0$:

$$
\begin{aligned}
l_{n+1}(t,u,x) &= \int_x^u \int_0^x l_n(t,w,z) f(x-z) g(u-w)\,dz\,dw \\
&\quad + g(u-x) \int_0^x \int_0^w l_n(t,w,z) f(x-z)\,dz\,dw \\
&\quad + \int_x^u \int_0^x l_n(u,w,z) f(x-z) g(t-w)\,dz\,dw \\
&\quad + g(t-x) \int_0^x \int_0^w l_n(u,w,z) f(x-z)\,dz\,dw.
\end{aligned} \tag{2.4}
$$

3. Solutions for an $M/G/2$ system

Assume now that the interarrival times are exponentially distributed, so that $f(t) = \lambda e^{-\lambda t}$. Any instant between busy periods is now a renewal point, and so the equations of Section 2 become valid for any busy period, not merely the first. Define

$$m_n(t,u,x) = e^{\lambda x} l_n(t,u,x)$$

so that (2.2), (2.3), (2.4) become

$$h_n(t) = e^{-\lambda t} \int_0^t \int_0^u m_n(t,u,x)\,dx\,du, \tag{3.1}$$

$$m_0(t,u,x) = g(t) e^{\lambda x} \delta_0(u) \delta_0(x) \tag{3.2}$$

$$
\begin{aligned}
&m_{n+1}(t,u,x) \\
&\quad = \lambda \int_x^u \int_0^x m_n(t,w,z) g(u-w)\,dz\,dw \\
&\qquad + \lambda g(u-x) \int_0^x \int_0^w m_n(t,w,z)\,dz\,dw + \lambda \int_x^u \int_0^x m_n(u,w,z) g(t-w)\,dz\,dw \\
&\qquad + \lambda g(t-x) \int_0^x \int_0^w m_n(u,w,z)\,dz\,dw.
\end{aligned} \tag{3.3}
$$

Assume that the service times have a density of the form (1.1). It is *not* assumed that the A_{ij} are non-negative. However, if they are, they must form a probability distribution. The interpretation is then that, with probability A_{ij}, a customer requires service which consists of j i.i.d. stages, each exponentially distributed with rate ν_i. In this case, if one conditions on the type of service, and number of stages remaining in the service, of that customer whose service is under way at time U_n, then it is easy to see that $m_n(t,u,x)$ is necessarily of the form

$$m_n(t,u,x) = \sum_{\substack{1 \leq i \leq I \\ 1 \leq k \leq j \leq J}} A_{ij} g_{ik}(t-u) r_{ijk}^{(n)}(u,x) \tag{3.4}$$

for some functions $r_{ijk}^{(n)}(u,x)$.

In the general case, (3.4) is established by induction. The results are summarized here; the details are in Wiens (1987). Define

$$\hat{r}_{ijk}^{(n)}(\alpha, \gamma) = \int_0^\infty \int_0^u \exp(-\alpha u - \gamma x) r_{ijk}^{(n)}(u, x)\,dx\,du,$$

$$b_{ijk}^{(\alpha)} = A_{ij} \int_0^\infty \exp(-\alpha x) g_{ik}(x)\,dx.$$

Put $L = IJ(J+1)/2$ and build up L-vectors $\boldsymbol{b}(\alpha)$, $\hat{\boldsymbol{r}}^{(n)}(\alpha, \gamma)$ as follows:

$$\boldsymbol{b}_{ij}^T(\alpha) = (b_{ij1}^{(\alpha)}, \cdots, b_{ijj}^{(\alpha)}), \quad \hat{\boldsymbol{r}}_{ij}^{(n)T}(\alpha, \gamma) = (\hat{r}_{ij1}^{(n)}(\alpha, \gamma), \cdots, \hat{r}_{ijj}^{(n)}(\alpha, \gamma)) : 1 \times j,$$

$$\boldsymbol{b}_j^T(\alpha) = (\boldsymbol{b}_{1j}^T(\alpha), \cdots, \boldsymbol{b}_{Ij}^T(\alpha)),$$

$$\hat{\boldsymbol{r}}_j^{(n)T}(\alpha, \gamma) = (\hat{\boldsymbol{r}}_{1j}^{(n)T}(\alpha, \gamma), \cdots, \hat{\boldsymbol{r}}_{Ij}^{(n)T}(\alpha, \gamma)) : 1 \times Ij,$$

$$\boldsymbol{b}^T(\alpha) = (\boldsymbol{b}_1^T(\alpha), \cdots, \boldsymbol{b}_J^T(\alpha)), \quad \hat{\boldsymbol{r}}^{(n)T}(\alpha, \gamma) = (\hat{\boldsymbol{r}}_1^{(n)T}(\alpha, \gamma), \cdots, \hat{\boldsymbol{r}}_J^{(n)T}(\alpha, \gamma)) : 1 \times L.$$

Then with

$$\text{(3.5)} \qquad \boldsymbol{r}(z; \alpha, \gamma) = \sum_{n=0}^{\infty} z^n \hat{\boldsymbol{r}}^{(n)}(\alpha, \gamma) \quad (\mathrm{Re}\,\alpha, \mathrm{Re}(\alpha+\gamma) \geqq 0, \ |z| \leqq 1)$$

we get from (2.1), (3.1), (3.4) that

$$\text{(3.6)} \qquad E[e^{-\varepsilon B} z^{N-1}] = \boldsymbol{b}^T(\varepsilon + \lambda) \boldsymbol{r}(z; \varepsilon + \lambda, 0).$$

From (3.2),

$$\hat{r}_{ijk}^{(0)}(\alpha, \gamma) = \begin{cases} 1, & k = j, \\ 0, & k < j. \end{cases}$$

The induction yields a recursion for $\hat{r}_{ijk}^{(n)}$ which, when substituted into (3.5), gives the equation

$$\text{(3.7)} \qquad \hat{\boldsymbol{r}}^{(0)} = \left[I - \frac{\lambda z}{\gamma} X(\alpha)\right] \boldsymbol{r}(z; \alpha, \gamma) + \frac{\lambda z X(\alpha)}{\gamma} [I - \gamma P^{-1}(\alpha + \gamma)] \boldsymbol{r}(z; \alpha + \gamma, 0).$$

Here, $X(\alpha)$ has entries formed from the Laplace transforms, with parameter α, of the products $g_{ij} g_{ab}$ and $g_{ij} g$. See Section 4 for explicit examples. The matrix $P(\alpha)$ is defined by

$$P(\alpha) = \alpha I + (I - R)D,$$

where

$$D = \bigoplus_{j=1}^{J} \bigoplus_{i=1}^{I} v_i I_j, \quad R = \bigoplus_{j=1}^{J} \bigoplus_{i=1}^{I} R_j,$$

$$R_j = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & 0 & \\ 0 & & & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} : j \times j, \quad (R_1 = 0).$$

To complete the derivation of $E[e^{-\varepsilon B}z^{N-1}]$ we must extract $\boldsymbol{r}(z; \varepsilon+\lambda, 0)$ from (3.7). In the examples considered in Section 4, this is done as follows. We first find the L characteristic roots $\xi_i(\alpha)$ of $X(\alpha)$, and the corresponding left characteristic vectors $\boldsymbol{w}_i^T(\alpha)$. We then show that:

for each $i \leqq L$, and all $\varepsilon \geqq 0$, there is a unique $\alpha_i = \alpha_i(\varepsilon, z) \geqq 0$ satisfying

$$\varepsilon + \lambda = \alpha_i + \lambda z \xi_i(\alpha_i). \tag{3.8}$$

Put $\alpha = \alpha_i(\varepsilon, z)$, $\gamma = \lambda z \xi_i(\alpha_i)$ in (3.7) to get

$$\boldsymbol{w}_i^T(\alpha_i)\hat{\boldsymbol{r}}^{(0)} = \boldsymbol{w}_i^T(\alpha_i)[I - (\varepsilon+\lambda-\alpha_i)P^{-1}(\varepsilon+\lambda)]\boldsymbol{r}(z; \varepsilon+\lambda, 0), \quad 1 \leqq i \leqq L. \tag{3.9}$$

Now with

$$W^T(\varepsilon, z) = (\boldsymbol{w}_1(\alpha_1), \cdots, \boldsymbol{w}_L(\alpha_L)), \quad A(\varepsilon, z) = \operatorname{diag}(\alpha_1, \cdots, \alpha_L), \tag{3.10}$$

(3.9) becomes

$$W(\varepsilon, z)\hat{\boldsymbol{r}}^{(0)} = [W(\varepsilon, z) - (I-R)D + A(\varepsilon, z)W(\varepsilon, z)]P^{-1}(\varepsilon+\lambda)\boldsymbol{r}(z; \varepsilon+\lambda, 0). \tag{3.11}$$

Put $\boldsymbol{c}^T = \boldsymbol{b}^T(\varepsilon+\lambda)P(\varepsilon+\lambda)D^{-1}$. An easy calculation shows that $\boldsymbol{c}$ is independent of $\varepsilon+\lambda$, and is built up as

$$\boldsymbol{c}_{ij}^T = (A_{ij}, 0, \cdots, 0) : 1 \times j; \; \boldsymbol{c}_j^T = (\boldsymbol{c}_{1j}^T, \cdots, \boldsymbol{c}_{Ij}^T) : 1 \times Ij; \; \boldsymbol{c}^T = (\boldsymbol{c}^T, \cdots, \boldsymbol{c}_J^T) : 1 \times L.$$

Then if $W(\varepsilon, z)$ is non-singular, (3.6) and (3.11) now give

$$E[e^{-\varepsilon B}z^{N-1}] = \boldsymbol{c}^T[I - R + W^{-1}(\varepsilon, z)A(\varepsilon, z)W(\varepsilon, z)D^{-1}]^{-1}\hat{\boldsymbol{r}}^{(0)}. \tag{3.12}$$

Even in the singular case, (3.11) appears to give enough information to determine $E[e^{-\varepsilon B}z^{N-1}]$ — see the treatment below of Erlang service times.

We have not obtained a general description of the characteristic roots and vectors of $X(\alpha)$, but remark that one such pair is $(\hat{g}(\frac{1}{2}\alpha), \boldsymbol{b}^T(\frac{1}{2}\alpha))$. For this pair, (3.8) becomes

$$\varepsilon = \alpha + \lambda z \hat{g}(\tfrac{1}{2}\alpha) - \lambda. \tag{3.13}$$

To show the existence of a unique, real such α, it suffices to show that (3.13) defines a function $\varepsilon(\alpha)$ $(\alpha \geqq 0)$, with $\varepsilon(0) \leqq 0$, $\varepsilon(\infty) = \infty$, $\varepsilon'(\alpha) > 0$. The first two of these conditions are immediate, and

$$\varepsilon'(\alpha) = 1 + \frac{\lambda z}{2}\hat{g}'(\tfrac{1}{2}\alpha) \geqq 1 + \frac{\lambda z}{2}\hat{g}'(0) = 1 - \rho z \geqq 1 - \rho > 0.$$

4. Special cases and examples

A. *Hyperexponential service times*. Put $J = 1$ in (1.1) and write $A_i = A_{i1}$, $g_i = g_{i1}$, so that $g(t) = \Sigma_{i=1}^I A_i g_i(t)$, with $g_i(t) = \nu_i \exp(-\nu_i t)$. We find

$$\boldsymbol{c} = (A_1, \cdots, A_I)^T, \quad \hat{\boldsymbol{r}}^{(0)} = (1, \cdots, 1)^T, \quad R = 0, \quad D = \operatorname{diag}(\nu_1, \cdots, \nu_I); \tag{4.1}$$

$$
(4.2)\qquad X_{ij}(\alpha)=\begin{cases}\hat{g}(\alpha+v_i)+A_i\hat{g}_i(\alpha+v_i)=\dfrac{2A_iv_i}{\alpha+2v_i}+\sum\limits_{k\neq i}\dfrac{A_kv_k}{\alpha+v_i+v_k}, & i=j;\\[2ex] A_j\hat{g}_j(\alpha+v_i)=\dfrac{A_jv_j}{\alpha+v_i+v_j}, & i\neq j.\end{cases}
$$

Cohen (1982) studies the matrix $X(\alpha)$ (his $M^T(\alpha)$) and shows that the eigenvalues $\xi_i(\alpha)$ are real.

Example 1. $I=2$, $g(t)=A_1v_1\exp(-v_1t)+A_2v_2\exp(-v_2t)$. With $X(\alpha)$ given by (4.2), the characteristic roots and vectors are

$$
(4.3)\qquad \begin{aligned}\xi_1(\alpha)&=\hat{g}(\tfrac{1}{2}\alpha),\quad \boldsymbol{w}_1^T(\alpha)=\tfrac{1}{2}\boldsymbol{b}^T(\tfrac{1}{2}\alpha)=\left(\frac{A_1v_1}{\alpha+2v_1},\frac{A_2v_2}{\alpha+2v_2}\right),\\ \xi_2(\alpha)&=\frac{A_1v_1+A_2v_2}{\alpha+v_1+v_2},\quad \boldsymbol{w}_2^T(\alpha)=(-(\alpha+2v_1),(\alpha+2v_2)).\end{aligned}
$$

Then $\alpha_1(\varepsilon,z)$ is defined by (3.13), $\alpha_2(\varepsilon,z)$ as the positive solution to (3.8). Equivalently,

$$
(4.4)\qquad \alpha_2^2+[v_1+v_2-\varepsilon-\lambda]\alpha_2-[(\varepsilon+\lambda(1-z))(v_1+v_2)+2\rho zv_1v_2]=0.
$$

Now (4.1) and (4.3) in (3.10) and (3.12) give the result. Define

$$
\psi_1(\alpha_2)=A_1v_1(v_2+\tfrac{1}{2}\alpha_2)(v_2+\alpha_2),\quad \psi_2(\alpha_2)=A_2v_2(v_1+\tfrac{1}{2}\alpha_2)(v_1+\alpha_2).
$$

A calculation yields

$$
(4.5)\qquad \begin{aligned}&E[e^{-\varepsilon B}z^{N-1}]\\ &=\frac{\hat{g}(\tfrac{1}{2}\alpha_1)(v_1+\tfrac{1}{2}\alpha_1)(v_2+\tfrac{1}{2}\alpha_1)(\psi_1(\alpha_2)+\psi_2(\alpha_2))+A_1v_1A_2v_2(v_1-v_2)^2\tfrac{1}{2}\alpha_1}{(v_1+\alpha_1)(v_2+\tfrac{1}{2}\alpha_1)\psi_1(\alpha_2)+(v_2+\alpha_1)(v_1+\tfrac{1}{2}\alpha_1)\psi_2(\alpha_2)}.\end{aligned}
$$

Put $\kappa=\alpha_2(0,1)$; then

$$
(4.6)\qquad \begin{aligned}E[B]&=\frac{E(S)}{1-\rho}\left\{1+\frac{[A_2\psi_1(\kappa)-A_1\psi_2(\kappa)][v_1^{-1}-v_2^{-1}]-A_1A_2(v_1-v_2)^2}{2E[S][\psi_1(\kappa)+\psi_2(\kappa)]}\right\},\\ E[N]&=\frac{1}{1-\rho}+\frac{\rho[v_2\psi_1(\kappa)+v_1\psi_2(\kappa)-A_1v_1A_2v_2(v_1-v_2)^2]}{E[S](1-\rho)v_1v_2[\psi_1(\kappa)+\psi_2(\kappa)]}.\end{aligned}
$$

In the important special case that $g(t)$ is the convolution of independent exponential densities with rates v_1 and v_2, so that $A_1=v_2/(v_2-v_1)=1-A_2$, we have $\kappa=\lambda$ and (4.6) becomes

$$
(4.7)\qquad \begin{aligned}E[B]&=\frac{E[S]}{1-\rho}\left\{1+\frac{\rho^2}{(v_1^{-1}+v_2^{-1})^2[v_1^2+(2+3\rho)v_1v_2+v_2^2]}\right\},\\ E[N]&=\frac{1}{1-\rho}+\frac{\rho(v_1+v_2+\tfrac{1}{2}\lambda)(v_1+v_2+\lambda)}{(1-\rho)(v_1+v_2)(v_1+v_2+\tfrac{3}{2}\lambda)}.\end{aligned}
$$

B. *Erlang service times*. Put $I=1$, $A_{1j}=\delta_{j,J}$, $v_J=\nu$ in (1.1), so that $g(t)=(\nu t)^{J-1}\nu e^{-\nu t}/(J-1)!$. Then

$$X(\alpha)=\begin{bmatrix} S_1(\alpha) & & & & U_1(\alpha) \\ & S_2(\alpha) & 0 & & U_2(\alpha) \\ 0 & & \ddots & & \vdots \\ & & & S_{J-1}(\alpha) & U_{J-1}(\alpha) \\ & & & & S_J(\alpha)+U_J(\alpha) \end{bmatrix},$$

where S_j and U_j are $j\times j$ matrices with (a, b)th elements

$$\binom{J+b-a-1}{b-a}\left(\frac{\nu}{\alpha+2\nu}\right)^{J+b-a} 1_{\{a\le b\}} \quad \text{and} \quad \binom{j+b-a-1}{j-a}\left(\frac{\nu}{\alpha+2\nu}\right)^{j+b-a}$$

respectively. The roots $\xi(\alpha)$ of $X(\alpha)$ are $(\nu/(\alpha+2\nu))^J$, repeated $\frac{1}{2}J(J-1)$ times, and the J roots of $S_J(\alpha)+U_J(\alpha)$. There is only one characteristic vector belonging to $\xi(\alpha)=(\nu/(\alpha+2\nu))^J$, so that $W(\varepsilon, z)$ has rank $J+1$ and (3.12) does not apply. However, since

$$\boldsymbol{b}^T(\alpha)=\left(\boldsymbol{0}^T, \frac{\nu}{\nu+\alpha}, \cdots, \left(\frac{\nu}{\nu+\alpha}\right)^J\right)=(\boldsymbol{0}^T, \boldsymbol{b}_J^T(\alpha)),$$

say, $E[e^{-\varepsilon B}z^{N-1}]$ depends only on the last J elements of $\boldsymbol{r}(z;\varepsilon+\lambda, 0)$. Denote these by $\boldsymbol{r}_J(z;\varepsilon+\lambda,0)$ so that $E[e^{-\varepsilon B}z^{N-1}]=\boldsymbol{b}_J^T(\varepsilon+\lambda)\boldsymbol{r}_J(z;\varepsilon+\lambda,0)$. Let $\xi_1(\alpha),\cdots,\xi_J(\alpha)$ be the roots of $S_J(\alpha)+U_J(\alpha)$, with corresponding left characteristic vectors $\boldsymbol{y}_1^T(\alpha),\cdots,\boldsymbol{y}_J^T(\alpha)$. Put $\boldsymbol{w}_i^T(\alpha)=(\boldsymbol{0}^T, \boldsymbol{y}_i^T(\alpha)): 1\times L$ so that $\boldsymbol{w}_i^T(\alpha)$ is a characteristic vector of $X(\alpha)$ belonging to $\xi_i(\alpha)$. Then (3.9) becomes

$$\boldsymbol{y}_i^T(\alpha_i)\boldsymbol{\hat{r}}_J^{(0)}=\boldsymbol{y}_i^T(\alpha_i)[I-(\varepsilon+\lambda-\alpha_i)P_J^{-1}(\varepsilon+\lambda)]\boldsymbol{r}_J(z;\varepsilon+\lambda,0), \quad i=1,\cdots,J;$$

where $\boldsymbol{\hat{r}}_J^{(0)}=(0,\cdots,0,1)^T: J\times 1$, $P_J(\varepsilon+\lambda)=(\varepsilon+\lambda)I+\nu(I-R_J)$, and the α_i are determined from (3.8). Put

$$W_J^T(\varepsilon, z)=(\boldsymbol{y}_1(\alpha),\cdots,\boldsymbol{y}_J(\alpha_J)), \quad A_J(\varepsilon, z)=\operatorname{diag}(\alpha_1,\cdots,\alpha_J).$$

Then, as in the passage from (3.9) to (3.12), we get

$$E[e^{-\varepsilon B}z^{N-1}]=\boldsymbol{c}_J^T[I-R_J+\nu^{-1}W_J^{-1}(\varepsilon,z)A_J(\varepsilon,z)W_J(\varepsilon,z)]^{-1}\boldsymbol{\hat{r}}^{(0)},$$

where $\boldsymbol{c}_J^T=(1,0,\cdots,0): 1\times J$. Thus

(4.8) $E[e^{-\varepsilon B}z^{N-1}]$ *is the* $(1, J)$th *element of*

$$[I-R_J+\nu^{-1}W_J^{-1}(\varepsilon,z)A_J(\varepsilon,z)W_J(\varepsilon,z)]^{-1}.$$

It is shown in Wiens (1987) that the $\xi_i(\alpha)$ are of the form $c_i\hat{g}(\frac{1}{2}\alpha)$, with c_i real, $|c_i|\leqq 1$. This implies the existence and uniqueness of solutions to (3.8), in the same manner that (3.13) was handled.

Example 2: $J=2$, $g(t)=\nu^2 te^{-\nu t}$. We have $S_2+U_2=\frac{1}{4}\hat{g}(\frac{1}{2}\alpha)E^{-1}(\alpha)KE(\alpha)$, where

$$E(\alpha) = \operatorname{diag}\left(\frac{\nu}{\alpha + 2\nu}, \left(\frac{\nu}{\alpha + 2\nu}\right)^2\right)$$

and $K = \binom{2\ 4}{1\ 2}$. Then

$$W = \begin{bmatrix} \dfrac{\nu}{\alpha_1 + 2\nu} & 2\left(\dfrac{\nu}{\alpha_1 + 2\nu}\right)^2 \\ \dfrac{\nu}{\alpha_2 + 2\nu} & -2\left(\dfrac{\nu}{\alpha_2 + 2\nu}\right)^2 \end{bmatrix},$$

where $\alpha_2 = \varepsilon + \lambda$ and α_1 is obtained from (3.13). From (4.8), $E[e^{-\varepsilon B}z^{N-1}]$ is the (1,2)th element of

$$\left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \nu^{-1}W^{-1}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} W\right]^{-1}.$$

The calculations give

$$E[e^{-\varepsilon B}z^{N-1}] = \frac{2\nu^2[3\varepsilon + 3\lambda + 4\nu - \alpha_1]}{\alpha_1^2[3\varepsilon + 3\lambda + 4\nu] + \alpha_1[14\nu^2 + 12\nu(\varepsilon + \lambda) + (\varepsilon + \lambda)^2] + 2\nu^2[3\varepsilon + 3\lambda + 4\nu]} \tag{4.9}$$

with

$$E[B] = \frac{E[S]}{1-\rho}\left(1 + \frac{\rho^2}{4(3\rho + 4)}\right), \qquad E[N] = \frac{\rho^3 + 6\rho^2 + 14\rho + 8}{2(3\rho + 4)(1 - \rho)}. \tag{4.10}$$

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